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Theory of *n*-Cocycles and *n*-Cohomology Groups in Commutative Rings[1)]

By

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Prof. Y. Kawada[2)] has investigated the structure of 2-cohomology groups in integral domains of $\mathfrak{p}$-adic number fields. Recently Prof. M. Moriya[3)] has discussed the extension theory of 2-cocycles in commutative rings and, connectedly, has developed the theory of 2-cohomology groups in complete fields with respect to a discrete valuation.

Now, in the first step (§§ 2 and 3), we shall study on some properties of n-cocycles and n-cohomology groups in commutative rings. In the next step (§4), we shall investigate the structure of n-cocycles and n-cohomology groups in integral domains of $\mathfrak{p}$-adic number fields.

§ 1. Definition of *n*-cocycle and *n*-cohomology group in commutative rings.

Let $\mathfrak{R}$ be a commutative ring and $\mathfrak{M}$ an $\mathfrak{R}$-left module. Then, a unique mapping f of a product space $\overbrace{\mathfrak{R}\times\cdots\times\mathfrak{R}}^{n \text{ terms}}$ into $\mathfrak{M}$ is called an *n-cocycle* of $\mathfrak{R}$ in $\mathfrak{M}$ if it satisfies:

i) For $A_1, A_2, \dots, A_n \in \mathfrak{R}$

$$f(A_1, A_2, \dots, A_n) = f(A_i, A_j, \dots, A_l),$$

where $i, j, \dots, l$ means any permutation of $1, 2, \dots, n$;

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1) This note has been completed by the encouragement of Prof. M. Moriya. The author wishes to express here his hearty thanks to Prof. M. Moriya for his kindness.

2) Y. Kawada: On the derivations in number fields, Annals of Math., **54** (1954), pp. 302–314.

3) M. Moriya: Zur Fortsetzung der 2-Cozyklen in einem kommutative Ring, Math. Journ., Okayama Univ., **4** (1954), pp. 1–19.

_______: Theorie der 2-Cohomologiegruppen in diskret bewerteten perfekten Körpern, Proceedings of Japan Academy, **30** (1954), pp. 787–790.

ii) For $A_1, B_1, A_2, \dots, A_n \in \mathfrak{R}$

$$f(A_1+B_1, A_2, \dots, A_n) = f(A_1, A_2, \dots, A_n) + f(B_1, A_2, \dots, A_n);$$

iii) For $A_1, A_2, \dots, A_n, A_{n+1} \in \mathfrak{R}$

$$A_1 f(A_2, \dots, A_n, A_{n+1}) + \sum_{i=1}^{n}(-1)^i f(A_1, A_2, \dots, A_i A_{i+1}, \dots, A_{n+1})$$
$$+ (-1)^{n+1} A_{n+1} f(A_1, A_2, \dots, A_n) = 0;$$

iv) For the two mappings f_1, f_2

$$f_1(A_1, A_2, \dots, A_n) + f_2(A_1, A_2, \dots, A_n) = (f_1+f_2)(A_1, A_2, \dots, A_n).$$

We shall denote $Z(\mathfrak{R};\mathfrak{M})$ the totality of all the n-cocycles of $\mathfrak{R}$ in $\mathfrak{M}$. Then $Z(\mathfrak{R};\mathfrak{M})$ is an additive group. Now, we consider a unique mapping g of a product space $\overbrace{\mathfrak{R}\times\cdots\times\mathfrak{R}}^{n-1 \text{ terms}}$ into $\mathfrak{M}$ with the following properties:

i) For $A_1, A_2, \dots, A_{n-1} \in \mathfrak{R}$

$$g(A_1, A_2, \dots, A_{n-1}) = g(A_i, A_j, \dots, A_t),$$

where $i, j, \dots, t$ means any permutation of $1, 2, \dots, n-1$;

ii) For $A_1, B_1, A_2, \dots, A_{n-1} \in \mathfrak{R}$

$$g(A_1+B_1, A_2, \dots, A_{n-1}) = g(A_1, A_2, \dots, A_{n-1}) + g(B_1, A_2, \dots, A_{n-1});$$

iii) For the two mappings g_1, g_2

$$g_1(A_1, A_2, \dots, A_{n-1}) + g_2(A_1, A_2, \dots, A_{n-1}) = (g_1+g_2)(A_1, A_2, \dots, A_{n-1}).$$

