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A Structure Theorem for Complete Quasi-Unitary Algebras

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A $*$-algebra $\boldsymbol{A}$ with an inner product $\langle x, y\rangle$ is called unitary [**4**] provided it is a pre-Hilbert space and the following axioms are satisfied:

(AU1) $\langle xy, z\rangle = \langle y, x^* z\rangle$ *for* $x, y, z \in \boldsymbol{A}$;

(AU2) $\langle x, x\rangle = \langle x^*, x^*\rangle$ *for* $x \in \boldsymbol{A}$;

(AU3) *for every* $x \in \boldsymbol{A}$, *the operator* $U_x : y \rightarrow xy$ *is continuous*;

(AU4) *the elements of the form* xy, $x, y \in \boldsymbol{A}$, *are everywhere dense in* $\boldsymbol{A}$.

Furthermore if $\boldsymbol{A}$ is complete, that is, a Hilbert space, then (AU1)—(AU4) are equivalent to (AU1), (AU2) and

(A5)' $x^* x = 0$ *implies* $x = 0$ *for* $x \in \boldsymbol{A}$.

And then $\boldsymbol{A}$ is merely a (proper) H^*-algebra of Ambrose [**1**] in which the inner product has been replaced by the one obtained from $\langle , \rangle$ multiplied by a positive constant in such a way that the multiplicative property $\|xy\| \leqq \|x\|\,\|y\|$ holds. He determined the structure of $\mathbf{H}^*$-algebras.

In his investigations on group algebras on locally compact groups, J. Dixmier [**2**] introduced the concept of quasi-unitary algebras, generalizing unitary ones. A $*$-algebra $\boldsymbol{A}$ with an inner product $\langle x, y\rangle$ is called quasi-unitary provided it is a pre-Hilbert space and an automorphism J: $x \rightarrow x^j$ is defined in $\boldsymbol{A}$ and the following axioms are satisfied:

(A1) $\langle x, x^j\rangle \geq 0$ *for* $x \in \boldsymbol{A}$;

(A2) $\langle x, x\rangle = \langle x^*, x^*\rangle$ *for* $x \in \boldsymbol{A}$;

(A3) $\langle xy, z\rangle = \langle y, x^{*j} z\rangle$ *for* $x, y, z \in \boldsymbol{A}$;

(A4) *for every* $x \in \boldsymbol{A}$, *the operator* U_x: $x \rightarrow xy$ *is continuous*;

(A5) *the elements of the form* $xy + (xy)^j$ *are everywhere dense in* $\boldsymbol{A}$.

In case $\boldsymbol{A}$ is finite-dimensional, he determined its structure ([2], Cor. 2, p. 287). The purpose of this paper is to generalize this result for the case where $\boldsymbol{A}$ is complete. To this end we first show that $\boldsymbol{A}$ is then equivalent to an H^*-algebra with an automorphism $J: x \rightarrow x^j$ shared with a certain property described in Theorem 1 below. Then along the line of his proof for the finite-dimensional case we determine the structure of complete quasi-unitary algebras (Theorem 2). Throughout this paper we shall assume the elementary results obtained in [2].

