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On the Mutual Connectedness of Elements in Lattices

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In this paper we shall first in § 1 extend the notion of the connected sets in topological spaces to the case for the lattices with a binary relation, next in §§ 2 and 3 prove the fundamental theorems concerning this notion, and finally in § 4 discuss the relations between this binary relation and a mapping in a complete lattice.

§ 1. Definitions and axioms

Let L be a lattice. Given in L a binary relation γ, we shall write $x\gamma y$ to express the fact that the elements x and y of L are in the γ-relation in this order. We shall not assume that $x\gamma y$ implies $y\gamma x$. And $x\rho y$ means that $x\gamma y$ and $y\gamma x$ simultaneously. Next we shall define ρ-irreducible elements of L as follows.

Definition. *An element a of L is said to be ρ-irreducible, if a is not expressible as $a=x\smile y$, $x\rho y$* and *$x, y\neq a$.*

This notion is an extension of that of connected sets in topological spaces.

As the axioms concerning this γ-relation, we shall consider the following conditions:[1]

(I) If $z\leqq x\smile y$ and $x\rho y$, then $(x\smile y)\frown z=(x\frown z)\smile(y\frown z)$.

(II) If $x\gamma y$, $x_1\leqq x$ and $y_1\leqq y$, then $x_1\gamma y_1$.

(III_1) If $x\gamma y_1$ and $x\gamma y_2$, then $x\gamma(y_1\smile y_2)$.

(III_2) If $x_1\gamma y$ and $x_2\gamma y$, then $(x_1\smile x_2)\gamma y$.

(IV) If $x\rho x$, then $x=0$. (0 denotes the zero element of L).

(IV*) If $x\leqq y$ and $x\rho y$, then $x=0$.

Remark. It is obvious that (IV) implies (IV*). Under the axiom (II), (IV*) implies (IV). For, by (II) we have $(x\frown y)\rho(x\frown y)$; and since $x\leqq y$, i.e.,

1) This concept corresponds to that of "Verkettung" which has been proposed by F. Riesz as a primitive concept of abstract space. F. Riesz, *Stetigkeitsbegriff und abstrakte Mengenlehre,* Atti del IV Congresso Internazionale dei Matematici, 2 (Rome, 1909), pp. 18-24.

$x \frown y = x$, hence $x \rho x$, and therefore by (IV) we have $x=0$. That is, under the axiom (II), (IV*) is equivalent to (IV).

§ 2. Theorem of addition

In this section we shall prove the Theorem of addition.

Lemma 1.[1] *Under the axioms (I) and (II); if a is a ρ-irreducible element of L, then* $a \leqq x \smile y$, $x \rho y$ *implies* $a \leqq x$ *or* $a \leqq y$.

Proof. By (I) the hypothesis implies $a = a \frown (x \smile y) = (a \frown y) \smile (a \frown y)$, and by (II) $x \rho y$ implies $(a \frown x) \rho (a \frown y)$. Hence, since a is ρ-irreducible, $a = a \frown x$ or $a = a \frown y$, that is to say $a \leqq x$ or $a \leqq y$.

Lemma 2. Under the axioms (I), (II), (III_1) and (III_2); if a is a ρ-irreducible element of L, then $a \leqq x_1 \smile x_2 \smile \cdots \smile x_n$, $x_i \rho x_j$ $(i \neq j)$ implies $a \leqq x_i$ for some i.

Proof. By (III_1) and (III_2), $x_i \rho x_j (i \neq j)$ implies $x_1 \rho (x_2 \smile \cdots \smile x_n)$; hence by Lemma 1, $a \leqq x_1$ or $a \leqq x_2 \smile \cdots \smile x_n$. Therefore $a \leqq x_i$ for some i.