Next a mapping δg of a product space $\overbrace{\mathfrak{R}\times\cdots\times\mathfrak{R}}^{n \text{ terms}}$ into $\mathfrak{M}$ is called an *n-coboundary* of $\mathfrak{R}$ in $\mathfrak{M}$ if it satisfies:

i)* For $A_1, A_2, \dots, A_{n-1}, A_n \in \mathfrak{R}$

$$\delta g(A_1, A_2, \dots, A_{n-1}, A_n) = A_1 g(A_2, \dots, A_{n-1}, A_n)$$
$$+ \sum_{i=1}^{n-1}(-)^i g(A_1, A_2, \dots, A_i A_{i+1}, \dots, A_{n-1}, A_n)$$
$$+ (-)^n A_n g(A_1, A_2, \dots, A_{n-1});$$

ii)* $$\delta g(A_1, A_2, \dots, A_n) = \delta g(A_i, A_j, \dots, A_l),$$

where $i, j, \dots, l$ means any permutation of $1, 2, \dots, n$.

We shall denote $B(\mathfrak{R};\mathfrak{M})$ as the totality of all the n-coboundaries δg of $\mathfrak{R}$ in $\mathfrak{M}$. Then $B(\mathfrak{R};\mathfrak{M})$ is a subgroup of $Z(\mathfrak{R};\mathfrak{M})$. Here, for even numbers of n (in addition, $n=1,3$ [4]), we shall define *n-cohomology group* of $\mathfrak{R}$ in $\mathfrak{M}$ as the factor group $H(\mathfrak{R}\,;\mathfrak{M})=Z(\mathfrak{R}\,;\mathfrak{M})/B(\mathfrak{R}\,;\mathfrak{M})$.

§ 2. On some properties of *n*-cocycle and *n*-cohomology group for a subring $\mathfrak{R}_0$ of $\mathfrak{R}$.

Let $\mathfrak{R}$ be a commutative ring with 1-element and $\mathfrak{R}_0$ a subring containing 1. We consider an $\mathfrak{R}$-left module $\mathfrak{M}$ with 1 as a unit operator. In this section, we shall denote the element of $\mathfrak{R}$ and $\mathfrak{R}_0$ by Latin capital letters $A_1, A_2, \dots, A_{n-1}, A_n$ and by a Latin small letter a, respectively.

We shall prove the following

THEOREM 1. *If n is an odd number, it holds that:*

$$f(l,^{5)}\, A_1, \dots, A_{n-1})=0 \qquad \textit{for every rational integer } l.$$

PROOF. From

$$1f(1, 1, \dots, 1, A, 1)-f(1, 1, \dots, 1, A, 1)+\cdots$$
$$-f(1, 1, \dots, 1, 1, A)+f(1, 1, \dots, 1, 1, A)=0,$$

we have:

$$f(1, 1, \dots, 1, A)=0.$$

Now, we shall assume that

$$f(A_1, A_2, \dots, A_i, 1, \dots, 1)=0 \quad \text{for} \quad i<n-1.$$

Then if i is an even number, from

$$1f(A_1, A_2, \dots, A_i, A_{i+1}, 1, \dots, 1)-f(A_1, A_2, \dots, A_i, A_{i+1}, 1, \dots, 1)$$
$$+\sum_{j=1}^{i}(-1)^{j+1}f(1, A_1, \dots, A_jA_{j+1}, \dots, A_{i+1}, 1, \dots, 1)$$
$$+f(1, A_1, \dots, A_i, A_{i+1}, 1, \dots, 1)-\cdots+f(A_1, A_2, \dots, A_i, A_{i+1}, 1, \dots, 1)=0,$$

we have:

$$f(A_1, A_2, \dots, A_i, A_{i+1}, 1, \dots, 1)=0 \quad \text{for} \quad i<n-1.$$

If i is an odd number, from

$$1f(1, 1, \dots, 1, A_1, \dots, A_i, A_{i+1})-f(1, 1, \dots, 1, A_1, \dots, A_i, A_{i+1})+\cdots$$
$$+f(1, 1, \dots, 1, A_1, \dots, A_i, A_{i+1})-f(1, 1, \dots, 1, A_1A_2, \dots, A_i, A_{i+1})+\cdots$$
$$-f(1, 1, \dots, 1, A_1, \dots, A_iA_{i+1})+A_{i+1}f(1, 1, \dots, 1, A_1, \dots, A_i)=0,$$

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4) If $n=1$, we shall define $\delta\varphi(A)=0$ for $A\in\mathfrak{R}$, and if $n=3$, we have always $\delta\varphi(A_1,A_2,A_3)=0$ for $A_1, A_2, A_3\in\mathfrak{R}$.

