JOURNAL OF SCIENCE OF THE HIROSHIMA UNIVERSITY, SER. A. VOL. 19, No. 1, JULY, 1955

On Center of Higher Dimensions

By

Minoru URABE and Yasutaka SIBUYA

(Received Jan. 19, 1955)

§ 1. Introduction.

In this note, we shall seek for the canonical form of the system of n differential equations

$$\frac{dx_\nu}{dt} = \xi_\nu(x) \qquad (\nu = 1, 2, \ldots, n) \tag{1.1}$$

in the neighborhood of the center. For the system of two differential equations, making use of a formal transformation of the variables, Hukuhara has deduced the canonical form of the system [1]. Here, by making use of the invariant integral of the group of transformations [2], we shall show that, by a suitable analytic transformation, the system of n differential equations can be reduced to that of the form analogous to that which Hukuhara has deduced for the system of two differential equations.

In this note, we agree that the center is a critical point such that the characteristics passing through any point lying in the neighborhood of that point are all closed. We assume that $\xi_\nu(x)$'s *are all analytic in the neighborhood of the center* and that *the period* $\omega(x)$ *of the characteristic passing through any point* (x_ν) *lying in the neighborhood of the center is bounded.*

§ 2. Reduction of the linear parts.

Without loss of generality, we may assume that the center is the origin. Then let the expansions of $\xi_\nu(x)$ be

$$\xi_\nu(x) = \sum_{\mu=1}^{n} c_{\nu\mu} x_\mu + \sum_{\mathfrak{p}}{}'' c_{\nu\mathfrak{p}} x_1^{p_1} \ldots x_n^{p_n}, \tag{2.1}$$

where $\sum_{\mathfrak{p}}''$ denotes the summation over $\mathfrak{p} = (p_1, p_2, \ldots, p_n)$ such that $s(\mathfrak{p}) = d_1 + p_2 + \cdots p_n \geqq 2$. Let the characteristic roots of the matrix $C = \|c_{\nu\mu}\|$ be

$\lambda_1, \lambda_2, \cdots, \lambda_n$. Then, since the origin is a center, it must be that λ_ν's are all pure imaginary inclusive of zero, consequently we put

$$\lambda_\nu = i\,\theta_\nu. \tag{2.2}$$

Let the transformation of the one-parameter group defined by (1.1) be

$$x'_\nu = \varphi_\nu(x, t), \tag{2.3}$$

then, it is evident that

$$\varphi_\nu(x, t) = e^{tX}(x_\nu) = x_\nu + t\xi_\nu(x) + \frac{t^2}{2!}X(\xi_\nu) + \frac{t^3}{3!}X^2(\xi_\nu) + \cdots, \tag{2.4}$$

where $X \equiv \sum_{\nu=1}^{n} \xi_\nu(\partial/\partial x_\nu)$. Evidently, $\varphi_\nu(x, t)$ is a system of the solutions of (1.1) such that $\varphi_\nu(x, 0) = x_\nu$. By the assumption,

$$\varphi_\nu(x, \omega(x)) = x_\nu. \tag{2.5}$$

Put

$$x_\nu = \rho l_\nu, \tag{2.6}$$

where

$$\rho = \sqrt{\sum_{\nu=1}^{n} x_\nu^2}. \tag{2.7}$$

Let one of the limits of $\omega(x)$ be ω_0 when $\rho \to 0$ l_ν being arbitrarily fixed. Since $\omega(x)$ is bounded, ω_0 must be finite. Then, from (2.5), we see that

$$\sum_\lambda c_{\nu\lambda}\, l_\lambda + \frac{\omega_0}{2!}\sum_{\mu,\lambda} c_{\nu\mu}\, c_{\mu\lambda}\, l_\lambda + \frac{{\omega_0}^2}{3!}\sum_{\mu_1,\mu_2,\lambda} c_{\nu\mu_1}\, c_{\mu_1\mu_2}\, c_{\mu_2\lambda}\, l_\lambda + \cdots = 0,$$

or, in the form of the matrix,

$$\left(C + \frac{\omega_0}{2!}C^2 + \frac{{\omega_0}^2}{3!}C^3 + \cdots\right)\vec{l} = 0, \tag{2.8}$$

where $\vec{l}$ is a column vector whose components are $l_1, l_2, \cdots, l_n$. We choose a suitable matrix S so that $SCS^{-1} = C_0$ may be of the canonical form as follows:

$$C_0 = SCS^{-1} = \sum \oplus \begin{pmatrix} \lambda & 0 & \cdots & \cdots & 0 \\ 1 & \lambda & & & \vdots \\ 0 & 1 & \lambda & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 1 & \lambda \end{pmatrix}, \tag{2.9}$$

where $\sum \oplus$ denotes the direct sum. Since C is real, the imaginary characteristic roots of C occur in conjugate pairs λ_ν and $\lambda_{\bar\nu}=\overline{\lambda_\nu}$, consequently we can choose $S=\|s_{\nu\mu}\|$ so that $s_{\bar\nu\mu}=\overline{s_{\nu\mu}}$ and, when λ_ν is real, $s_{\nu\mu}$'s may be real. In the sequel, we take such S. Put

$$S\vec{l}=\vec{l}'. \tag{2.10}$$

Then, from (2.8), it follows that

$$\left(C_0+\frac{\omega_0}{2\,!}C_0^2+\frac{\omega_0^2}{3\,!}C_0^3+\cdots\right)\vec{l}'=0,$$

namely,

$$\frac{1}{t}\left(e^{tC_0}-I\right)\Big|_{t=\omega_0}\cdot\ \vec{l}'=0, \tag{2.11}$$

