JOURNAL OF SCIENCE OF THE HIROSHIMA UNIVERSITY, SER. A, VOL. 19, NO. 1, JULY, 1955

On the Derivations of Lie Algebras

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(Received Feb. 25, 1955)

1. Introduction

Let $\mathfrak{L}$, $\mathfrak{M}$ be Lie algebras over a field K of characteristic 0. A linear mapping D of $\mathfrak{L}$ into $\mathfrak{M}$ is called a derivation of $\mathfrak{L}$ into $\mathfrak{M}$ if $D(x\circ y)= D(x)\circ y+x\circ D(y)$ for all x, y in $\mathfrak{L}$. A derivation of $\mathfrak{L}$ into itself is simply called a derivation of $\mathfrak{L}$. The set $\mathfrak{D}(\mathfrak{L})$ of all derivations of $\mathfrak{L}$ forms a Lie algebra with the commutator product $D_1\circ D_2=D_2D_1-D_1D_2$, which is called the derivation algebra of $\mathfrak{L}$. For any element x of $\mathfrak{L}$, the adjoint mapping $D_x: y\rightarrow y\circ x$ is a derivation of $\mathfrak{L}$. Such a derivation is called inner. It is easy to see that the inner derivations of $\mathfrak{L}$ form an ideal in $\mathfrak{D}(\mathfrak{L})$ which we denote by $\mathfrak{J}(\mathfrak{L})$. Let $\mathfrak{L}_1$ be a subalgebra of $\mathfrak{L}$. We shall denote by $D|\mathfrak{L}_1$ the restriction to $\mathfrak{L}_1$ of a derivation D of $\mathfrak{L}$ and, for any subset $\mathfrak{E}$ of $\mathfrak{D}(\mathfrak{L})$, denote by $\mathfrak{E}|\mathfrak{L}_1$ the set of $D|\mathfrak{L}_1$ for all D in $\mathfrak{E}$. A subset of $\mathfrak{L}$ is called characteristic if it is mapped into itself by every derivation of $\mathfrak{L}$. The radical $\mathfrak{R}$ of $\mathfrak{L}$ is a characteristic ideal [2] so that $\mathfrak{D}(\mathfrak{L})|\mathfrak{R}$ is a subalgebra of $\mathfrak{D}(\mathfrak{R})$. If there exists a subalgebra $\mathfrak{L}_2$ such that $\mathfrak{L}=\mathfrak{L}_1+\mathfrak{L}_2$ and $\mathfrak{L}_1\cap\mathfrak{L}_2=0$, then we say that $\mathfrak{L}$ splits over $\mathfrak{L}_1$ and that $\mathfrak{L}_2$ is a complement of $\mathfrak{L}_1$ in $\mathfrak{L}$.

The purpose of this paper is to study the relations between the derivation algebras of Lie algebras and their radicals. By a well-known theorem of E. Cartan, every derivation of a semi-simple Lie algebra is an inner derivation. We give a necessary and sufficient condition for a derivation of $\mathfrak{L}$ to be inner (Theorem 1) and show that every derivation of $\mathfrak{L}$ is inner if and only if $\mathfrak{D}(\mathfrak{L})|\mathfrak{R}=\mathfrak{J}(\mathfrak{L})|\mathfrak{R}$ (Theorem 2). Recently G. F. Leger [5] has proved that, if $\mathfrak{D}(\mathfrak{R})$ splits over $\mathfrak{J}(\mathfrak{R})$, $\mathfrak{D}(\mathfrak{L})$ splits over $\mathfrak{J}(\mathfrak{L})$. We show that, in order that $\mathfrak{D}(\mathfrak{L})$ may split over $\mathfrak{J}(\mathfrak{L})$, each of the following conditions is necessary and sufficient: (1) $\mathfrak{D}(\mathfrak{L})|\mathfrak{R}$ splits over $\mathfrak{J}(\mathfrak{L})|\mathfrak{R}$; (2) $\mathfrak{D}(\mathfrak{L})|\mathfrak{R}$ splits over $\mathfrak{J}(\mathfrak{R})$ (Theorem 3). We also generalize a result of G. Hochschild [2] and study the derivation algebras of reductive Lie algebras.