§1. The relation to H^*-algebras

First we note that, when $\boldsymbol{A}$ is complete, (A1)—(A5) are equivalent to (A1)—(A3) and (A5)$'$. (A5)$'$ holds for every quasi-unitary algebra. For suppose that $x^* x = 0$. $\langle x y , x^j y\rangle = \langle y , (x^* x)^j y\rangle = 0$ implies, by letting V_y (the operator $x \rightarrow xy$) converge strongly to the identity operator I, $\langle x , x^j\rangle = 0$ and in turn $x = 0$. (cf. [2], p. 278). Suppose that (A1)—(A3) and (A5)$'$ hold. (A4) follows from (A3) via the closed graph theorem and therefore by (A2) xy is a continuous function of x and y, that is, $\|xy\| \leq k\|x\|\,\|y\|$, k being a positive constant, where we may assume that $k=1$ since $k^2\langle x , y\rangle$ can be taken as an inner product instead of $\langle x , y\rangle$. Then $\boldsymbol{A}$ is a Banach $*$-algebra. Let $|x|$ stand for the operator norm of U_x. $x \neq 0$ implies $|x| > 0$ since $x x^* \neq 0$ by (A5)$'$. Clearly $\|x y\| \leq |x|\,\|y\|$. $|x|^2 = \underset{\|y\|\leq 1}{l.\,u.\,b.}\langle x y , x y\rangle \leq \frac{1}{\alpha}\underset{\|y\|\leq 1}{l.\,u.\,b.}\langle xy , x^j y^j\rangle$ $= \frac{1}{\alpha}\underset{\|y\|\leq 1}{l.\,u.\,b.}\langle x^*xy , y^j\rangle \leq \frac{1}{\alpha}|x^*x|\underset{\|y\|\leq 1}{l.\,u.\,b.}\|y^j\| \leq \frac{\beta}{\alpha}|x^*x|$, where positive constant α, β are taken such that $\alpha I \leq J \leq \beta I$. Owing to a theorem ([**5**], Theorem 16, p. 30) $\boldsymbol{A}$ is a dual A*-algebra of the 1st kind. Then the closure of $x\boldsymbol{A}$ contains x and therefore the elements of the form xy are everywhere dense in $\boldsymbol{A}$. Since $I + J$ has a bounded inverse it follows that (A5) holds.

Remark. Let $\boldsymbol{A}$ be a $*$-algebra with an inner product $\langle x , y\rangle$ under which $\boldsymbol{A}$ is a Hilbert space. The following axioms are assumed: (i) $\langle xy , z\rangle = \langle x , zy^*\rangle$; (ii) for every $x \in \boldsymbol{A}$, the operator U_x is continuous; (iii) $x^*x = 0$ implies $x = 0$. Then V_y is continuous by (i) via the closed graph theorem. We may assume $\|xy\| \leq \|x\|\,\|y\|$ by the same reason as above. Then $\boldsymbol{A}$ is a *right* H^*-algebra of Smiley [**8**]. Let $|x|$ stand for the operator norm of V_x. Then the above reasoning shows that $\boldsymbol{A}$ is a dual A^*-algebra of the 1st kind. The result was obtained by Smiley [8] following the argument of Ambrose

[1]. It is noted that in every A^*-algebra $x \to x^*$ is continuous ([7], Lemma 5.3).

Lemma 1. *A complete quasi-unitary algebra $\boldsymbol{A}$ can be renormed in such a way that it becomes an H^*-algebra with the same involution.*

Proof. If we introduce the inner product $\langle x, y\rangle' = \langle x^{j^{-1}}, x\rangle$, then $\boldsymbol{A}$ with this inner product will satisfy (i)—(iii) of the remark above, and the lemma follows from the result of Smiley [8]. For the sake of completeness, using the fact that $\boldsymbol{A}$ is a dual A^*-*algebra* of the 1st kind, we sketch the proof of the lemma. $\boldsymbol{A}$ is the closure of the direct sum of simple closed $*$-ideals P_ι. We show that $JP_\iota = P_\iota$, that is, each P_ι is a simple complete quasi-unitary algebra. To this end it is sufficient to show that $x^j x \neq 0$ for a non-zero self-adjoint element $x \in P_\iota$. Suppose the contrary. $\langle xy, xy\rangle = \langle x, x^j xy\rangle = 0$. Then $x\boldsymbol{A} = 0$. Since $\boldsymbol{A}$ is dual, we have $x = 0$, a contradiction. P_ι are orthogonal to each other. For $x \in P_\iota$, $y \in P_\kappa$, $\iota \neq \kappa$, we have $\langle xz, y\rangle = \langle z, x^{*j}y\rangle = 0$ for every $z \in \boldsymbol{A}$. Since the closure of $x\boldsymbol{A}$ contains x, we have $\langle x, y\rangle = 0$. Therefore $\boldsymbol{A}$ is the Hilbert space sum of P_ι.