where I is a unit matrix. Since

$$e^{tC_0}=\sum\oplus\begin{pmatrix} e^{t\lambda} & 0 & 0\cdots\cdots 0\\ \frac{t}{1\,!}e^{t\lambda} & e^{t\lambda} & 0 \quad\vdots\\ \frac{t^2}{2\,!}e^{t\lambda} & \frac{t}{1\,!}e^{t\lambda} & e^{t\lambda} \quad\vdots\\ \vdots & \cdot & \cdot\ \cdots\ddots e^{t\lambda}\end{pmatrix},$$

corresponding to any one of the blocks, from (2.11), there hold the relations of the forms as follows:

$$\begin{cases} E_\lambda(\omega_0)\,l_1'=0,\\ e^{\omega_0\lambda}\,l_1'+E_\lambda(\omega_0)\,l_2'=0,\\ \frac{\omega_0}{2\,!}\,e^{\omega_0\lambda}\,l_1'+e^{\omega_0\lambda}\,l_2'+E_\lambda(\omega_0)\,l_3'=0,\\ \cdots\cdots\cdots\cdots, \end{cases} \tag{2.12}$$

where

$$E_\lambda(t)=\frac{1}{t}\left(e^{t\lambda}-1\right). \tag{2.13}$$

If C_0 is not of the diagonal form, from (2.12), it must be that, when λ is imaginary, $l_1'=l_{\bar1}'=0$ and, when λ is real, $l_1'=0$. In the latter case, $\sum_\mu s_{1\mu}l_\mu=0$, that is to say that the direction $\vec{l}$ lies in a certain hyperplane. This contradicts the fact that the direction $\vec{l}$ is an arbitrary direction. In the former case, we have

$$\sum_{\mu} (s_{1\mu} + \overline{s_{1\mu}})\, l_{\mu} = \sum_{\mu} (s_{1\mu} - \overline{s_{1\mu}})\, l_{\mu} = 0,$$

then, as in the above case, we again obtain a contradiction. Thus it is seen that C_0 must be of the diagonal form.

Then, from (2.12), the relation (2.11) is written as follows:

$$\text{(2.14)} \qquad \begin{cases} E_{\lambda_1}(\omega_0)\, l_1' = 0, \\ E_{\lambda_2}(\omega_0)\, l_2' = 0, \\ \cdots\cdots\cdots\cdots \\ E_{\lambda_n}(\omega_0)\, l_n' = 0. \end{cases}$$

Since C is real, from (2.2), the characteristic roots of C occur in the forms as follows:

$$\text{(2.15)} \qquad \begin{cases} \lambda_k = i\,\theta_k, \quad \lambda_{\bar{k}} = -i\,\theta_k; & (\theta_k > 0, \quad k = 1, 2, \dots, m) \\ \lambda_p = 0. & (p = 2m+1, \dots, n) \end{cases}$$

Then, if $E_{\lambda_k}(\omega_0) \neq 0$ for at least one k, from (2.14), it must be that $l_k' = l_{\bar{k}}' = 0$. This contradicts the fact that $\vec{l}$ is arbitrary. Thus it must be that

$$E_{\lambda_k}(\omega_0) = 0, \qquad (k = 1, 2, \dots, m)$$

from which it follows that $\omega_0 \neq 0$. Then, from (2.13), it must be that $\exp(\lambda_k \omega_0) = 1$, consequently there holds the relation as follows:

$$\text{(2.16)} \qquad \omega_0 = \frac{2\pi}{\theta_1}\,\Theta_1 = \frac{2\pi}{\theta_2}\,\Theta_2 = \cdots = \frac{2\pi}{\theta_m}\,\Theta_m,$$

where Θ_k's are positive integers. Thus we have

THEOREM 1. *Under our assumption, the canonical form of C must be of the diagonal form, and the characteristic roots of C must be pure imaginary or zero, and moreover they must be relatively commensurable.*

COROLLARY. *If all the characteristic roots of C are zero, then C must be a zero matrix.*

In virtue of Theorem 1, as is well known, by means of a suitable real linear transformation, the differential equations (1.1) can be reduced to the equations of the forms as follows:

$$(2.17)\qquad \begin{cases} \dfrac{dx_k}{dt} = \qquad -\theta_k x_{\bar{k}} + [x]_2, \\ \dfrac{dx_{\bar{k}}}{dt} = \theta_k x_k \qquad + [x]_2, \end{cases} (k = 1, 2, \dots, m) \\ \dfrac{dx_p}{dt} = \qquad [x_2], \qquad (p = 2m+1, \dots, n)$$

where $[x]_2$ denotes the sum of the terms of the second and higher orders with respect to x_ν.

For the equations of the form (2.17), we shall seek for the solutions $\varphi_\nu(x, t)$ such that $\varphi_\nu(x, 0) = x_\nu$. By analyticity of the equations, $\varphi_\nu(x, t)$ can be expanded in the power series with respect to x_ν as follows:

$$(2.18)\qquad \varphi_\nu(x, t) = \varphi_{\nu 0}(t) + \sum_\mu \varphi_{\nu\mu}(t)\, x_\mu + \sum_{\mathfrak{p}}{}'' \varphi_{\nu\mathfrak{p}}(t)\, x_1^{p_1} \dots x_n^{p_n}.$$

From uniqueness of the solutions, it is easily seen that $\varphi_{\nu 0}(t) \equiv 0$. From $\varphi_\nu(x, 0) = x_\nu$, it follows that

$$(2.19)\qquad \varphi_{\nu\mu}(0) = \delta_{\nu\mu},$$

where $\delta_{\nu\mu}$ is a Kronecker's delta. Substituting (2.18) into (2.17), we have:

$$\frac{d\varphi_{k\mu}}{dt} = -\theta_k \varphi_{\bar{k}\mu}, \quad \frac{d\varphi_{\bar{k}\mu}}{dt} = \theta_k \varphi_{k\mu}, \quad \frac{d\varphi_{p\mu}}{dt} = 0.$$

