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The "Lorentz Transformations Without Rotation" and The New Fundamental Group of Transformations in Special Relativity and Quantum Mechanics.

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Introduction and Summary.

In the previous papers [**1, 2**], we have proposed to replace the "Lorentz transformations without rotation" by the new fundamental group of transformations, as representing the relations between the coordinates in two inertial systems one of which is moving with uniform velocity to the other. In this paper we shall investigate some property of the new fundamental group of transformations comparing with the "Lorentz transformations without rotation". In § 1,we shall state the character of the "Lorentz transformations without rotation" and the new fundamental group of transformations, and then show that the transformation of the new fundamental group is obtained by means of a suitable combination of the "Lorentz transformation without rotation" together with certain spatial rotation. The explicit expression of such spatial rotation will be obtained. In § 2, corresponding to the Thomas precession [**3, 4**], introduced in his theory of the kinematics of an electron with an axis on the basis of the Lorentz transformations without rotation, we shall calculate the precession caused by the successive transformations of the new group. In § 3, for the transformations of the new group, we shall give the tensor expression in the 4-dimensional space-time. In § 4, we shall investigate the transformation from the fixed system to the momentary rest system of a particle in arbitrary motion. As a special case we shall show that the successive rest systems of a particle in a periodic motion coincide with the rest systems after a period of the motion to each other.

§ 1. The "Lorentz transformations without rotation" and the new fundamental group of transformations.

The "Lorentz transformations without rotation" are defined by the following equations [**2**, **3**, **4**]:

$$\bar{x}^i = x^i + u^i\left[\frac{(ux)}{u^2}\left\{\frac{1}{\sqrt{1-u^2/c^2}} - 1\right\} - \frac{t}{\sqrt{1-u^2/c^2}}\right] \quad (i = 1, 2, 3)$$
$$\bar{t} = [t - (ux)/c^2]/\sqrt{1-u^2/c^2} \tag{1.1}$$

where $x^i (i = 1, 2, 3)$, t and $\bar{x}^i (i = 1, 2, 3)$, $\bar{t}$ denote the apace-time coordinates in the systems K and $\bar{K}$, the components of the uniform velocity of $\bar{K}$ relative to K being $u^h (h = 1, 2, 3)$. The round bracket of u and x: (ux) denotes the inner product of u^i and x^i. In (1.1), it is assumed that the component of the positional vector x^i in the dirction transverse to the velocity u^i undergoes no change. For, if v^i denotes the vector perpendicular to u^i, from the condition that $(uv) = 0$, (1.1) give the relation $(v\bar{x}) = (vx)$. Only the component of the positional vector x^i in the directin parallel to the velocity u^i, i. e. $(ux)/u$, undergoes the transformation of the form:

$$(u\bar{x})/u = [(ux)/u - ut]/\sqrt{1 - u^2/c^2}.$$

While we take the above assumption, the combination of two "Lorentz transformations without rotation" in general is not the same form but is equivalent to a Lorentz transformation with a rotation. Namely (1.1) do not form a *3*-parameter group regarding $u^h (h = 1, 2, 3)$ as parameters, although (1.1) make the form of $ds^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 - c^2 dt^2$ invariant. So, abandoning the above assumption, we modify the Lorentz transformations defined by (1.1) such that they form a *3*-parameter group (regarding $u^h (h = 1, 2, 3)$ as parameters) and make the form of ds^2 invariant. In the previous paper [**1**], we have described the equations of such modified transformations (i. e. the new fundamental group of transformations) in the following form:

$$x'^i = x^j\left[\delta^i_j - \frac{d^i - u^i/c}{1-(du)/c} d_j - d^i\left\{\frac{u_j/c}{\sqrt{1-u^2/c^2}} - \frac{d_j\sqrt{1-u^2/c^2}}{1-(du)/c}\right\}\right]$$
$$+ t\left[d^i \frac{u^2/c - (du)\{1-\sqrt{1-u^2/c^2}\}}{\{1-(du)/c\}\sqrt{1-u^2/c^2}} - \frac{u^i}{1-(du)/c}\right] \tag{1.2}$$
$$t' = [t - (ux)/c^2]/\sqrt{1-u^2/c^2} \qquad (i, j = 1, 2, 3)$$

which are the equations (1.2) in the previous paper [**1**]. Here $d^h = d_h$ $(h = 1, 2, 3)$ are any constants satisfying the condition: $(dd) = d_h d_h = 1$. The equations (1.2) are regarded as representing the relations between the coordinates in two inertial systems K and K', the velocity of K' relative to K being u^h.