5) By l we mean $l\cdot 1$.

we have:

$$f(A_1, A_2, \dots, A_i, A_{i+1}, 1, \dots, 1) = 0 \quad \text{for} \quad i < n-1.$$

Thus, by mathematical induction on the number i of the element A, we have:

$$(1) \qquad f(1, A_1, A_2, \dots, A_{n-1}) = 0.$$

From

$$f(0+0, A_1, \dots, A_{n-1}) = f(0, A_1, \dots, A_{n-1}) + f(0, A_1, \dots, A_{n-1}),$$

we have:

$$(2) \qquad f(0, A_1, A_2, \dots, A_{n-1}) = 0.$$

By (1) and (2) we can easily obtain our theorem.

THEOREM 2. *If n is an even number, there exists an n-cocycle f in any cohomology class of $H(\mathfrak{R}; \mathfrak{M})$ such that $f(a, A_1, \dots, A_{n-1}) = 0$ holds.*

PROOF. For an arbitrary $f \in Z(\mathfrak{R}; \mathfrak{M})$ we can take a g such that $f(\overbrace{1, 1, \dots, 1}^{n-1 \text{ terms}}, A) = g(\overbrace{1, 1, \dots, 1}^{n-2 \text{ terms}}, A)$. Then from

$$Af(1, 1, \dots, 1) - f(A, 1, \dots, 1) + \cdots - f(A, 1, \dots, 1) = 0,$$

we have:

$$g(A, 1, \dots, 1) = f(A, 1, \dots, 1) = Af(1, 1, \dots, 1) = Ag(1, 1, \dots, 1).$$

Then for the n-cocycle $f_0 = f - \delta g$ we have:

$$f_0(A, 1, \dots, 1) = f(A, 1, \dots, 1) - Ag(1, 1, \dots, 1)$$
$$+ g(A, 1, \dots, 1) - \cdots - g(A, 1, \dots, 1) = 0.$$

Thus, there exists an n-cocycle f_0 with $f_0(A, 1, \dots, 1) = 0$ in any cohomology class of $H(\mathfrak{R}; \mathfrak{M})$.

Here, we shall assume that $f_0(1, 1, \dots, 1, A_1, \dots, A_i) = 0$ for $i < n-1$. Then if i is an even number, from

$$1 f_0(1, \dots, 1, A_1, \dots, A_i, A_{i+1}) - f_0(1, \dots, 1, A_1, \dots, A_i, A_{i+1}) + \cdots$$
$$+ f_0(1, \dots, 1, A_1, \dots, A_i, A_{i+1}) - f_0(1, \dots, 1, A_1A_2, \dots, A_i, A_{i+1}) + \cdots$$
$$+ f_0(1, \dots 1, A_1, \dots, A_iA_{i+1}) - A_{i+1} f_0(1, \dots, 1, A_1, \dots, A_i) = 0,$$

we have:

$$f_0(1, \dots, 1, A_1, \dots, A_i, A_{i+1}) = 0 \quad \text{for} \quad i < n-1.$$

If i is an odd number, from

$$1f_0(A_1, A_2, \dots, A_i, A_{i+1}, 1, \dots, 1) - f_0(A_1, A_2, \dots, A_i, A_{i+1}, 1, \dots, 1)$$
$$+ f_0(1, A_1A_2, \dots, A_i, A_{i+1}, 1, \dots, 1) - \cdots + f_0(1, A_1, \dots, A_iA_{i+1}, 1, \dots, 1)$$
$$- f_0(1, A_1, \dots, A_i, A_{i+1}, 1, \dots, 1) + \cdots - f_0(1, A_1, \dots, A_i, A_{i+1}, 1, \dots, 1) = 0,$$

we have:

$$f_0(1, \dots, 1, A_1, \dots, A_i, A_{i+1}) = 0 \qquad \text{for} \qquad i < n-1.$$