Theorem 1. (Theorem of addition).[2] *Under the axioms (I) and (II); if a set A of ρ-irreducible elements of L contains an element* a_0 *such that* $a_0 \bar{\gamma} a$ *for every* $a \varepsilon A$ ($\bar{\gamma}$ *denotes the negation of* γ), *then the join* $\bigvee (a;\ a \varepsilon A)$ *(if it exists) is ρ-irreducible.*

Proof. Let us suppose that $\bigvee(a;\ a \varepsilon A) = x \smile y$, $x \rho y$, then we have $a \leqq x \smile y$, $x \rho y$ for every $a \varepsilon A$. Since a is ρ-irreducible, by Lemma 1 it results $a \leqq x$ or $a \leqq y$ for each $a \varepsilon A$. In particular, for a_0 we have $a_0 \leqq x$ or $a_0 \leqq y$; therefore we can suppose that $a_0 \leqq y$. Hence by (II) $x \rho y$ implies $a_0 \gamma x$. If $a_1 \leqq x$ for some $a_1 \varepsilon A$, then by (II) $a_0 \gamma x$ implies $a_0 \gamma a_1$, which contradicts the hypothesis that $a_0 \bar{\gamma} a$ for every $a \varepsilon A$. Therefore $a \leqq y$ for every $a \varepsilon A$; and hence $\bigvee(a;\ a \varepsilon A) \leqq y$. On the other hand, from $\bigvee(a; a \varepsilon A) = x \smile y$ it follows $\bigvee(a;\ a \varepsilon A) \geqq y$; hence we have $\bigvee(a;\ a \varepsilon A) = y$. Therefore $\bigvee(a;\ a \varepsilon A)$ is ρ-irreducible. This completes the proof.

§ 3. Theorem of decomposition

In this section we shall prove the Theorem of decomposition.

Lemma 3. *The axiom (IV*) is a necessary and sufficient condition that* (α) *be equivalent to* (β):

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1) This corresponds to Theorem III in B. Knaster and C. Kuratowski, *Sur les ensembles connexes*, Fund. Math., vol. 2 (1921), p. 209.

2) This theorem corresponds to Corollary V in B. Knaster and C. Kuratowski, loc. cit. p. 210.

(α) *an element a of L is ρ-irreducible,*

(β) *an element a of L is not expressible as $a=x\smile y$, $x\,\rho\,y$ and $x,y\neq 0$.*

Proof. (α) is equivalent to that a is not expressible as $a=x\smile y$, $x\,\rho\,y$, $x\ngeq y$ and $y\ngeq x$. For $a=x\smile y\neq x$ is equivalent to $x\ngeq y$. It is obvious that (β) implies (α). The necessary and sufficient condition that (α) imply (β) is that if $x\,\rho\,y$ and $x\leqq y$ then $x=0$; that is to say, the condition is the axiom (IV*).

Remark. Under the axiom (II); (IV*) is equivalent to (IV). (Cf. Remark in § 1.) Under the axiom (II) and (IV); if there exist two elements x and y such that $x\,\rho\,y$, then the lattice contains the zero element. For by (II) $x\,\rho\,y$ implies $(x\frown y)\,\rho\,(x\frown y)$. If L does not contain 0, the axiom (IV) means that $z\,\bar{\rho}\,z$ for $z\in L$; hence, in particular, $(x\frown y)\,\bar{\rho}\,(x\frown y)$. This is a contradiction. That is, either L contains 0 or $x\,\bar{\rho}\,y$ for all $x,y\in L$. In the latter case, every element of L is ρ-irreducible.

In the following we shall assume that L contains the zero element.

Lemma 4.[1] *Under the axioms (II) and (IV); if an element a is not expressible as the join of m ρ-irreducible elements, then there exist $m+1$ elements $x_1, x_2, \cdots, x_{m+1}$ such that*

$$a=x_1\smile x_2\smile \cdots \smile x_m,\ x_i\,\rho\,x_j,\ (i\neq j),\ x_i\neq 0,\ i,j=1,2,\cdots,m.$$

Proof. If $m=1$, then this assertion is obviously valid by Lemma 3. Suppose that this lemma holds for $m-1$, then we have

$$a=x_1\smile x_2\smile \cdots \smile x_m,\ x_i\,\rho\,x_j,\ (i\neq j) \text{ and } x_i\neq 0,\ i,j=1,2,\cdots,m.$$

Here, by the hypothesis, one of x_i, say x_m, is not ρ-irreducible; hence there exist two elements x_m^* and x_{m+1} such that $x_m=x_m^*\smile x_{m+1}$, $x_m^*\,\rho\,x_{m+1}$, and x_m^*, $x_{m+1}\neq 0$. Thus this lemma is proved by the mathematical induction.