By elimination of the four variables (x^i, t) between (1.1) and (1.2) we can show that $x'^i (= x'_i)$ and $\bar{x}^i (= \bar{x}_i)$ are connected by the following spatial rotation:

$$x'_i = D_{ij}\,\bar{x}_j, \quad \text{or} \quad \bar{x}_j = D_{ij}\,x'_i, \qquad (i,\ j = 1,\ 2,\ 3) \tag{1.3}$$

where

$$\begin{aligned} D_{ij} = \delta_{ij} - d_i\,d_j\,\frac{1-\sqrt{1-u^2/c^2}}{1-(du)/c} + \frac{u_i d_j}{c}\,\frac{1}{1-(du)/c} \\ - \frac{u_i u_j}{u^2}\,\frac{1-\sqrt{1-u^2/c^2}}{1-(du)/c} + d_i\,u_j\,\frac{2\{1-\sqrt{1-u^2/c^2}\}(du)/u^2 - 1/c}{1-(du)/c} \end{aligned} \tag{1.4}$$

(The time coordinate is unaltered). Namely (1.2) are equivalent with the combination of (1.1) and (1.3). (The condition of orthogonality: $D_{ij}D_{ik} = \delta_{jk}$ is easily verified by actual calculation) The axis of rotation of (1.3) is given by considering $X_i = \varepsilon_{ijk} u_j d_k$ since $D_{ij} X_j = X_i$, namely the axis of rotation is perpendicular to the plane determined by u_j and d_j. And the angle θ of rotation (measured from u_j to d_j) is given by the following equations:

$$\begin{aligned} \cos\theta &= \frac{1}{1-(du)/c}\left[\sqrt{1-\frac{u^2}{c^2}}\left\{1-\frac{(du)^2}{u^2}\right\} + (du)\left\{\frac{(du)}{u^2} - \frac{1}{c}\right\}\right] \\ \sin\theta &= \frac{\sqrt{u^2-(du)^2}}{1-(du)/c}\left[\frac{(du)}{u^2}\sqrt{1-\frac{u^2}{c^2}} - \left\{\frac{(du)}{u^2} - \frac{1}{c}\right\}\right] \end{aligned} \tag{1.5}$$

This is verified by considering the special case where $u_i = (u, 0, 0)$ and $d_i = (d_1, d_2, 0)$. From the other point of view, we can say that (1.3) represent the rotation of the Cartesian axis in K' which would give these axes the same orientation as the axes in K.

§ 2. The precession caused by the successive transformations of the new group.

Let us now consider three inertial systems K, K' and K'' of which K' moves with the velocity u^i relative to K while K'' moves with the velocity v^i relative to K'. The connexion between the coordinates (x, t) in K and (x', t') in K' is given by the equations (1.2) (not by (1.1)). In the same way, the connexion between (x', t') and the coordinates (x'', t'') in K'' is represented by the equations obtained from (1.2) after replacing (x, t, u) by (x', t', v) and (x', t') by (x'', t''). By elimination of the four variables (x', t') between these eight equations we obtain the connexion between (x, t) and (x'', t'') and, by the group property of the transformations (1.2), this relation must also be expressed by the equations obtained from (1.2) after replacing (x', t') by (x'', t'') and $u^i (i=1, 2, 3)$ by certain parameters, say w^i, (leaving x unreplaced). The $w^i (i=1, 2, 3)$ represent the components of the velocity of K'' relative to K. In the previous paper [**1**], we have shown that the actual form of w^i is given by the following equations:

$$w^i = \frac{\begin{bmatrix} v^i\{1-(du)/c\}+d^i[(uv)/c-(dv)\{1-\sqrt{1-u^2/c^2}\}] \\ +u^i\{1-(dv)/c\}\{1-(du)/c\}/\sqrt{1-u^2/c^2} \end{bmatrix}}{\dfrac{\{(du)/c-u^2/c^2\}(dv)/c+1-(du)/c}{\sqrt{1-u^2/c^2}}+\dfrac{(uv)-(du)(dv)}{c^2}} \tag{2.1}$$

which are the equations (1.4) in [**1**].