Integrating these equations, from (2.19), we have:

$$\begin{cases} \varphi_{k\mu} = \delta_{k\mu} \cos \theta_k t - \delta_{\bar{k}\mu} \sin \theta_k t, \\ \varphi_{\bar{k}\mu} = \delta_{k\mu} \sin \theta_k t + \delta_{\bar{k}\mu} \cos \theta_k t, \\ \varphi_{p\mu} = \delta_{p\mu}. \end{cases}$$

Consequently the solutions (2.18) are written as follows:

$$(2.20)\qquad \begin{cases} \varphi_k(x, t) = x_k \cos \theta_k t - x_{\bar{k}} \sin \theta_k t + \sum_{\mathfrak{p}}{}'' \varphi_{k\mathfrak{p}}(t)\, x_1^{p_1} \dots x_n^{p_n}, \\ \varphi_{\bar{k}}(x, t) = x_k \sin \theta_k t + x_{\bar{k}} \cos \theta_k t + \sum_{\mathfrak{p}}{}'' \varphi_{\bar{k}\mathfrak{p}}(t)\, x_1^{p_1} \dots x_n^{p_n}, \\ \varphi_p(x, t) = x_p + \sum_{\mathfrak{p}}{}'' \varphi_{p\mathfrak{p}}(t)\, x_1^{p_1} \dots x_n^{p_n}. \end{cases}$$

§ 3. Analyticity of the period.

First we consider the case where $|C| \neq 0$. When $|C| \neq 0$, the characteristic roots of C are all not zero, consequently $n = 2m$. Let one of the limits of $\omega(x)$

be ω_0 when $x_\nu \to 0$. From (2.7), $\sum_{\nu=1}^{n} l_\nu^2 = 1$. Consequently, if we take a suitable subsequence of (x_ν), there exists the number $l_{0\nu}$ to which l_ν tends as $x_\nu \to 0$. Then, for such ω_0 and $\vec{l}_0 = \{l_{0\nu}\}$, there holds the relation (2.8). Since $\sum_{\nu=1}^{n} l_{0\nu}^2 = 1$, at least one of $l_{0\nu}$'s is not zero, consequently, from (2.14), it must be that, for certain ν,

$$E_{\lambda_\nu}(\omega_0) = 0.$$

Now, in the present case, $\lambda_\nu \neq 0$. Thus we see that, for certain k, ω_0 must be of the form as follows:

$$\omega_0 = \frac{2\pi}{\theta_k} n_k, \tag{3.1}$$

where n_k is a positive integer.

Since $\omega(x)$ is bounded in the neighborhood of the origin, for fixed k, there exist only a finite number of n_k's for which (3.1) hold. Let the least common multiple of such n_k's be N_k. Then there exist the multiples of the period which tend to $2\pi N_k/\theta_k$ as $x_\nu \to 0$. Let these multiples be $\omega_k(x)$, and let us seek for a multiple of $\omega_k(x)$'s which tend to a fixed value as $x_\nu \to 0$. For this, it is only necessary to seek for the positive integers M_k's such that

$$\frac{M_1 N_1}{\theta_1} = \frac{M_2 N_2}{\theta_2} = \cdots = \frac{M_m N_m}{\theta_m}. \tag{3.2}$$

Now, by Theorem 1, θ_k's are relatively commensurable, consequently there exist relatively prime positive integers Θ_k's such that

$$\frac{\theta_1}{\Theta_1} = \frac{\theta_2}{\Theta_2} = \cdots = \frac{\theta_m}{\Theta_m} = \frac{2\pi}{\theta}, \tag{3.3}$$

where $\theta > 0$. Substituting (3.3) into (3.2), we have:

$$\frac{M_1}{\frac{\Theta_1}{N_1}} = \frac{M_2}{\frac{\Theta_2}{N_2}} = \cdots = \frac{M_m}{\frac{\Theta_m}{N_m}}. \qquad (= \sigma)$$

From this, M_k's can be determined. For our purpose, it is sufficient to take M_k's such that they are relatively prime. Then, corresponding to such M_k's, we have a multiple of the period which tends to

$$\omega_0 = \frac{2\pi}{\theta_1} M_1 N_1 = \frac{2\pi}{\theta_2} M_2 N_2 = \cdots = \frac{2\pi}{\theta_m} M_m N_m = \sigma\theta \tag{3.4}$$

as $x_\nu \to 0$. Let us call this multiple of the period *the universal period.* In the sequel, instead of the original period —— if necessary, we shall call the original period *the primitive period* ——, we consider the universal period as the period of all the characteristics and denote it by $\omega(x)$. Then, from the above results, it is seen that *the universal period* $\omega(x)$ *is continuous at the origin.*

In the sequel, we shall show that the universal period $\omega(x)$ becomes analytic at the origin. Put

$$x_\nu = \rho\, l_\nu, \tag{3.5}$$

then, for sufficiently small positive numbers δ and ε $(<\omega_0)$, in the domain $U_\varepsilon \times D_\delta$ where $U_\varepsilon : |t-\omega_0|<\varepsilon$, $D_\delta : |\rho l_\nu|<\delta$, the functions $(1/\rho)\varphi_\nu(x, t) \equiv \phi_\nu(\rho, l, t)$ are analytic with respect to the arguments and, in the domain D_δ, $|\omega(x)-\omega_0| = |\omega(\rho l)-\omega_0|<\varepsilon$. From (2.20), it follows that