Theorem 2.[2] **(Theorem of decomposition).** *Under the axioms (I), (II), (III$_1$), (III$_2$) and (IV); if $a, a_1, a_2, \cdots, a_m$ are ρ-irreducible elements of L such that*

$$a=(a_1\smile a_2\smile\cdots\smile a_m)\smile(b_1\smile b_2\smile\cdots\smile b_n),\ a_k\neq 0,\ b_i\neq 0, \text{ and } b_i\,\rho\,b_j\ (i\neq j),$$
$$k=1,2,\cdots,m;\ i,j,=1,2,\cdots,n,$$

1) This corresponds to Theorem 6 in C. Kuratowski, Topologie II, Warsaw 1950, p. 84.

2) This theorem is an extension of both Theorem 7 (in C. Kuratowski, loc. cit., p. 84) and Theorem 9.9 (in R. L Wilder, Topology of Manifolds, New York, 1949, Amer. Math. Soc. Colloq. Publ., vol. 32, p. 20).

then the elements $a_1 \smile a_2 \smile \cdots \smile a_m \smile b_i$ $(i=1,2,\cdots,n)$ *are the joins of* m ρ*-irreducible elements. (distinct or not).*

Proof. Suppose, on the contrary, that $a_1 \smile a_2 \smile \cdots \smile a_m \smile b_{i_0}$ for some i_0 is not expressible as the join of m ρ-irreducible elements of L. Then by Lemma 4, there exists a decomposition such that

$$a_1 \smile a_2 \smile \cdots \smile a_m \smile b_{i_0} = x_1 \smile x_2 \smile \cdots \smile x_{m+1},\ x_i \neq 0 \text{ and } x_i \,\rho\, x_j\ (i \neq j),\ i,j=1,2,\cdots,m+1. \tag{3.1}$$

First we shall prove that

$$x_j \,\bar{\rho}\, \bigvee(b_i\,;\, i \neq i_0) \qquad \text{for } j=1,2,\cdots,m+1. \tag{3.2}$$

If, on the contrary, $x_{j_0} \,\rho\, \bigvee(b_i\,;\, i=i_0)$ for some j_0, then, since $x_{j_0} \,\rho\, x_j$ for $j \neq j_0$, by (III_1) and (III_2) we have

$$x_{j_0} \,\rho \Big(\bigvee(x_j\,;\, j \neq j_0) \smile \bigvee(b_i\,;\, i \neq i_0)\Big). \tag{3.3}$$

By (3.1) the hypothesis implies that

$$a = x_{j_0} \smile \Big(\bigvee(x_j\,;\, j \neq j_0) \smile \bigvee(b_i\,;\, i \neq i_0)\Big),\ x_j \neq 0,\ j=1,\cdots,m+1. \tag{3.4}$$

This fact ((3.3) and (3.4)) contradicts the hypothesis that a is a ρ-irreducible element. Hence we have (3.2).

Next we shall prove that

$$b_{i_0} \ngeqq x_j \qquad \text{for } j=1,2,\cdots,m+1. \tag{3.5}$$

By (III_1) and (III_2), the hypothesis $b_{i_0} \,\rho\, b_i (i \neq i_0)$ implies $b_{i_0} \,\rho \bigvee(b_i\,;\, i \neq i_0)$. Here, if $x_{i_1} \leqq b_{i_0}$ for some j_1, then by (II) we have $x_{j_1} \,\rho \bigvee(b_i\,;\, i \neq i_0)$, contradicting (3.2). Therefore (3.5) is valid.