Now we consider the case where the transition from K' to K'' is an infinitesimal transformation, i. e. where $v^i (i=1, 2, 3)$ are infinitesimal quantities. In this case, the coefficients of the rotation of the Cartesian axes in K'' which would give these axes the same orientation as the axes in K', are obtained from (1.4) after replacing u^i by v^i and neglecting all terms of higher than the first order in v^i's. Namely

$$D_{ij} = \delta_{ij} + (v_i d_j - d_i v_j)/c \tag{2.2}$$

Here v_i is the vector in K'. Now putting

$$w_i = u_i + \delta u_i \tag{2.3}$$

in the rest system K, we shall investigate the above infinitesimal rotation. For this purpose we introduce the system (y') defined by the following equations:

$$y'_i = D_{ij} x_j \qquad (i,j = 1,2,3) \tag{2.4}$$

Then the systems (y') and K' are connected by the Lorentz transformation without rotation, the velocity of K' relative to (y') being given by $D_{ij} u_j$. The coefficients D_{ij} in (2.4) depend on u_i's viz. D_{ij} are functions $\mathrm{D}_{ij}(u)$ of u_i's. When u_i is changed into $u_i + \delta u_i$, we introduce the second system (y'') defined by

$$y''_i = \mathrm{D}_{ij}(u+\delta u) x_j \qquad (i,j = 1,2,3) \tag{2.5}$$

where $D_{ij}(u+\delta u)$ denote the expression obtained from (1.4) after replacing u_i by $u_i+\delta u_i$. Thus the transition from (y') to (y''):

$$y_i'' = D_{ij}(u+\delta u) D_{hj}(u) y_h' \qquad (i, j, h = 1, 2, 3) \tag{2.6}$$

corresponds to the transition from K' to K''. From (2.6), after a simple, but lengthy, calculation neglecting all terms of higher than first order in δu_i's, we have

$$y_i'' = (\delta_{ih} + L_{ih}) y_h' \qquad (i, h = 1, 2, 3) \tag{2.7}$$

with

$$L_{ih} = \frac{1}{1-\dfrac{(du)}{c}} \left[\begin{aligned} &\left\{ 2(1-\sqrt{1-u^2/c^2})\frac{(d\cdot u)}{u^2} - \frac{1}{c} \right\} (d_i \delta u_h - d_h \delta u_i) \\ &-2(1-\sqrt{1-u^2/c^2})\frac{(d\cdot\delta u)}{u^2}(d_i u_h - d_h u_i) \\ &-\left(\frac{1}{\sqrt{1-u^2/c^2}} - 1\right)\frac{(u\cdot\delta u)}{cu^2}(d_i u_h - d_h u_i) \\ &-(1-\sqrt{1-u^2/c^2})\frac{1}{u^2}(u_i \delta u_h - u_h \delta u_i) \end{aligned} \right] \tag{2.8}$$

These equations give the rotation of the axes of coordinates [from (y') to (y'')] caused by the infinitesimal change δu_i of the velocity u_i in K. If we put $L_{ij} = \varepsilon_{ijh}\Omega_h$ and consider the vector $\boldsymbol{\Omega}$ whose components being Ω_k, Ω_k is expressed as

$$\Omega_k = \tfrac{1}{2}\varepsilon_{ijk} L_{ij} \qquad (i, j, k, = 1, 2, 3) \tag{2.9}$$

which represents an infinitesimal rotation around the direction of the vector $\boldsymbol{\Omega}$, the angle of rotation being equal to the magnitude of the vector $\boldsymbol{\Omega}$. This fact does not occur in the Thomas precession deduced by L. H. Thomas from the Lorentz transformations without rotation [**3**, **4**].