Thus, by mathematical induction on the number i of the element A, we have:

$$f_0(1, A_1, \dots, A_{n-1}) = 0.$$

Now we shall consider a g_0 such that $g_0(aA_1, A_2, \dots, A_{n-1}) = f_0(a, A_1, A_2, \dots, A_{n-1})$, then we have:

$$g_0(a, A_2, \dots, A_{n-1}) = f_0(a, 1, A_2, \dots, A_{n-1}) = 0$$

and

$$g_0(A_1, A_2, \dots, A_{n-1}) = f_0(1, A_1, A_2, \dots, A_{n-1}) = 0.$$

Thus, for the n-cocycle $f_1 = f_0 + \delta g_0$ we have:

$$f_1(a, A_1, \dots, A_{n-1}) = f_0(a, A_1, \dots, A_{n-1}) + a\,g_0(A_1, \dots, A_{n-1})$$
$$- g_0(aA_1, A_2, \dots, A_{n-1}) + g_0(a, A_1A_2, \dots, A_{n-1}) - \cdots$$
$$+ A_{n-1}\, g_0(a, A_1, \dots, A_{n-2}) = 0.$$

This completes the proof of our theorem.

Therefore we shall denote $Z(\mathfrak{R}, \mathfrak{R}_0; \mathfrak{M})$ as the totality of all the *relative n-cocycles* f with $f(a, A_1, \dots, A_{n-1}) = 0$ for every $a \in \mathfrak{R}_0$ and $B(\mathfrak{R}, \mathfrak{R}_0; \mathfrak{M})$ as the totality of all the *relative n-coboundaries* δg with $g(a, A_1, \dots, A_{n-2}) = 0$ and $g(aA_1, \dots, A_{n-1}) = ag(A_1, \dots, A_{n-1})$.

Here, for even numbers of n (in addition, $n = 1, 3$), we shall define the *relative n-cohomology group* for a subring $\mathfrak{R}_0$ of $\mathfrak{R}$ as the factor group $H(\mathfrak{R}, \mathfrak{R}_0; \mathfrak{M}) = Z(\mathfrak{R}, \mathfrak{R}_0; \mathfrak{M})/B(\mathfrak{R}, \mathfrak{R}_0; \mathfrak{M})$.

Then we have

THEOREM 3. *For every* $f \in Z(\mathfrak{R}, \mathfrak{R}_0; \mathfrak{M})$ *we have:*

$$f(aA_1, A_2, \dots, A_n) = af(A_1, A_2, \dots, A_n).$$

PROOF. From

$$af(A_1, A_2, \dots, A_n) - f(aA_1, A_2, \dots, A_n) + \sum_{i=1}^{n-1}(-1)^{i+1} f(a, A_1, \dots, A_i A_{i+1}, \dots, A_n) + (-1)^{n+1} A_n f(a, A_1, \dots, A_{n-1}) = 0,$$

we have:

$$f(aA_1, A_2, \dots, A_n) = af(A_1, A_2, \dots, A_n), \quad \text{q. e. d.}$$

§ 3. On some properties of *n*-cocycle and *n*-cohomology group for a subring of $\mathfrak{R}_0$ of $\mathfrak{R} = \mathfrak{R}_0[\theta]$.

Let $\mathfrak{R}$ be a commutative ring with 1-element, and $\mathfrak{R}_0 (\neq \mathfrak{R})$ *a* subring containing 1. Moreover, we shall assume that $\mathfrak{R}$ has a linear independent $\mathfrak{R}_0$-base $1, \theta, \dots, \theta^{s-1}$ $(s > 1)$, i. e., any element of $\mathfrak{R}$ be uniquely represented as the linear form of $1, \theta, \dots, \theta^{s-1}$ with coefficients in $\mathfrak{R}_0$. Now let $F(x) = x^s + a_1 x^{s-1} + \cdots + a_s = 0$ be the irreducible defining equation of θ in $\mathfrak{R}_0[x]$, i. e., $\theta^s = -a_1\theta^{s-1} - \cdots - a_s$.