2. Preliminary lemmas

We know [2] that a Lie algebra $\mathfrak{L}$ over K is semi-simple if and only if every derivation of $\mathfrak{L}$ into an arbitrary Lie algebra $\mathfrak{M} \supsetneqq \mathfrak{L}$ can be extended to an inner derivation of $\mathfrak{M}$. Let $\mathfrak{T}$ be a semi-simple ideal of $\mathfrak{L}$ and let $\mathfrak{L}_0$ be the set of all elements x of $\mathfrak{L}$ such that $x \circ \mathfrak{T} = 0$. Then it is easy to see that $\mathfrak{L}_0$ is an ideal of $\mathfrak{L}$ which contains the radical of $\mathfrak{L}$, and that $\mathfrak{L}$ is the direct sum of $\mathfrak{T}$ and $\mathfrak{L}_0$. We shall first show the following

LEMMA 1. *A semi-simple ideal of $\mathfrak{L}$ and its complementary ideal in $\mathfrak{L}$ are both characteristic.*

PROOF. Let $\mathfrak{T}$ be a semi-simple ideal of $\mathfrak{L}$ and let $\mathfrak{L}_0$ be the complementary ideal of $\mathfrak{T}$ in $\mathfrak{L}$. Let D be any derivation of $\mathfrak{L}$. Then $D|\mathfrak{T}$ is a derivation of $\mathfrak{T}$ into $\mathfrak{L}$. Since $\mathfrak{T}$ is semi-simple, there exists an element x in $\mathfrak{L}$ such that $D|\mathfrak{T} = D_x|\mathfrak{T}$. Then we have $D(\mathfrak{T}) = \mathfrak{T} \circ x \subseteqq \mathfrak{T}$. Therefore $\mathfrak{T}$ is characteristic. Let $\mathfrak{R}$ be the radical of $\mathfrak{L}$ and let $\mathfrak{S}$ be a maximal semi-simple subalgebra of $\mathfrak{L}_0$. Since $\mathfrak{R}$ is also the radical of $\mathfrak{L}_0$, we have $\mathfrak{L}_0 = \mathfrak{S} + \mathfrak{R}$. We can now find an element y in $\mathfrak{L}$ such that $D|\mathfrak{S} = D_y|\mathfrak{S}$. Since $\mathfrak{L}_0$ is an ideal of $\mathfrak{L}$, it follows that $D(\mathfrak{S}) = \mathfrak{S} \circ y \subseteqq \mathfrak{L}_0$. This, together with $D(\mathfrak{R}) \subseteqq \mathfrak{R}$, gives $D(\mathfrak{L}_0) \subseteqq \mathfrak{L}_0$. Therefore $\mathfrak{L}_0$ is characteristic and the lemma is proved.

Let $\mathfrak{T}_1$, $\mathfrak{T}_2$ be semi-simple ideals of $\mathfrak{L}$. Let $\mathfrak{T}_3$ be the complementary ideal of $\mathfrak{T}_1$ in $\mathfrak{T}_1 + \mathfrak{T}_2$. Then $\mathfrak{T}_3$ is isomorphic to $(\mathfrak{T}_1 + \mathfrak{T}_2)/\mathfrak{T}_1$ and therefore to $\mathfrak{T}_2/\mathfrak{T}_1 \cap \mathfrak{T}_2$. Since $\mathfrak{T}_2$ is semi-simple and $\mathfrak{T}_1 \cap \mathfrak{T}_2$ is an ideal of $\mathfrak{T}_2$, it follows that $\mathfrak{T}_2/\mathfrak{T}_1 \cap \mathfrak{T}_2$ is semi-simple and therefore $\mathfrak{T}_3$ is also semi-simple. Hence $\mathfrak{T}_1 + \mathfrak{T}_2$ is a semi-simple ideal of $\mathfrak{L}$. Thus the sum of all semi-simple ideals of $\mathfrak{L}$ is the largest semi-simple ideal of $\mathfrak{L}$. We shall now prove the following lemma (see [3, Theorem 1.2]).