$$\begin{cases} \phi_k(\rho, l, t) = l_k \cos\theta_k t - l_{\bar{k}} \sin\theta_k t + [\rho\,;\, l, t]_1, \\ \phi_{\bar{k}}(\rho, l, t) = l_k \sin\theta_k t + l_{\bar{k}} \cos\theta_k t + [\rho\,;\, l, t]_1, \end{cases} \tag{3.6}$$

where $[\rho\,;\, l, t]_1$ denotes the sum of the terms of the first and higher orders with respect to ρ. Consequently we have:

$$\begin{cases} \text{(i)} \quad \phi_k(0, l^0, \omega_0) = l_k^0, \quad \dfrac{\partial}{\partial t}\phi_k(0, l^0, \omega_0) = -\theta_k l_{\bar{k}}^0\,; \\ \text{(ii)} \quad \phi_{\bar{k}}(0, l^0, \omega_0) = l_{\bar{k}}^0, \quad \dfrac{\partial}{\partial t}\phi_{\bar{k}}(0, l^0, \omega_0) = \theta_k l_k^0. \end{cases} \tag{3.7}$$

Let ρ and l_ν's take the complex values which lie in the domains

$$C_\eta: \quad |\rho|<\eta,$$

and $$D_{\bar{k}}: \quad \sigma \leqq |l_{\bar{k}}| \leqq \delta', \quad |l_\nu| \leqq \delta' \qquad (\nu \neq \bar{k})$$

or $$D_k: \quad \sigma \leqq |l_k| \leqq \delta', \quad |l_\nu| \leqq \delta', \qquad (\nu \neq k)$$

where $\eta\delta' < \delta$ and σ is an arbitrary positive number. Then, from (3.7) (i), it is seen that, in the sufficiently small neighborhood $C_{\eta_1(l^0)} \times V_{\bar{k}}(l^0) \times U_{\varepsilon_1(l^0)}$ where $\eta_1(l^0) \leqq \eta$, $V_{\bar{k}}(l^0) \subset D_{\bar{k}}$ and $\varepsilon_1(l^0) \leqq \varepsilon$, the equation

$$\phi_k(\rho, l, t) = l_k \tag{3.8 (i)}$$

has the unique solution

$$t = \omega_k(\rho, l), \tag{3.9 (i)}$$

such that $|\omega_k(\rho, l)-\omega_0|<\varepsilon_1(l^0)$ for $\rho \in C_{\eta_1(l^0)}$, $(l_\nu) \in V_{\bar{k}}(l^0)$. Evidently this unique solution is regular in the domain $C_{\eta_1(l^0)} \times V_{\bar{k}}(l^0)$. Now, from (3.6), it is evident that $\phi_k(0, l, \omega_0) = l_k$ for $(l_\nu) \in V_{\bar{k}}(l^0)$, consequently from uniqueness of the solution of (3.8) (i), it follows that

$$\omega_k(0, l) = \omega_0. \tag{3.10}$$

Let the function $\omega_k(\rho, l)$ defined corresponding to $(l_\nu^{0\prime}) \in D_{\bar{k}}$ be $\omega_k'(\rho, l)$ and assume that $V_{\bar{k}}(l^0) \cap V_{\bar{k}}(l^{0\prime}) \neq \emptyset$. Let any $(l_\nu) \in V_{\bar{k}}(l^0) \cap V_{\bar{k}}(l^{0\prime})$ be $(l_\nu^{0\prime\prime})$. Then there exists a small neighborhood $C_{\eta_1(l^{0\prime\prime})} \times V_{\bar{k}}(l^{0\prime\prime}) \times U_{\varepsilon_1(l^{0\prime\prime})}$ in which the solution of (3.8) (i) is unique. Since $\omega_k(\rho, l)$ and $\omega_k'(\rho, l)$ are continuous with respect to ρ, from (3.10), there exists a small neighborhood $C_{\eta'}$ in which $\omega_k(\rho, l^{0\prime\prime})$, $\omega_k'(\rho, l^{0\prime\prime}) \in U_{\varepsilon_1(l^{0\prime\prime})}$. If we take $C_{\eta'}$ so small that $C_{\eta'} \subset C_{\eta_1(l^{0\prime\prime})} \cap C_{\eta_1(l^0)} \cap C_{\eta_1(l^{0\prime})}$, then, from uniqueness of the solution of (3.8) (i), it must be that $\omega_k(\rho, l^{0\prime\prime}) = \omega_k'(\rho, l^{0\prime\prime})$ for $\rho \in C_{\eta'}$. Now $\omega_k(\rho, l^{0\prime\prime})$ and $\omega_k'(\rho, l^{0\prime\prime})$ are regular in $C_{\eta_1(l^0)} \cap C_{\eta_1(l^{0\prime})}$, consequently $\omega_k(\rho, l^{0\prime\prime}) = \omega_k'(\rho, l^{0\prime\prime})$ in $C_{\eta_1(l^0)} \cap C_{\eta_1(l^{0\prime})}$. But $(l_\nu^{0\prime\prime})$ is any point $\in V_{\bar{k}}(l^0) \cap V_{\bar{k}}(l^{0\prime})$. Thus we see that $\omega_k(\rho, l) = \omega_k'(\rho, l)$ in $(C_{\eta_1(l^0)} \cap C_{\eta_1(l^{0\prime})}) \times (V_{\bar{k}}(l^0) \cap V_{\bar{k}}(l^{0\prime}))$, in other words, each of $\omega_k(\rho, l)$ and $\omega_k'(\rho, l)$ is an analytic continuation of the other. Now, since $D_{\bar{k}}$ is compact, $D_{\bar{k}}$ is covered by a finite number of $\{V_{\bar{k}}(l^0)\}$. Thus we see that the elements of the functions $\omega_k(\rho, l)$ define a one-valued regular function in the ring domain $C_{\eta_1(\sigma)} \times D_{\bar{k}}$ where $C_{\eta_1(\sigma)} = \cap C_{\eta_1(l^0)}$. We shall express this one-valued regular function also by $\omega_k(\rho, l)$. Then, from the method of construction, it is evident that

$$\omega_k(0, l) = \omega_0 \quad \text{and} \quad |\omega_k(\rho, l) - \omega_0| < \varepsilon. \tag{3.11 (i)}$$