Then by Theorem 3, we have immediately

LEMMA 1. *Let it be* $A_1, A_2, \dots, A_n \in \mathfrak{R}$, *and* $A_i = \sum_{j_i=0}^{s-1} a_{ij_i}\theta^{j_i}$ $(i = 1, 2, \dots, n)$ *with* $a_{ij_i} \in \mathfrak{R}_0$. *Then for every* $f \in Z(\mathfrak{R}, \mathfrak{R}_0; \mathfrak{M})$, *it holds that*:

$$f(A_1, A_2, \dots, A_n) = \sum_{j_1=0}^{s-1}\sum_{j_2=0}^{s-1}\cdots\sum_{j_n=0}^{s-1} a_{1j_1} a_{2j_2} \cdots a_{nj_n} f(\theta^{j_1}, \theta^{j_2}, \dots, \theta^{j_n}).$$

Therefore every $f \in Z(\mathfrak{R}, \mathfrak{R}_0; \mathfrak{M})$ is uniquely determined by the values $f(\theta^\alpha, \theta^\beta, \dots, \theta^\gamma, \theta^\delta)$, where $1 \leq \alpha \leq s-1$, $1 \leq \beta \leq s-1$, ..., $1 \leq \gamma \leq s-1$, $1 \leq \delta \leq s-1$.

Now we shall prove the following

THEOREM 4. *For every n-cocycle* $f \in Z(\mathfrak{R}, \mathfrak{R}_0; \mathfrak{M})$,

$$f(\theta^\alpha, \theta^\beta, \dots, \theta^\gamma, \theta^\delta)\ (1 \leq \alpha \leq s-1, 1 \leq \beta \leq s-1, \dots, 1 \leq \gamma \leq s-1, 1 \leq \delta \leq s-1)$$

are uniquely represented as the linear form of $f(\theta, \theta, \dots, \theta, \theta^i)$ $(1 \leq i \leq s-1)$ *with coefficients in* $\mathfrak{R}_0[\theta]$.

PROOF. From

$$\theta f \theta^{\alpha-1}, \theta^\beta, \dots, \theta^\gamma, \theta^\delta) - f(\theta^\alpha, \theta^\beta, \dots \theta^\gamma, \theta^\delta) + f(\theta, \theta^{\alpha+\beta-1}, \dots, \theta^\gamma, \theta^\delta) - \cdots + (-1)^{n+1}\theta^\delta f(\theta, \theta^{\alpha-1}, \dots, \theta^\gamma) = 0,$$

we have:

$$f(\theta^{\alpha}, \theta^{\beta}, \dots, \theta^{\gamma}, \theta^{\delta}) = \theta f(\theta^{\alpha-1}, \theta^{\beta}, \dots, \theta^{\gamma}, \theta^{\delta}) + f(\theta, \theta^{\alpha+\beta-1}, \dots, \theta^{\gamma}, \theta^{\delta})$$
$$- \cdots + (-1)^{n+1}\theta^{\delta} f(\theta, \theta^{\alpha-1}, \dots, \theta^{\gamma}).$$

By this formula and $\theta^{s} = -\sum_{i=1}^{s} a_i \theta^{s-i}$, $f(\theta^{\alpha}, \theta^{\beta}, \dots, \theta^{\gamma}, \theta^{\delta})$ are uniquely represented as the linear form of $f(\theta, \theta^{\lambda}, \dots, \theta^{\mu}, \theta^{\nu})$ $(1 \leq \lambda \leq s-1, \dots, 1 \leq \mu \leq s-1, 1 \leq \nu \leq s-1)$ with coefficients in $\Re_0[\theta]$. Now we shall assume that $f(\theta^{\alpha}, \theta^{\beta}, \dots, \theta^{\gamma}, \theta^{\delta})$ are uniquely represented as the linear form of $f(\overbrace{\theta, \dots, \theta}^{i \text{ terms}}, \theta^{\sigma}, \theta^{\tau}, \dots \theta^{\xi}, \theta^{\eta})$ $(1 \leq \sigma \leq s-1, 1 \leq \tau \leq s-1, \dots, 1 \leq \xi \leq s-1, 1 \leq \eta \leq s-1)$ with coefficients in $\Re_0[\theta]$. Then,