LEMMA 2. *Let $\mathfrak{R}$, $\mathfrak{N}$ be the radical and the largest nilpotent ideal of $\mathfrak{L}$. Let $\mathfrak{S}$ be a maximal semi-simple subalgebra of the complementary ideal of the largest semi-simple ideal in $\mathfrak{L}$. Then no D_x for x in $\mathfrak{S}$ induces inner derivations of $\mathfrak{R}$ and of $\mathfrak{N}$.*

PROOF. Let $\mathfrak{T}$ be the largest semi-simple ideal of $\mathfrak{L}$ and let $\mathfrak{L}_0$ be its complementary ideal in $\mathfrak{L}$. Then we have $\mathfrak{L}_0 = \mathfrak{S} + \mathfrak{R}$. Since $\mathfrak{N}$ is characteristic [3], $D_x|\mathfrak{N}$ for any x in $\mathfrak{L}$ is a derivation of $\mathfrak{N}$. If we denote by $\mathfrak{S}_1$ the set of all elements x of $\mathfrak{S}$ such that $D_x|\mathfrak{N}$ is an inner derivation of $\mathfrak{N}$, then it is easy to verify that $\mathfrak{S}_1$ is an ideal of $\mathfrak{S}$. Hence $\mathfrak{S}_1$ is semi-simple. But it is clear that the mapping $x \in \mathfrak{S}_1 \rightarrow D_x|\mathfrak{N}$ is a nilpotent representation of $\mathfrak{S}_1$. Therefore we have $D_x|\mathfrak{N} = 0$ for all x in $\mathfrak{S}_1$. Then, from the fact that $D(\mathfrak{R}) \subseteqq \mathfrak{N}$

for every derivation D of $\mathfrak{L}$, it follows that $D_x^2(\mathfrak{R})=0$ for all x in $\mathfrak{S}_1$ and therefore the mapping $x\in\mathfrak{S}_1\rightarrow D_x|\mathfrak{R}$ gives a nilpotent representation of $\mathfrak{S}_1$. Since $\mathfrak{S}_1$ is semi-simple, we have $D_x|\mathfrak{R}=0$ for all x in $\mathfrak{S}_1$, i.e. $\mathfrak{R}\circ\mathfrak{S}_1=0$. Thus we see that $\mathfrak{S}_1$ is a semi-simple ideal of $\mathfrak{L}$. Then $\mathfrak{T}+\mathfrak{S}_1$ is also a semi-simple ideal of $\mathfrak{L}$ and, from the maximality of $\mathfrak{T}$, we conclude that $\mathfrak{S}_1=0$. In a similar manner we can show that, for any element x of $\mathfrak{S}$, $D_x|\mathfrak{R}$ is not an inner derivation of $\mathfrak{R}$.

LEMMA 3. *$\mathfrak{J}(\mathfrak{L})|\mathfrak{R}$ coincides with $\mathfrak{J}(\mathfrak{R})$ if and only if $\mathfrak{L}$ is the direct sum of a semi-simple ideal and the radical.*

PROOF. Let $\mathfrak{T}$ be the largest semi-simple ideal and let $\mathfrak{S}$ be a maximal semi-simple subalgebra of the complementary ideal of $\mathfrak{T}$ in $\mathfrak{L}$. If $\mathfrak{J}(\mathfrak{L})|\mathfrak{R}=\mathfrak{J}(\mathfrak{R})$, then we have $D_{\mathfrak{S}}|\mathfrak{R}\subseteqq\mathfrak{J}(\mathfrak{R})$, where $D_{\mathfrak{S}}|\mathfrak{R}$ denotes the set of $D_x|\mathfrak{R}$ for all x in $\mathfrak{S}$. It follows from Lemma 2 that $\mathfrak{S}=0$. Therefore $\mathfrak{L}$ is the direct sum of $\mathfrak{T}$ and $\mathfrak{R}$. The converse is evident.

LEMMA 4. *$\mathfrak{J}(\mathfrak{L})|\mathfrak{R}$ splits over $\mathfrak{J}(\mathfrak{R})$.*

PROOF. Suppose that $\mathfrak{T}$, $\mathfrak{S}$ have the same meanings as in the proof of Lemma 3. Then we have $\mathfrak{L}=\mathfrak{T}+\mathfrak{S}+\mathfrak{R}$. It follows that $\mathfrak{J}(\mathfrak{L})|\mathfrak{R}=D_{\mathfrak{S}}|\mathfrak{R}+\mathfrak{J}(\mathfrak{R})$. Lemm 2 tells us that $D_{\mathfrak{S}}|\mathfrak{R}\cap\mathfrak{J}(\mathfrak{R})=0$. Thus $\mathfrak{J}(\mathfrak{L})|\mathfrak{R}$ splits over $\mathfrak{J}(\mathfrak{R})$ as the lemma asserts.