In the same way, starting from (3.7) (ii), we obtain a one-valued regular function $\omega_{\bar{k}}(\rho, l)$ in the ring domain $C_{\eta_2(\sigma)} \times D_k$ such that $\omega_{\bar{k}}(\rho, l)$ is a unique solution of the equation

$$\phi_{\bar{k}}(\rho, l, t) = l_{\bar{k}} \tag{3.8 (ii)}$$

in $C_{\eta_2(l^0)} \times V_k(l^0) \times U_{\varepsilon_2(l^0)}$ where $\eta_2(l^0) \leqq \eta$, $V_k(l^0) \subset D_k$ and $\varepsilon_2(l^0) \leqq \varepsilon$. As in $\omega_k(\rho, l)$, it is valid that

$$\omega_{\bar{k}}(0, l) = \omega_0 \quad \text{and} \quad |\omega_{\bar{k}}(\rho, l) - \omega_0| < \varepsilon. \tag{3.11 (ii)}$$

Now, since $\omega(x)$ is a universal period, it is valid that

$$\varphi_\nu(x, \omega(x)) = x_\nu, \tag{3.12}$$

consequently, since $\omega(0) = \omega_0$, for real ρ and l_ν's such that $\rho \in C_\eta$ and $|l_\nu| \leqq \delta'$, it is valid that

(3.13) $$\phi_\nu(\rho, l, \omega(\rho l)) = l_\nu.$$

Since $\omega(x) = \omega(\rho l)$ is continuous at the origin, for any real $(l_\nu) \in D_{\bar{k}}$, it is possible to take $\xi_1(l^0)$ so that, for real $\rho \in C_{\xi_1(l^0)}$, $\omega(\rho l) \in U_{\varepsilon_1(l^0)}$ where (l_ν^0) is a point such that $(l_\nu) \in V_{\bar{k}}(l^0)$. Since, for $\rho \in C_{\xi_1(l^0)} \cap C_{\eta_1(\sigma)}$, $(l_\nu) \in V_{\bar{k}}(l^0)$ and $t \in U_{\varepsilon_1(l^0)}$, the solution of the equation (3.8) (i) is unique, it must be that $\omega(\rho l) = \omega_k(\rho, l)$ for real $\rho \in C_{\xi_1(l^0)} \cap C_{\eta_1(\sigma)}$ and real $(l_\nu) \in V_{\bar{k}}(l^0)$. Now, as is remarked before, $D_{\bar{k}}$ is covered by a finite number of $\{V_{\bar{k}}(l^0)\}$. Consequently, if we write $C_{\xi_1} = \cap C_{\xi_1(l^0)}$, we see that $\omega(\rho l) = \omega_k(\rho, l)$ for real $\rho \in C_{\xi_1} \cap C_{\eta_1(\sigma)}$ and real $(l_\nu) \in D_{\bar{k}}$. Reasoning on the equation (3.8) (ii), in the same way as on (3.8) (i), we see that, for real $\rho \in C_{\xi_2} \cap C_{\eta_2(\sigma)}$ and real $(l_\nu) \in D_k$, $\omega(\rho l) = \omega_{\bar{k}}(\rho, l)$. Thus we see that, for real $\rho \in C_{\eta_0(k)} = (C_{\xi_1} \cap C_{\xi_2}) \cap (C_{\eta_1(\sigma)} \cap C_{\eta_2(\sigma)})$ and real $(l_\nu) \in \Delta_k = D_{\bar{k}} \cap D_k$,

(3.14) $$\omega(\rho l) = \omega_k(\rho, l) = \omega_{\bar{k}}(\rho, l).$$

Now the functions $\omega_k(\rho, l)$ and $\omega_{\bar{k}}(\rho, l)$ are regular in the domain $C_{\eta_0(k)} \times \Delta_k$, therefore, from (3.14), we see that the functions $\omega_k(\rho, l)$ and $\omega_{\bar{k}}(\rho, l)$ coincide with each other in the domain $C_{\eta_0(k)} \times \Delta_k$. Then the Laurent expansions of $\omega_k(\rho, l)$ and $\omega_{\bar{k}}(\rho, l)$ in $C_{\eta_0(k)} \times \Delta_k$ must coincide with each other, consequently their expansions cannot contain the negative powers of $l_{\bar{k}}$ and l_k, because $\omega_k(\rho, l)$ and $\omega_{\bar{k}}(\rho, l)$ are regular in $C_{\eta_0(k)} \times D_{\bar{k}}$ and $C_{\eta_0(k)} \times D_k$ respectively. Thus we see that the function $\omega_k(\rho, l) = \omega_{\bar{k}}(\rho, l)$ can be continuated to $l_{\bar{k}} = l_k = 0$, in other words, in the domain $C_{\eta_0(k)} \times \Delta$ (Δ: $|l_\nu| \leqq \delta'$), a regular function $\tilde{\omega}_k(\rho, l)$ can be constructed so that $\tilde{\omega}_k(\rho, l)$ may coincide with $\omega_k(\rho, l)$ and $\omega_{\bar{k}}(\rho, l)$ in $C_{\eta_0(k)} \times D_{\bar{k}}$ and $C_{\eta_0(k)} \times D_k$ respectively. From (3.11) and the proof of (3.14), it is seen that $\tilde{\omega}(0, l) = \omega_0$ and, for real ρ and l_ν's, $\tilde{\omega}_k(\rho, l)$ takes a real value which cincides with $\omega(\rho l)$ when $\rho \in C_{\eta_0(k)}$ and $(l_\nu) \in D_{\bar{k}} \cup D_k$.