Case 1. For even numbers of i

From

$$\theta f(\overbrace{\theta, \theta, \dots, \theta}^{i \text{ terms}}, \theta^{\sigma-1}, \theta^{\tau}, \dots, \theta^{\xi}, \theta^{\eta}) - f(\theta^2, \theta, \dots, \theta, \theta^{\sigma-1}, \theta^{\tau}, \dots, \theta^{\xi}, \theta^{\eta}) + \cdots$$
$$+ f(\theta, \theta, \dots, \theta^2, \theta^{\sigma-1}, \theta^{\tau}, \dots, \theta^{\xi}, \theta^{\eta}) - f(\overbrace{\theta, \theta, \dots, \theta}^{i \text{ terms}}, \theta^{\sigma}, \theta^{\tau}, \dots, \theta^{\xi}, \theta^{\eta})$$
$$+ f(\overbrace{\theta, \theta, \dots, \theta}^{i+1 \text{ terms}}, \theta^{\sigma+\tau-1}, \dots, \theta^{\xi}, \theta^{\eta}) - \cdots + (-1)^{n+1}\theta^{\eta} f(\overbrace{\theta, \theta, \dots, \theta}^{i+1 \text{ terms}}, \theta^{\sigma-1}, \theta^{\tau}, \dots, \theta^{\xi}) = 0,$$

we have:

$$f(\overbrace{\theta, \theta, \dots, \theta}^{i \text{ terms}}, \theta^{\sigma}, \theta^{\tau}, \dots, \theta^{\xi}, \theta^{\eta}) = \theta f(\overbrace{\theta, \theta, \dots, \theta}^{i \text{ terms}}, \theta^{\sigma-1}, \theta^{\tau}, \dots, \theta^{\xi}, \theta^{\eta})$$
$$+ f(\overbrace{\theta, \theta, \dots, \theta}^{i+1 \text{ terms}}, \theta^{\sigma+\tau-1}, \dots, \theta^{\xi}, \theta,^{\eta}) - \cdots + (-1)^{n+1}\theta^{\eta} f(\overbrace{\theta, \theta, \dots, \theta}^{i+1 \text{ terms}}, \theta^{\sigma-1}, \theta^{\tau}, \dots, \theta^{\xi}).$$

Case 2. For odd numbers of i

From

$$\theta f(\theta^{\tau}, \overbrace{\theta, \dots, \theta}^{i \text{ terms}}, \theta^{\sigma-1}, \dots, \theta^{\xi}; \theta^{\eta}) - f(\theta^{\tau+1}, \theta, \dots, \theta, \theta^{\sigma-1}, \dots, \theta^{\xi}, \theta^{\eta})$$
$$+ f(\theta, \theta^{\tau+1}, \theta, \dots, \theta, \theta^{\sigma-1}, \dots, \theta^{\xi}, \theta^{\eta}) - f(\theta, \theta^{\tau}, \theta^2, \dots, \theta, \theta^{\sigma-1}, \dots, \theta^{\xi}, \theta^{\eta}) + \cdots$$
$$+ f(\theta, \theta^{\tau}, \theta, \dots, \theta^2, \theta^{\sigma-1}, \dots, \theta^{\xi}, \theta^{\eta}) - f(\theta, \theta^{\tau}, \dots, \theta, \theta^{\sigma}, \dots, \theta^{\xi}, \theta^{\eta})$$
$$+ f(\theta, \theta^{\tau}, \dots, \theta, \theta^{\sigma+\cdots}, \dots, \theta^{\xi}, \theta^{\eta}) - \cdots + (-1)^{n+1}\theta^{\eta} f(\theta, \theta^{\tau}, \dots, \theta, \theta^{\sigma-1}, \dots, \theta^{\xi}) = 0,$$

we have:

$$f(\overbrace{\theta, \dots, \theta}^{i \text{ terms}}, \theta^{\sigma}, \theta^{\tau}, \dots, \theta^{\xi}, \theta^{\eta}) = \theta f(\overbrace{\theta, \dots, \theta}^{i \text{ terms}}, \theta^{\sigma-1}. \theta^{\tau}, \dots, \theta^{\xi}, \theta^{\eta})$$
$$+ f(\overbrace{\theta, \dots, \theta}^{i+1 \text{ terms}}, \theta^{\sigma+\cdots}, \theta^{\tau}, \dots, \theta^{\xi}, \theta^{\eta}) - \cdots + (-1)^{n+1}\theta^{\eta} f(\overbrace{\theta, \dots, \theta}^{i+1 \text{ terms}}, \theta^{\sigma-1}, \theta^{\tau}, \dots, \theta^{\xi}).$$