§ 3. The tensor expression of the transformations of the new fundamental group in the 4-dimensional space-time.

For future investigation, we shall express the transformations (1.2) in a tensor form in the 4-dimensional space-time. In the previous paper [**2**] we have shown that the new fundamental group of transformations (1.2) is generated by the infinitesimal transformations of the form:

$$x_\mu' = x_\mu + (k_\mu \delta w_\nu - k_\nu \delta w_\mu) x_\nu \qquad (\mu, \nu = 1, \dots, 4) \tag{3.1}$$

Here $x_\mu(\mu=1, \dots, 4)$ denote the coordinates $x_h(h=1, 2, 3)$ and ict, and k_μ is the four-vector having the components

$$k_\mu = (d_1, d_2, d_3, i) \tag{3.2}$$

And $\delta w_\mu(\mu=1, \dots, 4)$ are arbitrary infinitesimal quantities.

So, for rewriting (1.2) in a tensor form in the 4-dimensional space-time, we shall directly obtain the tensor form of the transformations generated by (3.1). For this purpose we introduce the four-velocity: $U_\lambda = dx_\lambda/d\tau$, τ being the proper time defined by $-c^2 d\tau^2 = (dx_1)^2 + (dx_2)^2 + (dx_3)^2 - c^2 dt^2$, viz.

$$U_\lambda = (u^h/\sqrt{1-u^2/c^2}, \quad ic/\sqrt{1-u^2/c^2}) \tag{3.3}$$

where u^h is the usual three dimensional velocity vector. Then we can show that the transformations generated by (3.1) are expressed in the form:

$$x'_\lambda = A_{\lambda\mu} x_\mu \qquad (\lambda, \mu = 1, \dots, 4) \tag{3.4}$$

with

$$A_{\lambda\mu} = \delta_{\lambda\mu} + k_\lambda \frac{U_\mu - U'_\mu}{(kU')} + \frac{U'_\lambda - U_\lambda}{(kU)} k_\mu + \frac{(UU') - (UU)}{(kU)(kU')} k_\lambda k_\mu \tag{3.5}$$

Here U'_λ is defined by $U'_\lambda = dx'_\lambda/d\tau$ and, for brevity, the notations (kU), (kU') etc. are used in place of $k_\nu U_\nu$, $k_\nu U'_\nu$ etc. Accordingly $(UU) = -c^2$. (The calculation for deduction of (3.5) is described in the Note at the end of this paper).

Now, let $u^h (h=1, 2, 3)$ be the velocity of K' relative to K. The corresponding four-velocity U_λ is then given by (3.3), and the components of this four-velocity relative to K', i. e. U'_λ, are

$$U'_\lambda = (0, 0, 0, ic) \tag{3.6}$$

since U_λ is the four-velocity of a point at rest in K', which means that the vector U_λ lies in the direction of the x'_4-axis. Therefore, *the transformations* (3.4) *accompanied by* (3.5) *and* (3.6), *are the desired tensor forms of the transformations* (1.2) *of the new group.* Hereafter, in (3.5) we always take U'_λ as (3.6), and regard $A_{\lambda\mu}$ as the function $A_{\lambda\mu}(U)$ of $U_1, \dots, U_4$.

From (3.4), we can easily see that the components of the vector k_μ relative to the system K' viz. k'_μ are given by

$$k^\lambda = k_\lambda (kU)/(kU') \quad \text{or} \quad k'_\lambda = k_\lambda [1-(du)/c]/\sqrt{1-u^2/c^2} \tag{3.7}$$

since $(kU) = [(du) - c]/\sqrt{1-u^2/c^2}$ and $(kU') = -c$. (The vector $-ck_\mu/(kU)$ is independent on the multiple of k_μ and is equal to $k_\mu \sqrt{1-u^2/c^2}/[1-(du)/c]$ which is introduced in the momentum mass vector defined in the previous paper [**1**]. Cf. the foot-notes at the end of the previous paper [**2**])