For real ρ such that $0 < |\rho| < \eta_0(k)$, consider the function

(3.15) $$\tilde{\omega}_k(\rho, x/\rho) = \tilde{\omega}_{k\rho}(x).$$

Then $\tilde{\omega}_{k\rho}(x)$ is regular in the domain $\tilde{D}_\rho$: $|x_\nu| \leqq |\rho|\delta'$ and takes a real value $\omega(\rho l) = \omega(x)$ for real x_ν's such that $|\rho|\sigma \leqq (|x_{\bar{k}}|$ or $|x_k|) \leqq |\rho|\delta'$ and $|x_\nu| \leqq |\rho|\delta'$ ($\nu \neq \bar{k}$ or k). Then, since $\tilde{\omega}_{k\rho_1}(x)$ and $\tilde{\omega}_{k\rho_2}(x)$ coincides with each other for real x_ν's such that $\sigma \max(|\rho_1|, |\rho_2|) \leqq (|x_{\bar{k}}|, |x_k|) \leqq \delta' \min(|\rho_1|, |\rho_2|)$ and $|x_\nu| \leqq \delta' \min(|\rho_1|, |\rho_2|)$ ($\nu \neq k, \bar{k}$), $\tilde{\omega}_{k\rho}(x)$ determines a function $\tilde{\omega}_k(x)$ regular in the domain D^k: $|x_\nu| < \eta_0(k)\delta'$ such that $\tilde{\omega}_k(x) = \tilde{\omega}_{k\rho}(x)$ in $\tilde{D}_\rho$. Put $\min_k \eta_0(k) = \eta_0$ and let us consider the domain D: $|x_\nu| < \eta_0\delta'$. Then the functions

$\tilde{\omega}_k(x)$'s $(k=1, 2, \dots, m)$ are regular in D and moreover they coincide with $\omega(x)$ for real x_ν's such that $|\rho|\sigma \leqq (|x_{\bar{\nu}}|, |x_\nu|) \leqq |\rho|\delta'$ for $0<|\rho|<\eta_0$. Consequently they coincide with one another in D, in other words, in the domain D, they determine a regular funtion $\tilde{\omega}(x)$ which coincides with $\omega(x)$ for real x_ν's such that, for certain k, $|\rho|\sigma \leqq (|x_{\bar{k}}|$ or $|x_k|) \leqq |\rho|\delta'$ and $|x_\nu| \leqq |\rho|\delta'$ $(\nu \neq \bar{k}$ or $k)$ for $0<|\rho|<\eta_0$.

Take real (x_ν) arbitrarily in D. Then, if the point (x_ν) is not the origin, we take $\rho = x_{\bar{k}}/\delta'$ or $\rho = x_k/\delta'$ according as $|x_{\bar{k}}| = \max_\nu |x_\nu|$ or $|x_k| = \max_\nu |x_\nu|$. Since the point (x_ν) is not the origin, $\rho \neq 0$, consequently it is valid that $|\rho|\sigma < (|x_{\bar{k}}|$ or $|x_k|) = |\rho|\delta'$, $|x_\nu| \leqq |\rho|\delta'$ $(\nu \neq \bar{k}$ or $k)$ and $0<|\rho|<\eta_0$. Then, from the characteristic property of $\tilde{\omega}(x)$, it holds that $\tilde{\omega}(x) = \omega(x)$. Now $\tilde{\omega}(x)$ and $\omega(x)$ are both continuous at the origin, consequently the equality $\tilde{\omega}(x) = \omega(x)$ holds also at the origin, namely the equality $\tilde{\omega}(x) = \omega(x)$ holds for any real $(x_\nu) \in D$. Since $\tilde{\omega}(x)$ is regular in D, ultimately we have:

When $|C| \neq 0$, *there exists a universal period* $\omega(x)$ *and it is analytic at the origin.*

When $|C|=0$, there may not exist a universal period, namely a multiple of the primitive period continuous at the origin. But, when $C \neq 0$, namely when at least one of the characteristic roots of C does not vanish, as in the case where $|C| \neq 0$, it is readily seen that, not in the whole space but in the subspace R such that, for a small positive number $\sigma' < 1/\sqrt{n}$, $x_{\bar{k}}/\sqrt{\sum_\nu x_\nu^2}$ or $x_k/\sqrt{\sum_\nu x_\nu^2} \geqq \sigma'$ for at least one k, there exists a multiple of the primitive period continuous at the origin. Let this multiple be $\omega(x)$, then, making use of this $\omega(x)$ instead of the universal period and taking σ so that $l_{\bar{k}}/\sqrt{\sum_\nu l_\nu^2}$ or $l_k/\sqrt{\sum_\nu l_\nu^2} \geqq \sigma/\sqrt{n}\,\delta' > \sigma'$, namely so that the point (x_ν) corresponding to any real $(\rho, l_\nu) \in C_\eta \times (D_{\bar{k}} \cup D_k)$ may belong to R for any k, we see that there exists a function $\tilde{\omega}(x)$ regular in D such that, for any real $(x_\nu) \in D$, $\tilde{\omega}(x) = \omega(x)$ provided that x_ν's satisfy the conditions as follows:

$$
(3.16)\qquad \begin{cases} \text{for certain } k, \\ \qquad |\rho|\sigma \leqq (|x_{\bar{k}}| \text{ or } |x_k|) \leqq |\rho|\delta' \\ \qquad \text{and } |x_\nu| \leqq |\rho|\delta' \qquad (\nu \neq \bar{k} \text{ or } k) \\ \text{for } \rho \text{ such that } 0<|\rho|<\eta_0. \end{cases}
$$