Let $\{e_{\iota,\alpha}\}$ be a maximal family of orthogonal self-adjoint primitive idempotents of P_ι. Let $\{e_{\iota,\alpha\beta}\}$ be a system of matrix units associated with $\{e_{\iota,\alpha}\}$, that is, $e_{\iota,\alpha\alpha} = e_{\iota,\alpha}$, $e_{\iota\alpha\beta} = e^*_{\iota,\beta\alpha}$, $e_{\iota,\alpha\beta}\, e_{\iota,\beta\gamma} = e_{\iota,\alpha\gamma}$, and $e_{\iota,\alpha\beta}\, e_{\iota,\gamma\delta} = 0$ for $\beta \neq \gamma$. Let $z = \sum'_{\iota,\alpha,\beta} \lambda_{\iota,\alpha\beta}\, e_{\iota,\alpha\beta}$ be any finite linear combination of $e_{\iota,\alpha\beta}$. Then for some constants c, c', c'', c''' such that $\dfrac{1}{\alpha} \geq c$, $c''' \geq \dfrac{1}{\beta}$ and $\alpha \leq c'$, $c'' \leq \beta$, we have

$$
\begin{aligned}
(1) \qquad \|z\|^2 &= \langle z, z\rangle = c\langle z, z^j\rangle \\
&= c\sum_\iota{}'\sum_\alpha{}'\langle \sum_\beta{}'\lambda_{\iota,\alpha\beta}\, e_{\iota,\alpha\beta},\ \sum_\beta{}'\ \lambda_{\iota,\alpha\beta}\, e^j_{\iota,\alpha\beta}\rangle \\
&= c\,c'\sum_\iota{}'\sum_\alpha{}'\langle \sum_\beta{}'\lambda_{\iota,\alpha\beta}\, e_{\iota,1\beta},\ \sum_\beta{}'\lambda_{\iota,\alpha\beta}\, e_{\iota,1\beta}\rangle \\
&= c\,c'\sum_\iota\sum_\alpha{}'\langle \sum_\beta{}'\bar{\lambda}_{\iota,\alpha\beta}\, e_{\iota,\beta 1},\ \sum_\beta{}'\bar{\lambda}_{\iota,\alpha\beta}\, e_{\iota,\beta 1}\rangle \\
&= c\,c'\,c''\,c'''\sum_{\iota,\alpha,\beta}{}'\ |\lambda_{\iota,\alpha\beta}|^2\ \|e_{\iota,1}\|^2.
\end{aligned}
$$

For $x, y \in \boldsymbol{A}$ we put $e_{\iota,\alpha}\, x e_{\iota,\beta} = \lambda_{\iota,\alpha\beta}\, e_{\iota,\alpha\beta}$, $e_{\iota,\alpha}\, y e_{\iota,\beta} = \mu_{\iota,\alpha\beta}\, e_{\iota,\alpha\beta}$. Then $\sum \lambda_{\iota,\alpha\beta}\, e_{\iota,\alpha\beta}$ is summable to x. Then the the equation (1) shows that

$$
(2) \qquad \frac{\alpha^2}{\beta^2}\ \|x\|^2 \leq \sum_{\iota,\alpha,\beta} |\lambda_{\iota,\alpha\beta}|^2\ \|e_{\iota,1}\|^2 \leq \frac{\beta^2}{\alpha^2}\ \|x\|^2.
$$

If we put $\langle\!\langle x, y\rangle\!\rangle = \sum \lambda_{\iota,\alpha\beta}\, \bar{\mu}_{\iota,\alpha\beta}\, \|e_{\iota,1}\|^2$, (2) shows that $\boldsymbol{A}$ is a Hilbert space with an inner product $\langle\!\langle x, y \rangle\!\rangle$. Owing to $\|e_{\iota,1}\| \geq 1$ simple computation will show that $\boldsymbol{A}$ is an H^*-algebra for $\langle\!\langle x, y \rangle\!\rangle$, as desired.

Theorem 1. (i) *Let $\boldsymbol{A}$ be a complete quasi-unitary algebra, and $\langle\!\langle x, y \rangle\!\rangle$ be an inner product under which $\boldsymbol{A}$ is an H^*-algebra with the same involution. Then a positive definite operator $M \in \boldsymbol{R}^d$ (right ring of $\boldsymbol{A}$) is uniquely determined in such a way that*

$$(\alpha) \qquad \langle x, y \rangle = \langle\!\langle x, MM'y \rangle\!\rangle, \quad J = MM'^{-1},$$

where $M' = SMS$ *and S is the involution* $Sx = x^*$.