In concluding this section, we shall give the composition law of two four-velocities. Since the inverse transformation of (3.4) is given by

$$x_\mu = A_{\lambda\mu} x'_\lambda \tag{3.8}$$

putting $dx_\mu/d\tau = W_\mu$ and $dx'_\lambda/d\tau = V_\lambda$, from the above we have $W_\mu = A_{\lambda\mu} V_\lambda$, or, using (3.5),

$$W_\mu = V_\mu + (kV)\frac{U_\mu - U'_\mu}{(kU')} + \frac{(U'V)-(UV)}{(kU)} k_\mu + \frac{(UU')+c^2}{(kU)\,(kU')}(kV)\, k_\mu \tag{3.9}$$

Substituting (3.6) for U'_μ in the above, we have the formula for sum W_μ of four-velocities U_μ and V_μ.

§ 4. Successive transformations of the new group.

By the similar way by which C. Mϕller has treated the successive Lorentz transformations without rotation [**4**], we shall investigate the successive transformations of the new group. As in § 2, let us consider three inertial systems K, K' and K'', and let

$$x'_\lambda = A_{\lambda\mu}(U)x_\mu \quad (K \to K'), \qquad x''_\lambda = A_{\lambda\mu}(V)\, x'_\mu \quad (K' \to K'') \tag{4.1}$$

be two successive transformations of the new group (3.4). Here $A_{\lambda\mu}(V)$ means the expression obtained from (3.5) after replacing U_ω by V_ω, V_ω being the four-velocity (corresponding to v^h) of K'' relative to K'. By the group property of (3.4), the coefficients of the resultant transformation:

$$x''_\lambda = A_{\lambda\mu}(V)\, A_{\mu\nu}(U)\, x_\nu \tag{4.2}$$

are again given by the expression obtained from (3.5) after replacing U_ω by certain four-velocity, say W_ω, viz.

$$A_{\lambda\mu}(V)\, A_{\mu\nu}(U) = A_{\lambda\nu}(W) \tag{4.3}$$

W_ω represents the four-velocity of K'' relative to K. In the special case where the transformation from K' to K'' is an infinitesimal transformation, i. e. where v_1, v_2, v_3 are infinitesimal quantities, we have, neglecting second-order terms in these quantities,

$$V_h = v_h \ (h = 1, 2, 3), \quad V_4 = ic \tag{4.4}$$

Accordingly, by (3.5), $A_{\lambda\mu}(V)$ reduces to

$$A_{\lambda\mu}(V) = \delta_{\lambda\mu} + \omega_{\lambda\mu} \tag{4.5}$$

with

$$\omega_{\lambda\mu} = [-k_\lambda (V_\mu - U'_\mu) + (V_\lambda - U'_\lambda) k_\mu]/c \tag{4.6}$$

Hence, according to (4.3) and (4.5),

$$A_{\lambda\nu}(W) = (\delta_{\lambda\mu} + \omega_{\lambda\mu}) A_{\mu\nu}(U) = A_{\lambda\nu}(U) + \omega_{\lambda\mu} A_{\mu\nu}(U) \tag{4.7}$$

Let $x_\lambda = f_\lambda(\tau)$ ($\lambda = 1, \ldots, 4$) represent the time track of a particle in arbitrary motion in K, τ being the proper time of the particle. We shall now try to determine the successive rest systems of the particle such that the coordinates of two consecutive rest systems at any time are connected by the transformation of the new group (3.4). Let K' and K'' in (4.1) be momentary rest systems of the particle at the times τ and $\tau + d\tau$, respectively. The four-velocity of K' relative to K is then

$$U_\lambda(\tau) = dx_\lambda / d\tau = \dot{f}_\lambda(\tau) \tag{4.8}$$

Similarly, the four-velocity of K'' relative to K is

$$W_\lambda = U_\lambda(\tau) + dU_\lambda(\tau) = U_\lambda(\tau) + \dot{U}_\lambda(\tau)\, d\tau = \dot{f}_\lambda(\tau) + \ddot{f}_\lambda(\tau)\, d\tau \tag{4.9}$$