We remark here that, if the adjoint representation of $\mathfrak{L}$ is splittable, an analogue of Lemma 4 for the largest nilpotent ideal $\mathfrak{N}$ can be established. In fact, if $\mathfrak{J}(\mathfrak{L})$ is splittable, then there exists an abelian subalgebra $\mathfrak{A}$ such that $\mathfrak{R}=\mathfrak{N}+\mathfrak{A}$, $\mathfrak{N}\cap\mathfrak{A}=0$ and that, for any element x of $\mathfrak{A}$, D_x is a semi-simple matrix [6]. From this fact and Lemma 2 it follows immediately that $\mathfrak{J}(\mathfrak{L})|\mathfrak{N}$ splits over $\mathfrak{J}(\mathfrak{N})$.

3. A necessary and sufficient condition for a derivation to be inner

In this section, making use of Lemma 2, we shall show the following theorem.

THEOREM 1. *Let $\mathfrak{L}$ be a Lie algebra over K and let $\mathfrak{R}$ be its radical. Then a derivation D of $\mathfrak{L}$ is inner if and only if there exists an elment x in $\mathfrak{L}$ such that $D|\mathfrak{R}=D_x|\mathfrak{R}$.*

PROOF. The necessity is evident. We shall show the sufficiency. Let $\mathfrak{T}$ be the largest semi-simple ideal of $\mathfrak{L}$. Let $\mathfrak{L}_0$ be the complementary ideal of

$\mathfrak{T}$ in $\mathfrak{L}$ and let $\mathfrak{S}$ be a maximal semi-simple subalgebra of $\mathfrak{L}_0$. Then we have $\mathfrak{L}_0 = \mathfrak{S} + \mathfrak{R}$. If we write $\mathfrak{S}_0 = \mathfrak{T} + \mathfrak{S}$, then it is clear that $\mathfrak{S}_0$ is a semi-simple subalgebra of $\mathfrak{L}$. Therefore there exists an element y in $\mathfrak{L}$ such that $D|\mathfrak{S}_0 = D_y|\mathfrak{S}_0$. We put $D_1 = D - D_y$. Then D_1 is a derivation of $\mathfrak{L}$ such that $D_1(\mathfrak{S}_0) = 0$, and it is sufficient to show the statement for D_1.

Suppose now that D is a derivation of $\mathfrak{L}$ such that $D(\mathfrak{S}_0) = 0$ and satisfies the condition of the theorem. Let $\mathfrak{Z}$ denote the center of $\mathfrak{R}$. Since $\mathfrak{Z} \circ \mathfrak{S} \subseteqq \mathfrak{Z}$ and since every representation of a semi-simple Lie algebra is completely reducible, we can find a subspace $\mathfrak{U}$ such that $\mathfrak{R} = \mathfrak{Z} + \mathfrak{U}$, $\mathfrak{Z} \cap \mathfrak{U} = 0$ and $\mathfrak{U} \circ \mathfrak{S} \subseteqq \mathfrak{U}$. Let x be an element of $\mathfrak{L}$ such that $D|\mathfrak{R} = D_x|\mathfrak{R}$, where we may suppose that x is in $\mathfrak{S} + \mathfrak{U}$. Then we have $D(s \circ r) = D_x(s \circ r)$ for all s in $\mathfrak{S}$ and r in $\mathfrak{R}$. Since $D(\mathfrak{S}_0) = 0$, it follows that $D_x(s) \circ r = 0$. If we put $x = s' + u$ with s' in $\mathfrak{S}$ and u in $\mathfrak{U}$, then we have $D_{s \circ s'}(r) = D_{u \circ s}(r)$ for all r in $\mathfrak{R}$. Since $u \circ s$ is in $\mathfrak{R}$, by Lemma 2 we see that $s \circ s' = 0$. Thus $\mathfrak{S} \circ s' = 0$ so that $s' = 0$, whence x is in $\mathfrak{U}$. Then it follows that $D_x(\mathfrak{S}) \subseteqq \mathfrak{S} \circ \mathfrak{U} \subseteqq \mathfrak{U}$ and that $D_x(\mathfrak{S}) \subseteqq \mathfrak{Z}$. Therefore we have $D_x(\mathfrak{S}) \subseteqq \mathfrak{Z} \cap \mathfrak{U} = 0$. Now it is obvious that $D_x(\mathfrak{S}_0) = 0$, which shows that $D = D_x$. Thus the theorem is proved.