Conversely (ii) *if $\boldsymbol{A}$ is an H^*-algebra with an inner product $\langle\!\langle x, y \rangle\!\rangle$, and M is a positive definite operator $\in \boldsymbol{R}^d$, then J defined by (α) is an automorphism of $\boldsymbol{A}$, and $\boldsymbol{A}$ is a complete quasi-unitary algebra with an inner product $\langle x, y \rangle$ defined by (α) and the automorphism J.*

Proof. Ad (i). Define J_1 and K by $\langle x, Jy \rangle = \langle\!\langle x, J_1 y \rangle\!\rangle$ and $\langle x, y \rangle = \langle\!\langle x, Ky \rangle\!\rangle$. It is clear that J_1, K are positive operators. We show that K is positive definite and $SKS = K$. Define L by $\langle x, Ly \rangle = \langle\!\langle x, y \rangle\!\rangle$. $\langle x, LKy \rangle = \langle\!\langle x, Ky \rangle\!\rangle = \langle x, y \rangle$ implies that $LK = I$. Similarly $KL = I$. Therefore K and L are positive definite with respect to both inner products $\langle x, y \rangle$ and $\langle\!\langle x, y \rangle\!\rangle$, and $K = L^{-1}$. $\langle\!\langle x, SKSy \rangle\!\rangle = \overline{\langle\!\langle Sx, KSy \rangle\!\rangle} = \overline{\langle Sx, Sy \rangle} = \langle x, y \rangle = \langle\!\langle x, Ky \rangle\!\rangle$ implies $SKS = K$. $\langle\!\langle x, J_1 y \rangle\!\rangle = \langle x, Jy \rangle = \langle\!\langle x, KJy \rangle\!\rangle$ implies $J_1 = KJ$. Hence J_1 has a bounded inverse and $SJ_1S = SKSSJS = KJ^{-1} = J_1 J^{-2}$. Since $\langle\!\langle x, zJ_1 y \rangle\!\rangle = \langle\!\langle z^* x, J_1 y \rangle\!\rangle = \langle z^* x, Jy \rangle = \langle x, J(zy) \rangle = \langle\!\langle x, J_1(zy) \rangle\!\rangle$, it follows that $J_1 \in \boldsymbol{R}^s$ and $J_1 J^{-2} \in \boldsymbol{R}^s$, and therefore $J_1(J_1 J^{-2}) = (J_1 J^{-2}) J_1$, that is, J_1 and J^{-2} are commutative, and so also for J_1 and J. Therefore K, J and J_1 are commutative with each other. Put $M = J_1^{\frac{1}{2}}$, and $M' = SMS = (SJ_1S)^{-\frac{1}{2}}$, then $J = (J_1 SJ_1^{-1} S)^{\frac{1}{2}} = MM'^{-1}$, and $K = J_1 J^{-1} = MM'$. Thus (α) holds. That M is uniquely determined is clear from $M^2 = KJ$.

Ad (ii). Since $\langle x, x^J \rangle = \langle x, Jx \rangle = \langle\!\langle x, M^2 x \rangle\!\rangle$, J is positive definite and has a bounded inverse. $J(xy) = MM'^{-1}(xy) = M'^{-1}M(U_x y) = M'^{-1}U_x My$ since $M \in \boldsymbol{R}^d$. On the other hand, $\langle\!\langle (MM'^{-1}x)M'^{-1}z, y \rangle\!\rangle = \langle\!\langle M'^{-1}z, (SMM'^{-1}x)y \rangle\!\rangle = \langle\!\langle (M'^{-1}z)y^*, SMM'^{-1}x \rangle\!\rangle = \langle\!\langle M'^{-1}(zy^*), M'M^{-1}x^* \rangle\!\rangle = \langle\!\langle zy^*, M^{-1}x^* \rangle\!\rangle = \langle\!\langle (M'^{-1}x)z, y \rangle\!\rangle$. This yields $M'^{-1}U_x = U_{M'^{-1}x} = U_{MM'^{-1}x}M'^{-1}$. Hence $J(xy) = M'^{-1}U_x My = (MM'^{-1}x)(M'^{-1}My) = (Jx)(Jy)$. Thus J is an automorphism of $\boldsymbol{A}$.