Now $\omega(x)$ is a multiple of the primitive period, consequently it is valid that $\varphi_\nu(x, \tilde{\omega}(x)) = x_\nu$ for real (x_ν) satisfying (3.16). But, from (3.11), $|\tilde{\omega}(x) - \omega_0| < \varepsilon$

for $(x_\nu)\in D$, consequently the functions $\varphi_\nu(x, \tilde{\omega}(x))$ $(\nu=1, 2, \dots, n)$ are regular in D because $|x_\nu|<\eta_0\delta'\leqq\eta\delta'<\delta$. Thus we see that

$$\varphi_\nu(x, \tilde{\omega}(x)) = x_\nu \tag{3.17}$$

for any $(x_\nu)\in D$. Now, from the characteristic property of $\tilde{\omega}(x)$, $\tilde{\omega}(x)$ is real for real x_ν's and, from $|\tilde{\omega}(x)-\omega_0|<\varepsilon<\omega_0$, $\tilde{\omega}(x)>0$ for real x_ν's, Thus, from (3.17), we see:

When $|C|=0$, *if* $C\neq 0$, *there also exists a universal period* $\tilde{\omega}(x)$ *analytic at the origin.*

§ 4. The canonical form of the system.

In this paragraph, we consider the case where there exists a universal period analytic at the origin. By § 3, the case where $C\neq 0$ falls under our case. Let the universal period be $\omega(x)$ and let the relations $\omega(0)=\omega_0$ and θ_k's be (2.16).

Suggested by the invariant integral of the group of transformations (2.3) defined by (1.1) [2], we consider the integrals as follows:

$$\begin{cases} F_k(x) = \dfrac{1}{\omega(x)}\displaystyle\int_0^{\omega(x)}\Big[\varphi_k(x,\tau)\cos\frac{2\theta_k\pi}{\omega(x)}\tau + \varphi_{\bar{k}}(x,\tau)\sin\frac{2\theta_k\pi}{\omega(x)}\tau\Big]\,d\tau, \\[2ex] F_{\bar{k}}(x) = \dfrac{1}{\omega(x)}\displaystyle\int_0^{\omega(x)}\Big[-\varphi_k(x,\tau)\sin\frac{2\theta_k\pi}{\omega(x)}\tau + \varphi_{\bar{k}}(x,\tau)\cos\frac{2\theta_k\pi}{\omega(x)}\tau\Big]\,d\tau, \\[2ex] F_p(x) = \dfrac{1}{\omega(x)}\displaystyle\int_0^{\omega(x)}\varphi_p(x,\tau)\,d\tau. \end{cases} \tag{4.1}$$

Since, for $|x_\nu|\ll 1$, $\omega(x)$ is expand as

$$\omega(x) = \omega_0 + [x]_1, \tag{4.2}$$

where $[x]_1$ is the sum of the terms of the first and higher orders with respect to x_ν, $2\theta_k\pi/\omega(x)$ becomes

$$\frac{2\theta_k\pi}{\omega(x)} = \frac{2\theta_k\pi}{\omega_0+[x]_1} = \theta_k + [x]_1.$$

Consequently, from (2.20), for $|x_\nu|\ll 1$, it follows that

$$F_k(x) = \frac{1}{\omega(x)} \int_0^{\omega(x)} \Big[(x_k \cos \theta_k \tau - x_{\bar{k}} \sin \theta_k \tau + [x]_2)(\cos \theta_k \tau + [x]_1)$$

$$+ (x_k \sin \theta_k \tau + x_{\bar{k}} \cos \theta_k \tau + [x]_2)(\sin \theta_k \tau + [x]_1) \Big] d\tau$$

$$= \frac{1}{\omega(x)} \int_0^{\omega(x)} (x_k + [x]_2)\, d\tau$$

$$= x_k + [x]_2,$$

where $[x]_N$ denotes the sum of the terms of the N-th and higher orders with respect to x_ν. In the same manner, $F_{\bar{k}}(x)$ and $F_p(x)$ are calculated. Summarizing these results, we can write them as follows:

$$F_\nu(x) = x_\nu + [x]_2. \qquad (\nu = 1, 2, \dots, n) \tag{4.3}$$

Put

$$y_\nu = F_\nu(x), \tag{4.4}$$

then, from (4.3), this gives an analytic transformation of the variables.

Now, since $\omega(x)$ is an invariant of the equations (1.1), $\omega[\varphi(x, t)] = \omega(x)$. Then, by means of the property that $\varphi_\nu[\varphi(x, t), \tau] = \varphi_\nu(x, t+\tau)$, $F_k[\varphi(x, t)]$'s are calculated in the following way:

$$F_k[\varphi(x, t)] = \frac{1}{\omega(x)} \int_0^{\omega(x)} \Big[\varphi_k(x, t+\tau) \cos \frac{2\theta_k \pi}{\omega(x)} \tau$$

$$+ \varphi_{\bar{k}}(x, t+\tau) \sin \frac{2\theta_k \pi}{\omega(x)} \tau \Big] d\tau$$

$$= \frac{1}{\omega} \int_t^{t+\omega} \Big[\varphi_k(x, \tau) \cos \frac{2\theta_k \pi}{\omega} (\tau - t) + \varphi_{\bar{k}}(x, \tau) \sin \frac{2\theta_k \pi}{\omega} (\tau - t) \Big] d\tau$$

$$= \frac{1}{\omega} \int_t^{t+\omega} \Big[(\varphi_k \cos \frac{2\theta_k \pi}{\omega} \tau + \varphi_{\bar{k}} \sin \frac{2\theta_k \pi}{\omega} \tau) \cos \frac{2\theta_k \pi}{\omega} t$$