The components of these four-velocities in K' are

$$U'_\lambda(\tau) = A_{\lambda\mu}(U)\, U_\mu = (0, 0, 0, ic) \tag{4.10}$$

$$V_\lambda = W'_\lambda = U'_\lambda + dU'_\lambda = A_{\lambda\mu}(U)[U_\mu + dU_\mu] = U'_\lambda + A_{\lambda\mu}(U)\, dU_\mu \tag{4.11}$$

Since the transformation from K' to K'' was supposed to be an infinitesimal transformation, the coefficients $A_{\lambda\nu}(W)$ in the transformation from K to K'' are obtained from (4.7) and (4.6), V_λ in (4.6) being given by (4.11). Since, from (4.11) $V_\lambda - U'_\lambda = A_{\lambda\mu}(U)dU_\mu$, (4.6) is expressed as

$$\omega_{\lambda\mu} = [-k_\lambda A_{\mu\omega}(U)\, dU_\omega + A_{\lambda\omega}(U)\, dU_\omega k_\mu]/c \tag{4.12}$$

Thus, from (4.7) and (4.12)

$$A_{\lambda\nu}(W) - A_{\lambda\nu}(U) = [-k_\lambda dU_\nu + A_{\lambda\omega}(U)\, dU_\omega k_\nu (kU')/(kU)]/c \tag{4.13}$$

since $A_{\mu\omega} A_{\mu\nu} = \delta_{\omega\nu}$ and $k_\mu A_{\mu\nu}(U) = k_\nu (kU')/(kU)$. Further, if we use the relations (3.7) or $k_\lambda = A_{\lambda\omega}(U) k_\omega (kU')/(kU)$ and $(kU') = -c$, (4.13) is rewritten as

$$A_{\lambda\nu}(W) - A_{\lambda\nu}(U) = A_{\lambda\omega}(U)\,[k_\omega dU_\nu - k_\nu dU_\omega]/(kU) \qquad (4.14)$$

The coefficients $A_{\lambda\nu}(U)$ can now be regarded as functions $A_{\lambda\nu}(\tau)$ of τ, $A_{\lambda\nu}(W)$ being then equal to $A_{\lambda\nu}(\tau+d\tau)$. Hence we get the following differential equations for the functions $A_{\lambda\nu}(\tau)$:

$$dA_{\lambda\nu}(\tau)/d\tau = A_{\lambda\omega}(\tau)\,[k_\omega \dot{U}_\nu - k_\nu \dot{U}_\omega]/(kU) \qquad (4.15)$$

In fact, these equations are directly deduced from the expression (3.5) by regarding U_ω ($\omega = 1, \ldots, 4$) as functions of τ. This result is different from the one deduced from the Lorentz transformations without rotation [**4**].

Next, let us attach a space vector $\boldsymbol{e}'(\tau)$ of unit length to the particle considered in such a way that the components $\boldsymbol{e}'$ with respect to the spatial axes of $K'(\tau)$ have the same values at all times. In 4-dimensional space this vector is represented by a space-like four-vector with components given by $e'_\lambda = (\boldsymbol{e}', 0)$ in K'. Its components in K are given by $e_\nu = e'_\lambda A_{\lambda\nu}(\tau)$, $A_{\lambda\nu}(\tau)$ being given by (3.5) regarding U_ω ($\omega = 1, \ldots, 4$) as functions of τ. These components satisfy the following differential equations:

$$de_\nu/d\tau = e'_\lambda\, dA_{\lambda\nu}(\tau)/d\tau = e_\omega\,[k_\omega \dot{U}_\nu - k_\nu \dot{U}_\omega]/(kU)$$

i. e. $$de_\nu/d\tau = [(ek)\,\dot{U}_\nu - k_\nu (e\dot{U})]/(kU)$$

which is also different from the result deduced from the Lorentz transformations without rotation [**4**].