$$\langle x, x \rangle = \langle\!\langle x, MM'x \rangle\!\rangle = \langle\!\langle SMM'x, x^* \rangle\!\rangle$$
$$= \langle\!\langle M'Mx^*, x^* \rangle\!\rangle = \langle x^*, x^* \rangle.$$

$$\langle x y , z \rangle = \langle\!\langle x y , M M' y \rangle\!\rangle = \langle\!\langle y , x^* M M' y \rangle\!\rangle = \langle\!\langle y , x^* M^2 J^{-1} z \rangle\!\rangle$$
$$= \langle\!\langle y , M^2 J^{-1} ((J x^*) z) \rangle\!\rangle = \langle\!\langle y , M M'^{-1} x^{*j} z \rangle\!\rangle = \langle y , x^{*j} z \rangle.$$

These equations show that (A2) and (A3) hold. That (A5)′ holds follows from the property of H^*-algebras. The proof is completed.

§2. The structure of complete quasi-unitary algebras.

Let $\mathfrak{H}$ be a Hilbert space and let S_P stand for the ordinary trace of operators on $\mathfrak{H}$. An operator T is of Hilbert-Schmidt type if and only if $S_P(T T^*) < +\infty$. Let $\boldsymbol{H}_{\mathfrak{H}}$ be the set of such operators. $\boldsymbol{H}_{\mathfrak{H}}$ is a simple H^*-algebra with an inner product $(T , T_1) = S_P(T T_1^*)$. Every simple H^*-algebra with an inner product $\langle\!\langle x , y \rangle\!\rangle$ is $*$-isomorphic with an $\boldsymbol{H}_{\mathfrak{H}}$ and $\langle x , y \rangle = \alpha (T , T_1)$ if T and T_1 correspond to x and y respectively under this $*$-isomorphism where α is a positive constant ≥ 1. An H^*-algebra $\boldsymbol{A}$ is the closure of a direct sum of simple ideals $\boldsymbol{A}_\iota$ where $\boldsymbol{A}_\iota$ are mutually orthogonal H^*-algebras and $\boldsymbol{A}$ is the Hilbert space sum of $\boldsymbol{A}_\iota$. To each $\boldsymbol{A}_\iota$ correspond $\boldsymbol{H}_{\mathfrak{H}_\iota}$ and α_ι. We may take $\mathfrak{H}_\iota$ mutually orthogonal. Let $\mathfrak{H}$ be the Hilbert space sum of $\mathfrak{H}_\iota$. Take $T_\iota \in \boldsymbol{H}_{\mathfrak{H}_\iota}$ such that $\sum \alpha_\iota S_P(T_\iota T_\iota^*) < +\infty$. Then there exists an operator T on $\mathfrak{H}$ such that the restriction of T on $\mathfrak{H}_\iota$ is T_ι. The set $\boldsymbol{H}$ of such T is an H^*-algebra with an inner produt $(T , T_1) = \sum \alpha_\iota Sp(T_\iota T_{1\iota}^*)$. $\boldsymbol{A}$ and $\boldsymbol{H}$ are $*$-isomorphic under the obvious correspondence and $\langle\!\langle x , y \rangle\!\rangle = (T , T_1)$ if x and y correspond to T and T_1 respectively. Thus $\boldsymbol{A}$ has the same structure as $\boldsymbol{H}$. This is a result of Ambrose [**1**]. Let $\boldsymbol{M}_\iota$ be the set of operators on $\mathfrak{H}_\iota$. Take C_ι from each $\boldsymbol{M}_\iota$ in such a way that $\{\| C_\iota \|\}$ is bounded. There exists a unique operator C on $\mathfrak{H}$ such that the restriction on $\mathfrak{H}_\iota$ is C_ι. Such C is characterized as an operator commutative with each projection P_ι on $\mathfrak{H}_\iota$. The set of such operators is a ring of operators in the sense of J. v. Neumann. Let U_C be the operator on $\boldsymbol{H}$ defined by $U_C T = C T$. It is easy to see that the correspondence U is $*$-isomorphic and normal in the sense of Dixmier [**3**]. The image of U is contained in the left ring of $\boldsymbol{H}$ and contains the left multiplications by the elements of $\boldsymbol{H}$. Owing to a theorem due to Dixmier [**3**] the image of U coincides with the left ring of $\boldsymbol{H}$. $S U_C S T = T C^*$. Let C be a positive definite operator $\in \boldsymbol{M}$. Then $T \to C^{-1} T C$ is an automorphism J of $\boldsymbol{H}$. Put $\langle T , T_1 \rangle = (T , C T_1 C)$. It follows from Theorem 1 that $\boldsymbol{H}$ with an inner product $\langle T , T_1 \rangle$ and an automorphism J is a complete quasi-unitary algebra and every complete quasi-unitary algebra has the same structure as such an $\boldsymbol{H}$.