4. Derivation algebras of Lie algebras and their radicals

There are intimate connexions between the derivation algebras of Lie algebras and those of their radicals. As an immediate consequence of Theorem 1 we have first the following

THEOREM 2. *Let $\mathfrak{L}$ be a Lie algebra over K and let $\mathfrak{R}$ be the radical of $\mathfrak{L}$. Then $\mathfrak{D}(\mathfrak{L}) = \mathfrak{J}(\mathfrak{L})$ if and only if $\mathfrak{D}(\mathfrak{L})|\mathfrak{R} = \mathfrak{J}(\mathfrak{L})|\mathfrak{R}$.*

We shall denote $\mathfrak{D}_1(\mathfrak{L}) = \mathfrak{D}(\mathfrak{L})$, $\mathfrak{D}_2(\mathfrak{L}) = \mathfrak{D}(\mathfrak{D}_1(\mathfrak{L}))$, ..., $\mathfrak{D}_{n+1}(\mathfrak{L}) = \mathfrak{D}(\mathfrak{D}_n(\mathfrak{L}))$, ... Now let $\mathfrak{L}$ be neither semi-simple nor solvable, and let the center of $\mathfrak{L}$ be zero. Then a Lie algebra isomorphic to $\mathfrak{D}_n(\mathfrak{L})$ for a sufficiently large n gives an example of a Lie algebra whose derivations are all inner [1] and which is neither semi-simple nor solvable. It is also to be noted that a 2-dimensional non-abelian Lie algebra over K is a solvable Lie algebra whose derivations are all inner [2].

COROLLARY 1. *If every derivation of $\mathfrak{R}$ can be extended to an inner derivation of $\mathfrak{L}$, then all derivations of $\mathfrak{L}$ are inner.*

COROLLARY 2. *Let $\mathfrak{L}$ be a Lie algebra over K and let $\mathfrak{R}$ be the radical of $\mathfrak{L}$. Then, in order that every derivation of $\mathfrak{R}$ may be inner, each of the following conditions* *is necessary and sufficient:*

(1) *Every derivation of* $\mathfrak{L}$ *induces an inner derivation of* $\mathfrak{R}$.

(2) *Every derivation of* $\mathfrak{L}$ *is inner and* $\mathfrak{L}$ *is the direct sum of* $\mathfrak{R}$ *and a semi-simple ideal.*

PROOF. This follows immediately from Theorem 2 and Lemma 3.

We know a theorem of G. F. Leger on the derivation algebras of Lie algebras [5]. But the result can be generalized. Along the same line as in [5], we can prove that, if $\mathfrak{D}(\mathfrak{L})|\mathfrak{R}$ splits over $\mathfrak{J}(\mathfrak{R})$, then $\mathfrak{D}(\mathfrak{L})$ splits over $\mathfrak{J}(\mathfrak{L})$. Owing to Theorem 1, we are now able to show the converse of this generalized result. These are contained in the following

THEOREM 3. *Let* $\mathfrak{L}$ *be a Lie algebra over* K *and let* $\mathfrak{R}$ *be the radical of* $\mathfrak{L}$. *Then the following conditions are equivalent:*

(1) $\mathfrak{D}(\mathfrak{L})$ *splits over* $\mathfrak{J}(\mathfrak{L})$.

(2) $\mathfrak{D}(\mathfrak{L})|\mathfrak{R}$ *splits over* $\mathfrak{J}(\mathfrak{L})|\mathfrak{R}$.

(3) $\mathfrak{D}(\mathfrak{L})|\mathfrak{R}$ *splits over* $\mathfrak{J}(\mathfrak{R})$.

PROOF. $(1)\rightarrow(2)$. Let $\mathfrak{E}$ be a complement of $\mathfrak{J}(\mathfrak{L})$ in $\mathfrak{D}(\mathfrak{L})$, i. e. $\mathfrak{D}(\mathfrak{L})=\mathfrak{J}(\mathfrak{L})+\mathfrak{E}$ and $\mathfrak{J}(\mathfrak{L})\cap\mathfrak{E}=0$. Then we have $\mathfrak{D}(\mathfrak{L})|\mathfrak{R}=\mathfrak{J}(\mathfrak{L})|\mathfrak{R}+\mathfrak{E}|\mathfrak{R}$. Since $\mathfrak{J}(\mathfrak{L})\cap\mathfrak{E}=0$, it follows from Theorem 1 that $\mathfrak{J}(\mathfrak{L})|\mathfrak{R}\cap\mathfrak{E}|\mathfrak{R}=0$. Therefore $\mathfrak{D}(\mathfrak{L})|\mathfrak{R}$ splits over $\mathfrak{J}(\mathfrak{L})|\mathfrak{R}$.