Lastly we consider the special case where $x_\lambda = f_\lambda(\tau)$ represent the time track of a partcle in a periodic motion in K. Let the period be T and the corresponding proper period be l, i.e. $f_h(\tau+l) = f_h(\tau)$ ($h = 1, 2, 3$), $f_4(\tau+l) = f_4(\tau) + icT$, T and l being constants. Since $U_\omega(\tau+l) = U_\omega(\tau)$ and the solution $A_{\lambda\nu}(\tau)$ of (4.15) is the functions of $U_\omega(\tau)$ ($\omega = 1, \ldots, 4$) regarding $U_\omega(\tau)$ as (4.8), we have $A_{\lambda\nu}(\tau+l) = A_{\lambda\nu}(\tau)$. This means that two momentary rest systems of the particle at the times τ and $\tau+l$ coincide with each other. This is not the case for the Lorentz transformations without rotation. [**4**].

Note. Deduction of the equations (3.5) from (3.1).

In order to obtain the finite form of the transformations generated by the infinitesimal transformations (3.1), we put $\delta w_\mu = w_\mu \varepsilon$ (for a time regarding w_μ fixed and ε as parameter) and solve the following differential equations:

$$dx'_\mu/d\varepsilon = (k_\mu w_\nu - k_\nu w_\mu)\,x'_\nu \qquad (\text{N.1})$$

with the initial condition that $x'_\mu = x_\mu$ when $\varepsilon = 0$. For brevity, we use the notations (kx'), (wx'), etc. in place of $k_\nu x'_\nu$, $w_\nu x'_\nu$, etc. Then (N.1) are expressed as

$$d x'_\mu / d\varepsilon = k_\mu (w x') - w_\mu (k x') \tag{N.2}$$

We can easily see that the following equations:

$$\varepsilon_{\lambda\mu\nu\xi}\, x'_\lambda\, k_\mu\, w_\nu = \varepsilon_{\lambda\mu\nu\xi}\, x_\lambda\, k_\mu w_\nu \tag{N.3}$$

$$(k x') = (k x) \exp\left[-(k w)\,\varepsilon\right] \tag{N.4}$$

$$2\,(kw)\,(wx') - (ww)\,(kx') = \left[2\,(kw)\,(wx) - (ww)\,(kx)\right] \exp\left[(kw)\,\varepsilon\right] \tag{N.5}$$

are the integrals of the equations (N.2) satisfying the initial condition, the latter two equations of the above being deduced from the relations: $d(kx')/d\varepsilon = -(kw)\,(kx')$ and $d(wx')/d\varepsilon = (kw)\,(wx') - (ww)\,(kx')$. Then we can solve (N.3), (N.4) and (N.5) for x'_λ in the following manner: Obtaining the relations:

$$x'_\lambda = x_\lambda + A\, k_\lambda + B\, w_\lambda \tag{N.6}$$

from (N.3), A and B being arbitrary, and, substituting (N.6) into (N.4) and (N.5), we can determine A and B. The resulting equations are given by

$$x'_\lambda = x_\lambda + \frac{k_\lambda}{2(kw)^2}\left[\begin{array}{l} -2(wx)\,(kw) + 2(ww)\,(kx) - (ww)\,(kx)\exp\left[-(kw)\varepsilon\right] \\ +\left\{2\,(kw)\,(wx) - (ww)\,(kx)\right\}\exp\left[(kw)\varepsilon\right] \end{array}\right]$$
$$+ (kx)\left\{\exp\left[-(kw)\varepsilon\right] - 1\right\} w_\lambda/(kw) \tag{N.7}$$

This is the finite form of the transformations generated by the infinitesimal transformations (3.1), regarding $w_\mu \varepsilon$ as parameters.