Furthermore, from (3.1) it follows that

$$x_j \leqq a_1 \smile a_2 \smile \cdots \smile a_m \smile b_{i_0}. \tag{3.6}$$

By using of (I),[1] we have from (3.6)

$$x_j = x_j \frown x_j \leqq (a_1 \frown x_j) \smile \cdots \smile (a_m \frown x_j) \smile (b_{i_0} \frown x_j)\,; \tag{3.7}$$

1) Under the axioms (I), (III_1) and (III_2) we can prove by the mathematical induction that if $z \leq x_1 \smile x_2 \smile \cdots \smile x_n$ and $x_i \,\rho\, x_j\ (i \neq j)$, then $(x_1 \smile x_2 \smile \cdots \smile x_n) \frown z = (x_1 \frown z) \smile \cdots \smile (x_n \frown z)$. For it is obvious for $n=2$ by (I); next by (III_1) and (III_2), we have, $x_1 \,\rho\, (x_2 \smile \cdots \smile x_n)$, so by (I) $(x_1 \smile \cdots \smile x_n) \frown z = (x_1 \frown z) \smile ((x_2 \smile \cdots \smile x_n) \frown z)$. If we write, $z' = (x_2 \smile \cdots \smile x_n) \frown z$, then $z' \leq x_2 \smile \cdots \smile x_n$; hence by the hypothesis of the mathematical induction we have $(x_2 \smile \cdots \smile x_n) \frown z' = (z_2 \frown z') \smile \cdots \smile (x_n \frown z')$. And since $x_2 \frown z' = x_2 \frown z, \cdots, z_n \frown z' = x_n \frown z$. this assertion is proved.

2) B. Knaster and C. Kratowski, loc. cit., pp. 206-255. C. Kuratowski, loc. cit., pp. 79-160. R. L. Wilder, loc. cit., pp. 16-27.

if $a_1 \frown x_j = \cdots = a_m \frown x_j = 0$, then from (3.7) we have $x_j \leqq b_{i_0} \frown x_j$, i. e., $x_j \leqq b_{i_0}$, contradicting (3.5). Therefore there exists at least one element a_p for each x_q such that $a_p \frown x_q \neq 0$.

Moreover, from (3.1) we have

$$a_p \leqq x_1 \smile x_2 \smile \cdots \smile x_{m+1}, \qquad x_i \,\rho\, x_j \, (i \neq j); \tag{3.8}$$

since a_p is ρ-irreducible, by Lemma 2 we have $a_p \leqq x_{i_1}$ for some i_1. And $x_i \,\rho\, x_{i_1}$ for $i \neq i_1$, hence by (III) $(x_i \frown x_{i_1}) \,\rho\, (x_i \frown x_{i_1})$; by (IV) we have $x_i \frown x_{i_1} = 0$. Since $a_p \leqq x_{i_1}$ implies $a_p \frown x_i \leqq x_{i_1} \frown x_i$, we have $a_p \frown x_i = 0$ for $i \neq i_1$. Therefore $a_p \frown x_q \neq 0$ implies $a_p \leqq x_q$. Since $a_p \neq 0$, we have $a_p \neq x_r$ for $r \neq q$. But q runs over 1, 2,$\cdots$, $m+1$ and p runs over 1, 2,$\cdots$, m. This is obviously a contradiction. Thus the proof is completed.

Thus we can obtain the two fundamental theorems—Theorem of addition and Theorem of decomposition—on ρ-irreducible elements of L, which correspond to the fundamental theorems in the theory of connected sets in topological spaces. And since many of the theorems in the classical theory of connected sets result from the fundamental theorems, we can regard the axioms (I), (II), (III_1), (III_2) and (IV) as the axioms concerning the mutual connectedness of elements in lattices, where $x \bar{\gamma} y$ means that x is connected with y in this order.

§ 4. Relations between a binary relation and a mapping in a complete lattice

Let L be a complete lattice with a binary relation γ.

Theorem 3. *There exists a mapping $x \rightarrow x^\alpha$ in L such that the relation $x \gamma y$ is given by $y \leqq x^\alpha$, if and only if the relation γ satisfies the following condition:*

$$(III_1^*) \quad \textit{If } y \leqq \bigvee (u;\ x \gamma u), \textit{ then } x \gamma y.$$

And then the mapping is uniquely determined as follows:

$$x \rightarrow x^\gamma = \bigvee (u;\ x \gamma u).$$

Remark. Under the axiom (II); (III^*) is equivalent to

(III_{1a}) If $x \gamma u$ for every $u \in A$, (A is a subset of L); then $x \gamma \bigvee (u;\ u \in A)$.