$(2)\rightarrow(3)$ follows immediately from Lemma 4.

The proof of $(3)\rightarrow(1)$ will be omitted.

COROLLARY. *Assume that every derivation of* $\mathfrak{R}$ *can be extended to a derivation of* $\mathfrak{L}$. *Then* $\mathfrak{D}(\mathfrak{L})$ *splits over* $\mathfrak{J}(\mathfrak{L})$ *if and only if* $\mathfrak{D}(\mathfrak{R})$ *splits over* $\mathfrak{J}(\mathfrak{R})$.

5. Derivation algebras of reductive Lie algebras

A Lie algebra $\mathfrak{L}$ over K is called to be reductive if $\mathfrak{J}(\mathfrak{L})$ is semi-simple. $\mathfrak{L}$ is reductive if and only if it is the direct sum of a semi-simple ideal and the center. It is easily seen that, if the radical of $\mathfrak{L}$ is 1-dimensional, then $\mathfrak{L}$ is reductive.

It is well known [4] that, if a linear Lie algebra $\mathfrak{g}$ over K is completely reducible, it contains no ideals composed of nilpotent matrices, and that $\mathfrak{g}$ is completely reducible if and only if it is the direct sum of a semi-simple ideal and the center whose elements are semi-simple matrices. The following lemma is an easy generalization of Theorem 4. 4 in [2].

LEMMA 5. $\mathfrak{D}(\mathfrak{L})$ *is completely reducible if and only if* $\mathfrak{L}$ *is a reductive Lie algebra whose center is at most* 1*-dimensional. Then* $\mathfrak{D}(\mathfrak{L})$ *is isomorphic to* $\mathfrak{L}$.

PROOF. Suppose that $\mathfrak{D}(\mathfrak{L})$ is completely reducible. Let $\mathfrak{R}$ be the radical of $\mathfrak{L}$ and let $\mathfrak{N}$ be the largest nilpotent ideal of $\mathfrak{L}$. Then the set $D_{\mathfrak{N}}$ of D_x for all x in $\mathfrak{N}$ is obviously an ideal of $\mathfrak{D}(\mathfrak{L})$ composed of nilpotent matrices. Since $\mathfrak{D}(\mathfrak{L})$ is completely reducible, it follows that $D_{\mathfrak{N}}=0$ i. e. $\mathfrak{L}\circ\mathfrak{N}=0$. Then we have $D_x^2(\mathfrak{L})=0$ for every element x of $\mathfrak{R}$, since $\mathfrak{L}\circ\mathfrak{R}\subseteqq\mathfrak{N}$. Therefore $D_{\mathfrak{R}}$ is an ideal of $\mathfrak{D}(\mathfrak{L})$ composed of nilpotent matrices, whence $D_{\mathfrak{R}}=0$. Thus $\mathfrak{R}$ is the center of $\mathfrak{L}$ so that $\mathfrak{L}$ is reductive. If $\mathfrak{T}$ denotes the largest semi-simple ideal of $\mathfrak{L}$, then by Lemma 1 we see that $\mathfrak{D}(\mathfrak{L})$ is isomorphic to the direct sum of $\mathfrak{D}(\mathfrak{T})$ and $\mathfrak{D}(\mathfrak{R})$. Since $\mathfrak{D}(\mathfrak{T})$ is semi-simple and $\mathfrak{D}(\mathfrak{R})$ is the Lie algebra of all linear mappings of $\mathfrak{R}$ into itself, it follows immediately that $\mathfrak{R}$ is zero or 1-dimensional.

Conversely let $\mathfrak{L}$ be a reductive Lie algebra whose center $\mathfrak{Z}$ is at most 1-dimensional. If $\mathfrak{Z}=0$, then $\mathfrak{L}$ is semi-simple, whence $\mathfrak{D}(\mathfrak{L})=\mathfrak{J}(\mathfrak{L})$. It follows that $\mathfrak{D}(\mathfrak{L})$ is isomorphic to $\mathfrak{L}$ and therefore is semi-simple. If $\mathfrak{Z}$ is 1-dimensional, then $\mathfrak{D}(\mathfrak{L})$ is isomorphic to the direct sum of $\mathfrak{D}(\mathfrak{T})$ and $\mathfrak{D}(\mathfrak{Z})$, where $\mathfrak{T}$ is the largest semi-simple ideal of $\mathfrak{L}$. Since $\mathfrak{D}(\mathfrak{T})$ is semi-simple and $\mathfrak{D}(\mathfrak{Z})$ is generated by the identical mapping of $\mathfrak{Z}$ into itself, we see that $\mathfrak{D}(\mathfrak{L})$ is completely reducible. The second part of the lemma is now evident. Thus the lemma is established.