$$- (-\varphi_k \sin \frac{2\theta_k \pi}{\omega} \tau + \varphi_{\bar{k}} \cos \frac{2\theta_k \pi}{\omega} \tau) \sin \frac{2\theta_k \pi}{\omega} t \Big] d\tau$$

$$= F_k(x) \cos \frac{2\theta_k \pi}{\omega(x)} t - F_{\bar{k}}(x) \sin \frac{2\theta_k \pi}{\omega(x)} t,$$

since the integrands are periodic with the period $\omega(x)$. In the same way, $F_{\bar{k}}[\varphi(x, t)]$ and $F_p[\varphi(x, t)]$ are calculated. Thus

$$(4.5)\qquad \begin{cases} F_k[\varphi(x,t)] = F_k(x)\cos\dfrac{2\theta_k\pi}{\omega(x)}t - F_{\bar{k}}(x)\sin\dfrac{2\theta_k\pi}{\omega(x)}t, \\ F_{\bar{k}}[\varphi(x,t)] = F_k(x)\sin\dfrac{2\theta_k\pi}{\omega(x)}t + F_{\bar{k}}(x)\cos\dfrac{2\theta_k\pi}{\omega(x)}t, \\ F_p[\varphi(x,t)] = F_p(x). \end{cases}$$

These formulas can be written in terms of y_ν as follows:

$$(4.6)\qquad \begin{cases} \psi_k(y,t) = y_k\cos\dfrac{2\theta_k\pi}{\tilde{\omega}(y)}t - y_{\bar{k}}\sin\dfrac{2\theta_k\pi}{\tilde{\omega}(y)}t, \\ \psi_{\bar{k}}(y,t) = y_k\sin\dfrac{2\theta_k\pi}{\tilde{\omega}(y)}t + y_{\bar{k}}\cos\dfrac{2\theta_k\pi}{\tilde{\omega}(y)}t, \\ \psi_p(y,t) = y_p, \end{cases}$$

where $\tilde{\omega}(y) = \omega\{F^{-1}(y)\}$ and $\psi_\nu(y,t) = F[\varphi\{F^{-1}(y),t\}]$. Differentiating (4.6) with respect to t, we have:

$$\begin{cases} \dfrac{d\psi_k}{dt} = -\dfrac{2\theta_k\pi}{\tilde{\omega}(y)}\psi_{\bar{k}}, \\ \dfrac{d\psi_{\bar{k}}}{dt} = \dfrac{2\theta_k\pi}{\tilde{\omega}(y)}\psi_k, \\ \dfrac{d\psi_p}{dt} = 0. \end{cases}$$

Since $\tilde{\omega}(y) = \tilde{\omega}[\psi(y,t)]$, the above equations can be written as follows:

$$(4.7)\qquad \begin{cases} \dfrac{dy_k}{dt} = -\dfrac{2\theta_k\pi}{\tilde{\omega}(y)}y_{\bar{k}}, \\ \dfrac{dy_{\bar{k}}}{dt} = \dfrac{2\theta_k\pi}{\tilde{\omega}(y)}y_k, \\ \dfrac{dy_p}{dt} = 0. \end{cases}$$

Thus we see that, by the analytical transformation (4.4), the system (2.17) is reduced to the system (4.7). Here $\tilde{\omega}(y)$ is an invariant for the reduced system and, for $|y_\nu| \ll 1$, from (4.2), it is expressed as

$$(4.8)\qquad \tilde{\omega}(y) = \omega_0 + [y]_1,$$

where $\omega_0 > 0$ when $m > 0$. Now the system (2.17) is a reduced system of (1.1) by a linear transformation. Thus we have the conclusion:

When there exists a universal period analytic at the origin, by the analytic trans-

formation leaving the origin fixed, the system (1.1) *can be reduced to the system of the form* (4.7) *where* $\tilde{\omega}(y)$ *is an invariant of the form* (4.8).

When $C \neq 0$, by §3, the condition of the above conclusion is fulfilled, consequently *the conclusion is valid.*

From the method of constructing (4.7), it is evident that the general solutions of (4.7) are the functions given by (4.6). Consequently we see that *the motion described by the original system* (1.1) *is equivalent to a direct sum of the plane rotations* and that, *in the* x_ν*-space, any point on the variety* $V: y_1 = y_{\bar{1}} = \cdots = y_m = y_{\bar{m}} = 0$ *is fixed.* The variety V is a point-wise invariant variety and any point on V is a center. When $0 < 2m < n$, such variety V contains a continuum of points, consequently *the center can not be isolated.* When $2m = n$, i.e. $|C| \neq 0$, the variety V consists of only one point, consequently *the center is isolated.*

REMARK. The case on which Hukuhara has studied [1] is the case where $n = 2$ and $|C| \neq 0$. Our results are a generalization of those of Hukuhara, though the methods taken are entirely different.

REFERENCES

[1] M. Hukuhara, *Sur un système de deux équations différentielles ordinaires nonlinéaires à coefficients réels*, J. Fac. Sci. Univ. Tôkyô, 6 (1951), 295-317.

[2] On the application of the invariant integral of the group of transformations, cf.:

M. Urabe, *Application of majorized group of transformations to functional equations*, J. Sci. Hiroshima Univ., Ser. A 16 (1952), 267-283,

do., *Yûkai Henkangun no Kansû-Hôteishiki eno Ôyô* (*Application of majorized group of transformations to functional equations*), Sûgaku, 6 (1954), 65-73. (In Japanese)

Department of Math., Fac. Sci., Hiroshima Univ. and
Department of Math., Fac. Sci., University of Tôkyô.