Consider the ring of operators generated by $\boldsymbol{H}$. It is easy to see that it coincides with $\boldsymbol{M}$. Put for $C\in \boldsymbol{M}^+$, $\Psi(C) = \sum \alpha_\iota S_p(C_\iota)$ where C_ι is the restriction of C on $\mathfrak{H}_\iota$. $\psi(C)$ is evidently a faithful, normal, essential pseudo-trace of $\boldsymbol{M}$ and $\boldsymbol{H}$ is the set of normed operators T, that is, $\psi(TT^*) < +\infty$. Here ψ has the following property: $\psi(P) \geq 1$ for every non-zero projection $P\in \boldsymbol{M}$. A ring of operators (containing 1) on a Hilbert space is $*$-isomorphic with a direct sum of factors of type I if and only if it has a faithful, normal, essential pseudo-trace ψ such that $\psi(P)\geq 1$, for every non-zero projection P of the ring [**6**]. Clearly $\boldsymbol{M}$ is a direct sum of factors of type I. Conversely let $\boldsymbol{M}$ be a ring of operators (containing 1) on a Hilbert space such that there exists a faithful, normal, essential pseudo-trace ψ such that $\Psi(P)\geq 1$, for every non-zero projection $P\in \boldsymbol{M}$. The set $\boldsymbol{H}$ of normed operators $T\in \boldsymbol{M}$, that is, $\psi(TT^*) < +\infty$, is an H^*-algebra with an inner product $(T, T_1) = \psi(TT_1^*)$ [**6**]. And $\boldsymbol{M}$ is $*$-isomorphic under the mapping U with the left ring of $\boldsymbol{H}$. Thus we have the following theorem.

Theorem 2. *Let $\boldsymbol{M}$ be a ring of operators (containing 1) on a Hilbert space $\mathfrak{H}$, with a faithful, normal, essential pseudo-trace ψ such that $\psi(P)\geq 1$, for every non-zero projection $P\in \boldsymbol{M}$. The set $\boldsymbol{H}$ of normed operators $T\in \boldsymbol{M}$ forms an H^*-algebra with an inner product $(T, T_1) = \psi(TT_1^*)$. Let C be a positive definite operator $\in \boldsymbol{M}$. Then J: $T\to C^{-1}TC$ is an automorphism of $\boldsymbol{H}$. Put $\langle T, T_1\rangle = (T, CT_1C)$. Then $\boldsymbol{H}$ is a complete quasi-unitary algebra with an inner product $\langle T, T_1\rangle$ and an automorphism J. Every complete quasi-unitary algebra has the same structure as such an $\boldsymbol{H}$.*

We note that if $\mathfrak{H}$ is finite-dimensional, this theorem gives the result obtained by Dixmier ([**2**], Cor. 2. p. 287).

Remark. For right H^*-algebras results analogous to Theorem 1 and 2 hold. To obtain the structure theorem we need only to define $\langle T, T_1\rangle = (T, CT_1)$ and adjust C such that $\|TT_1\| \leq \|T\|\,\|T_1\|$ holds. For example, let $\boldsymbol{H}$ be the H^*-algebra of operators of Hilbert-Schmidt type on a Hilbert space. Put $\langle T, T_1\rangle = Sp(CTT_1^*)$, C being a positive definite operator. In order that $\boldsymbol{H}$ becomes a Banach algebra with the norm $\langle T, T\rangle^{\frac{1}{2}}$, it is necessary and sufficient that $\|C\|\geq 1$. This will be proved by considering the spectral resolution of C and by noting that for primitive projections P $Sp(P) = 1$. A $*$-algebra $\boldsymbol{H}$ with $\langle T, T_1\rangle = Sp(CTT_1^*)$, $\|C\|\geq 1$, is a simple right H^*-algebra and conversely every simple right H^*-algebra is equivalent to such an $\boldsymbol{H}$. The result was obtained by Smiley [**8**] in the form of the matric right H^*-algebra.

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