Proof. Suppose that the relation $x \gamma y$ is given by $y \leqq x^\alpha$. If $y \leqq \bigvee (u;\ x \gamma u)$, then $y \leqq \bigvee (u;\ u \leqq x^\alpha)$; and since $\bigvee (u;\ u \leqq x^\alpha) \leqq x^\alpha$, hence we have $y \leqq x^\alpha$, that is to say, $x \gamma y$. Therefore the relation satisfies (III_1^*).

Conversely, suppose that the relation γ satisfies (III_1^*); if we define $x^\gamma = \bigvee(u;\ x\gamma u)$, then $x\gamma y$ is given by $y \leqq x^\gamma$. For $y \leqq x^\gamma$ is equivalent to $y \leqq \bigvee(u;\ x\gamma u)$; by ($\mathrm{III}_1^*$), $y \leqq \bigvee(u;\ x\gamma u)$ implies $x\gamma y$. Conversely if $x\gamma y$, then $y \leqq \bigvee(u;\ x\gamma u)$, i. e., $y \leqq x^\gamma$. Therefore $y \leqq x^\gamma$ is equivalent to $x\gamma y$.

If $x\gamma y$ is given by $y \leqq x^\alpha$, then $y \leqq x^\gamma$ is equaivalent to $y \leqq x^\alpha$, that is, $y \leqq x^\gamma$ if and only if $y \leqq x^\alpha$. Hence $x^\alpha = x^\gamma$ for every $x \in L$. Thus this theorem is completely proved.

Theorem 4. *Under the axiom (III_1^*), let $x \to x^\gamma$ be such a mapping in L that $x\gamma y$ is given by $y \leqq x$. Then*

(1) *$x_1 \leqq x$ implies $x_1^\gamma \geqq x^\gamma$, if and only if the relation γ satisfies (II).*

(2) *$(x_1 \smile x_2)^\gamma \geqq x_1^\gamma \frown x_2^\gamma$, if and only if the relation γ satisfies (III_2).*

Proof. (1) $x\gamma y$ is equivalent to $y \leqq x^\gamma$, (by Theorem 3). If the relation γ satisfies (II), then $x_1 \leqq x$, $x\gamma y$ implies $x_1\gamma y$, that is, $y \leqq x^\gamma$ implies $y \leqq x_1^\gamma$; therefore we have $x^\gamma \leqq x_1^\gamma$. Conversely, if $x_1 \leqq x$ implies $x_1^\gamma \geqq x^\gamma$, then $y \leqq x^\gamma$ implies $y \leqq x_1^\gamma$, that is, $x\gamma y$, $x_1 \leqq x$ implies $x_1\gamma y$.

(2) Suppose that the relation γ satisfies (III_2), then $x_1\gamma y$, $x_2\gamma y$ implies $(x_1 \smile x_2)\gamma y$; hence $y \leqq x_1^\gamma$, $y \leqq x_2^\gamma$ implies $y \leqq (x_1 \smile x_2)^\gamma$. Therefore we have $(x_1 \smile x_2)^\gamma \geqq x_1^\gamma \frown x_2^\gamma$. Conversely if $(x_1 \smile x_2)^\gamma \geqq x_1^\gamma \frown x_2^\gamma$, then $y \leqq x_1^\gamma$, x_2^γ implies $y \leqq (x_1 \smile x_2)^\gamma$; hence $x_1\gamma y$, $x_2\gamma y$ implies $(x_1 \smile x_2)\gamma y$. Thus the theorem is proved.