COROLLARY (Hochschild). *$\mathfrak{D}(\mathfrak{L})$ is semi-simple if and only if $\mathfrak{L}$ is semi-simple.*

PROOF. If $\mathfrak{D}(\mathfrak{L})$ is semi-simple, it is completely reducible. It follows from Lemma 5 that $\mathfrak{L}$ is isomorphic to $\mathfrak{D}(\mathfrak{L})$ and therefore is semi-simple. The converse is evident.

We can now prove the following

THEOREM 4. *Let $\mathfrak{L}$ be a Lie algebra over K. For arbitrary positive integers m and n, $\mathfrak{D}_m(\mathfrak{L})$ is completely reducible if and only if $\mathfrak{D}_n(\mathfrak{L})$ is completely reducible. And this is the case if and only if $\mathfrak{L}$ is a reductive Lie algebra whose center is at most 1-dimensional. Then $\mathfrak{L}$ and all $\mathfrak{D}_n(\mathfrak{L})$'s are isomorphic to each other.*

PROOF. On account of Lemma 5, it is sufficient to show that $\mathfrak{D}_2(\mathfrak{L})$ is completely reducible if and only if $\mathfrak{D}(\mathfrak{L})$ is completely reducible. If $\mathfrak{D}(\mathfrak{L})$ is completely reducible, by Lemma 5 we see that the center of $\mathfrak{D}(\mathfrak{L})$ is zero or 1-dimensional and therefore that $\mathfrak{D}_2(\mathfrak{L})$ is completely reducible. Conversely, suppose that $\mathfrak{D}_2(\mathfrak{L})$ is completely reducible. Then it follows from Lemma 5 that $\mathfrak{D}(\mathfrak{L})$ is reductive and its center is at most 1-dimensional. If the center of $\mathfrak{D}(\mathfrak{L})$ is zero, then $\mathfrak{D}(\mathfrak{L})$ is semi-simple. Therefore we consider the case where the center of $\mathfrak{D}(\mathfrak{L})$ is 1-dimensional. Let $\mathfrak{L}=\mathfrak{S}+\mathfrak{R}$ be a Levi decomposition of $\mathfrak{L}$, where $\mathfrak{R}$ is the radical and $\mathfrak{S}$ is a semi-simple subalgebra of $\mathfrak{L}$. Then it is clear that $D_{\mathfrak{S}}$ is a semi-simple subalgebra and $D_{\mathfrak{R}}$ is a solvable ideal of $\mathfrak{D}(\mathfrak{L})$. Therefore we have $D_{\mathfrak{S}\circ\mathfrak{R}} = D_{\mathfrak{S}} \circ D_{\mathfrak{R}} = 0$, whence $\mathfrak{S} \circ (\mathfrak{S} \circ \mathfrak{R}) = 0$. From the fact that every representation of a semi-simple Lie algebra is completely reducible, it follows that $\mathfrak{S} \circ \mathfrak{R} = 0$. Let $\mathfrak{Z}$ denote the center of $\mathfrak{L}$. Then $\mathfrak{R}/\mathfrak{Z}$ is at most 1-dimensional, since it is isomorphic to $D_{\mathfrak{R}}$. Now it is easy to see that $\mathfrak{R} = \mathfrak{Z}$. Hence $\mathfrak{L}$ is reductive. Since $\mathfrak{D}(\mathfrak{L})$ is isomorphic to the direct sum of $\mathfrak{D}(\mathfrak{S})$ and $\mathfrak{D}(\mathfrak{Z})$ and since $\mathfrak{D}(\mathfrak{S})$ is semi-simple, we see that $\mathfrak{D}(\mathfrak{Z})$ is reductive. Then it is obvious that $\mathfrak{Z}$ is 1-dimensional. By Lemma 5 we conclude that $\mathfrak{D}(\mathfrak{L})$ is completely reducible. The proof is completed.

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