Next, we express (N.7) in terms of the four-vectors $U_\lambda = dx_\lambda/d\tau$ and $U'_\lambda = dx'_\lambda/d\tau$. The relations between U_λ and U'_λ are expressed by the equations obtained from (N.7) after replacing x and x' by U and U', viz.

$$U'_\lambda = U_\lambda + k_\lambda\left[\cdots\cdots\right]/\,2\,(kw)^2 + (kU)\left\{\exp\left[-(kw)\,\varepsilon\right] - 1\right\} w_\lambda/(kw) \tag{N.8}$$

Our problem is to eliminate w_μ, ε from (N.7) and (N.8). From (N.8), dividing all terms by (kU), we express the last term: $\{\exp\left[-(kw)\varepsilon\right] - 1\} w_\lambda/(kw)$ by the remaining terms, and substitute this expression into the coefficient of (kx) in the last term of (N.7). Then the resulting equations become

$$x'_\lambda = x_\lambda - k_\lambda\left\{1 - \exp\left[(kw)\,\varepsilon\right]\right\}(wx)/(kw) +$$

$$+\left[\frac{U'_\lambda - U_\lambda}{(kU)} + \frac{k_\lambda(wU)}{(kw)(kU)}\{1 - \exp[(kw)\varepsilon]\}\right](kx) \tag{N.9}$$

Further, in order to express $(wx)/(kw)$ in the above in terms of U's and U''s, we multiply (N.8) by x_λ and sum up the resulting equations for $\lambda = 1, \ldots, 4$. Then we have

$$(U'x) = (Ux) + (kx)[\cdots\cdots]/2(kw)^2 + (kU)\{\exp[-(kw)\varepsilon] - 1\}(wx)/(kw) \tag{N.10}$$

But the last term is expressed as

$$(kU)\{\exp[-(kw)\varepsilon] - 1\}(wx)/(kw) = (kU')\{1 - \exp[(kw)\varepsilon]\}(wx)/(kw) \tag{N.11}$$

since $(kU') = (kU)\exp[-(kw)\varepsilon]$ from (N.4). Using (N.10) and (N.11), we can see that (N.9) are expressed in the following form:

$$x'_\lambda = A_{\lambda\mu}x_\mu \tag{N.12}$$

with

$$A_{\lambda\mu} = \delta_{\lambda\mu} - k_\lambda\frac{U'_\mu - U_\mu}{(kU')} + \frac{U'_\lambda - U_\lambda}{(kU)}k_\mu + \rho k_\lambda k_\mu \tag{N.13}$$

ρ being certain factor. We can determine ρ, from the condition of orthogonality: $A_{\lambda\mu}A_{\lambda\nu} = \delta_{\mu\nu}$, as follows: $\rho = [(UU') - (UU)]/(kU)(kU')$.

References

[1] T. Shibata: Definition of Momentum and Mass as an Invariant Vector of the new Fundamental Group of Transformations in Special Relativity and Quantum Mechanics. This journal Vol. **16**, No. 3 (1953), 487.

[2] T. Shibata: On Lorentz Transformations and Continuity Equation of Angular Momentum in Relativistic Quantum Mechanics. This journal Vol. **18**, No. 3 (1955), 391.

[3] L. H. Thomas: The Kinematics of an Electron with an Axis. Phil. Mag. (7), **3, 1** (1927), 1.

[4] C. Møller: *The Theory of Relativity.* (Oxford at the Clarendon Press 1952)

Note on the previous papers (on the suggestion given by H. P. Robertson)

In the previous paper entitled "Some results deduced from the new fundamental group of transformations in special relativity and quantum mechanics"

(J. Sci. Hiroshima Univ. Vol. 17 (1953) 67–73), we have shown that a light beam in the direction d^i suffers no aberration under the condition that an observer moves transverse to the direction of the beam. But, without using this condition in the process of the proof, the same result can be obtained. This is pointed out by H. P. Robertson from the standpoint of the group of transformations in Math. Reviews Vol. 15, No. 8 (1954) p. 752. (This result is also shown by the equations (3.7) in the previous paper [**2**] of the above references) Further, he has suggested that the new term appeared in the energy momentum vector defined in the previous paper [**1**] is interpretable as a beam of radiant energy along the given null-ray. I think that this suggestion has interesting meaning connected with the fact mentioned in the foot-notes at the end of the previous paper [**2**].

I wish to express my hearty thanks to Prof. H. P. Robertson for his valuable remarks concerning the contents of the previous papers.