Theorem 5. *There exists such a mapping $x \to x^\alpha$ in L that the relation $x\gamma y$ is given by $y \leqq x^\alpha$, $x \leqq y^\alpha$; if and only if the relation γ satisfies the following condition:*

(*III**) *If $y \leqq \bigvee(u;\ x\gamma u)$ and $x \leqq \bigvee(v;\ v\gamma y)$, then $x\gamma y$.*

(*V*) *If $x\gamma y$, then $y\gamma x$.*

And then one of the mapping $x \to x^\alpha$ is given by $x \to x^\gamma = \bigvee(u;\ x\gamma u)$; let $x \to x^\alpha$ be any mapping such that $x\gamma y$ is given by $y \leqq x^\alpha$, $x \leqq y^\alpha$, then $x^\gamma \leqq x^\alpha$.

Proof. Suppose that the relation $x\gamma y$ is given by $y \leqq x^\alpha$, $x \leqq y^\alpha$, then (V) is obviously valid. If $y \leqq \bigvee(u;\ x\gamma u)$, $x \leqq \bigvee(v;\ v\gamma y)$, then $y \leqq \bigvee(u;\ u \leqq x^\alpha, x \leqq u^\alpha)$, $x \leqq \bigvee(v; y \leqq v^\alpha, v \leqq y^\alpha)$; and obviously $\bigvee(u;\ u \leqq x^\alpha) \leqq x^\alpha$, $\bigvee(v;\ v \leqq y^\alpha) \leqq y^\alpha$. Hence we have $y \leqq x^\alpha$, $x \leqq y^\alpha$, that is to say, $x\gamma y$. Therefore we see that the relation γ satisfies (III*).

Conversely, suppose that the relation γ satisfies (III*) and (V), and if we define, as before, $x^\gamma = \bigvee(u;\ x\gamma u)$ then, $x\gamma y$ is given by $y \leqq x^\gamma$, $x \leqq y^\gamma$. Indeed, since $y \leqq x^\gamma$ is equivalent to $y \leqq \bigvee(u;\ x\gamma u)$, hence $y \leqq x^\gamma$, $x \leqq y^\gamma$ is equivalent to $y \leqq \bigvee(u;\ x\gamma u)$, $x \leqq \bigvee(v;\ y\gamma v)$, and moreover by (V)

$\bigvee(v;\ y\gamma v)=\bigvee(v;\ v\gamma y)$. Therefore, by (III*), $y\leqq x^{\gamma}$, $x\leqq y^{\gamma}$ implies $x\gamma y$. Conversely if $x\gamma y$, then $y\leqq\bigvee(u;\ x\gamma u)$, $x\leqq\bigvee(v;\ v\gamma y)$; and by (V) $\bigvee(v;\ v\gamma y)=\bigvee(v;\ y\gamma v)$. Therefore $x\gamma y$ implies $y\leqq x^{\gamma}$, $x\leqq y^{\gamma}$. Thus $x\gamma y$ is equivalent to $y\leqq x^{\gamma}$, $x\leqq y^{\gamma}$.

If $x\gamma y$ is given by $y\leqq x^{\alpha}$, $x\leqq y^{\alpha}$, then $x^{\gamma}=\bigvee(u;\ x\gamma u)=\bigvee(u;\ u\leqq x^{\alpha},\ x\leqq u^{\alpha})\leqq x^{\alpha}$. That is, $x^{\gamma}\leqq x^{\alpha}$.

Thus the theorem is completely proved.

Theorem 6. *Under the axiom (III*) and (V), let* $x\to x^{\gamma}$ *be such a mapping that* $x^{\gamma}=\bigvee(u;\ x\gamma u)$. *Then*

(1) $x_1\leqq x$ *implies* $x_1^{\gamma}\geqq x^{\gamma}$, *if and only if the relation* γ *satisfies (II).*

(2) $(x_1\smile x_2)^{\gamma}\geqq x_1^{\gamma}\frown x_2^{\gamma}$, *if and only if the relation satisfies*

*(III**)* $\bigvee(u;\ x_1\gamma x)\frown\bigvee(v;\ x_2\gamma v)\leqq\bigvee(w;\ (x_1\smile x_2)\gamma w)$.

(2′) *If* $(x_1\smile x_2)^{\gamma}\geqq x_1^{\gamma}\frown x_2^{\gamma}$, *then the relation* γ *satisfies* (III_1).

Proof. (1) By Theorem 5, $x\gamma y$ is equivalent to $y\leqq x^{\gamma}$, $x\leqq y^{\gamma}$. If $x_1\leqq x$ implies $x_1^{\gamma}\geqq x^{\gamma}$, then $x_1\leqq x$, $x\gamma y$ implies $x_1\leqq x\leqq y^{\gamma}$, $y\leqq x^{\gamma}\leqq x_1^{\gamma}$, i.e., $x_1\gamma y$. Conversely, if the relation γ satisfies (II), then, since $x^{\gamma}=\bigvee(u;\ x\gamma u)$ and $x_1^{\gamma}=\bigvee(v;\ x_1\gamma v)$, we have $x_1^{\gamma}\geqq x^{\gamma}$.

(2). $x_1^{\gamma}\frown x_2^{\gamma}\leqq(x_1\smile x_2)^{\gamma}$ is equivalent to $\bigvee(u;\ x_1\gamma u)\frown\bigvee(v;\ x_2\gamma v)\leqq\bigvee(w;\ (x_1\frown x_2)\gamma w)$.

(2′). Suppose that $(x_1\smile x_2)^{\gamma}\geqq x_1^{\gamma}\frown x_2^{\gamma}$, and if $y\gamma x_1$, $y\gamma x_2$, then $y\leqq x_1^{\gamma}$, $x\leqq y^{\gamma}$ and $y\leqq x_2^{\gamma}$, $x_2\leqq y^{\gamma}$; therefore $y\leqq x_1^{\gamma}\frown x_2^{\gamma}\leqq(x_1\smile x_2)^{\gamma}$ and $x_1\smile x_2\leqq y^{\gamma}$, which is to say, $y\gamma(x_1\smile x_2)$.

Thus the theorem is proved.

Example. Let L be the lattice 2^S of all subsets of an infinite set S, and by $x\gamma y$ it is meant that the intersection $x\frown y$ of subsets x and y of S is finite set; then this relation γ satisfies the axioms (II) (III_1) and (III_2), but satisfies neither (III_1^*), nor (III_2^*). Therefore this relation can be derived neither from such a mapping $x\to x^{\alpha}$ that $x\gamma y$ means $y\leqq x^{\alpha}$, nor from such a mapping $x\to x^{\alpha}$ that $x\gamma y$ means $y\leqq x^{\alpha}$, $x\leqq y^{\alpha}$.

Let L be a complete Boolean algebra, then $y\leqq x^{\alpha}$ is equivalent to $x^{\alpha c}\frown y=0$, (x^c means the complement of x). So, if we write $\beta=\alpha c$, then we obtain the following theorems from the above theorems.

Theorem 3′. *In a complete Boolean algebra L, there exists such a mapping* $x\to x^{\beta}$ *in L that the relation* $x\gamma y$ *is given by* $x^{\beta}\frown y=0$, *if and only if the relation* γ *satisfies the axiom (III*). And then the mapping is uniquely determined as* $x\to x^{\delta}=x^{\gamma c}=\bigwedge(u^c;\ x\gamma u)$.

Theorem 5′. *In a complete Boolean algebra L, there exists such a mapping $x \to x^\beta$ in L that the relation $x \gamma y$ is given by $x^\beta \frown y=0$, $x \frown y^\beta=0$, if and only if the relation γ satisfies the axioms (III*) and (V). And then one of the mapping $x \to x^\beta$ is given by $x \to x^\delta = \bigwedge (u^c\,;\; x \gamma u)$; let $x \to x^\beta$ be any mapping such that $x \gamma y$ is given by $x^\beta \frown y=0$, $x \frown y^\beta=0$, then $x^\delta \geqq x^\beta$.*

Theorem 4′. *In a complete Boolean algebra L, suppose the axiom (III*), and let $x \to x^\delta$ be such a mapping in L that $x \gamma y$ is given by $x^\delta \frown y=0$. Then*

(1) $x_1 \leqq x$ *implies* $x_1^\delta \leqq x^\delta$, *if and only if the relation* γ *satisfies (II).*

(2) $(x_1 \smile x_2)^\delta \leqq x_1^\delta \smile x_2^\delta$, *if and only if the relation* γ *satisfies* (III_2).

(3) $x \leqq x^\delta$, *if and only if the relation* γ *satisfies*

(IV**) $x \gamma y$ *implies* $x \frown y=0$.

(4) $0^\delta=0$, *if and only if the relation* γ *satisfies* $0 \gamma 1$.

(5) $x^{\delta\delta}=x^\delta$, *if and only if the relation* γ *satisfies*

(IV_1) $x \gamma y$, *if and only if* $\bigwedge (z^c\,;\; x \gamma z) \gamma y$.

Proof. (1) and (2) are obvious from Theorem 4.

(3) $x \leqq x^\delta$, i.e., $x \frown x^{\delta c}=0$ is equivalent to that $y \leqq x^{\delta c}$ implies $x \frown y=0$. Thus $x \leqq x^\delta$ is equivalent to that $x \gamma y$ implies $x \frown y=0$. (IV**). (Under the axiom (II), (IV**) is equivalent to (IV)).

(4) $0^\delta=0$, i.e., $0^\delta \frown 1=0$ is equivalent to $0 \gamma 1$.

(5) $x^{\delta\delta}=x^\delta$, i.e., $x^{\delta\delta c}=x^{\delta c}$ is equivalent to that $y \leqq x^{\delta\delta c}$ if and only if $y \leqq x^{\delta c}$; hence $x^\delta \gamma y$ if and only if $x \gamma y$. Therofore $x^{\delta\delta}=x^\delta$ is equivalent to that $\bigwedge (z^c\,;\; x \gamma z) \gamma y$, if and only if $x \gamma y$.

Theorem 6′. *In a complete Boolean algebra L, suppose the axiom (III*) and (V), and let $x \to x^\delta$ be such a mapping that $x^\delta = \bigwedge (u^c\,;\; x \gamma u)$. Then*

(1) $x_1 \leqq x$ *implies* $x_1^\delta \leqq x^\delta$, *if and only if the relation* γ *satisfles (II).*

(2) $(x_1 \smile x_2)^\delta \leqq x_1^\delta \smile x_2^\delta$, *if and only if the relation* γ *satisfies (III**).*

(2′) *If* $(x_1 \smile x_2)^\delta \leqq x_1^\delta \smile x_2^\delta$, *then the relation* γ *satisfies* (III_1).

(3) $x \leqq x^\delta$, *if and only if the relation* γ *satisfies (IV**).*

(4) $0^\delta=0$, *if and only if* $0 \gamma 1$.

(5) $x^{\delta\delta}=x^\delta$, *if and only if* $\bigvee (u\,;\; x \gamma u) = \bigvee (v\,;\; \bigwedge (w^c\,;\; x \gamma w) \gamma v)$.

Proof. (1), (2) and (2′) are obvious from Theorem 6.

(3) Suppose $x \leqq x^\delta$, (i.e., $x \frown x^{\delta c}=0$), then $y \leqq x^{\delta c}$ implies $x \frown y=0$. If $x \gamma y$, then $y \leqq x^{\delta c}$ and $x \leqq y^{\delta c}$; hence $x \gamma y$ implies $x \frown y=0$. Conversely, if $x \gamma y$ implies $x \frown y=0$, (i.e., $x \leqq y^c$), then $x^\delta = \bigwedge (u^c\,;\; x \gamma u) \geqq x$.

(4) Suppose $0^{\delta}=0$, then $0^{\delta}\frown 1=0$; and since $0\frown 1^{\delta}=0$, we have $0\,\gamma\, 1$. Conversely, if $0\,\gamma\, 1$, then $0^{\gamma}=\bigwedge(u^{c};\ 0\,\gamma\, u)=0$.

(5) $x^{\delta\delta}=x^{\delta}$ (i. e., $x^{\delta\delta c}=x^{\delta c}$, is equivalent to $\bigvee(u;\ x\,\gamma\, u)=\bigvee(v;\ x^{\delta}\,\gamma\, v)$ $=\bigvee(v;\bigwedge(w^{c};\ x\,\gamma\, w)\,\gamma\, v)$.

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