Thus, by both the mathematical induction on the number i of θ in $f(\theta, \dots, \theta, \theta^{\sigma}, \theta^{\tau}, \dots, \theta^{\xi}, \theta^{\eta})$ and the relation $\theta^{s} = -\sum_{i=1}^{s} a_i \theta^{s-i}$, $f(\theta^{\alpha}, \theta^{\beta}, \dots, \theta^{\gamma}, \theta^{\delta})$ $(1 \leq \alpha \leq s-1,\ 1 \leq \beta \leq s-1, \dots,\ 1 \leq \gamma \leq s-1,\ 1 \leq \delta \leq s-1)$ are uniquely represented as the linear form of $f(\theta, \dots, \theta, \theta^{t})$ $(1 \leq t \leq s-1)$ with coefficients in $\mathfrak{R}_0[\theta]$, q. e. d.

THEOREM 5. *If n is an odd number, the condition $f \in Z(\mathfrak{R}, \mathfrak{R}_0; \mathfrak{M})$ is the following:*

$$f(\theta, \dots, \theta, \theta^{i}) = i\theta^{i-1}\lambda \qquad \textit{for every natural number } i.$$

and

$$f(\theta, \dots, \theta, F(\theta)) = F'(\theta)\lambda = 0,$$

where $f(\theta, \dots, \theta) = \lambda$ is an arbitrary element of $\mathfrak{M}$.

PROOF. From

$$\theta f(\theta, \theta, \dots, \theta, \theta^{i-1}) - f(\theta^2, \theta, \dots, \theta, \theta^{i-1}) + \cdots + f(\theta, \theta, \dots, \theta^2, \theta^{i-1})$$
$$- f(\theta, \theta, \dots, \theta, \theta^{i}) + \theta^{i-1} f(\theta, \theta, \dots, \theta) = 0 \quad \text{for every natural number } i,$$

we have:

$$(1) \qquad f(\theta, \theta, \dots, \theta, \theta^{i}) = \theta f(\theta, \dots, \theta, \theta^{i-1})$$
$$+ \theta^{i-1} f(\theta, \dots, \theta, \theta) \qquad \text{for every natural number } i.$$

Let us put $f(\theta, \dots, \theta) = \lambda$, λ being an arbitrary element of $\mathfrak{M}$, then (1) is equivalent to

$$f(\theta, \dots, \theta, \theta^{i}) = i\theta^{i-1}\lambda \qquad \text{for every natural number } i.$$

By using $F(\theta) = 0$, we have:

$$f(\theta, \dots, \theta, F(\theta)) = F'(\theta)\lambda = 0, \quad \text{q. e. d.}$$

THEOREM 6. *If n is an even number, then there exists an n-cocycle f in any cohomology class of $H(\mathfrak{R}, \mathfrak{R}_0; \mathfrak{M})$ with $f(\theta, \dots, \theta, \theta^{i}) = 0$ for $1 \leq i < s-1$.*

PROOF. For an arbitrary $f \in Z(\mathfrak{R}, \mathfrak{R}_0; \mathfrak{M})$, we can take a g such that $f(\theta, \theta, \dots, \theta) = g(\theta^2, \theta, \dots, \theta)$. Then we have:

$$f(1, \theta, \dots, \theta) = g(\theta, \theta, \dots, \theta) = 0.$$

For the relative n-cocycle $f_1 = f + \delta g$, we have:

$$f_1(\theta, \dots, \theta) = f(\theta, \dots, \theta) + \theta g(\theta, \dots, \theta) - g(\theta^2, \theta, \dots, \theta)$$
$$+ \cdots - g(\theta, \dots, \theta^2) + \theta g(\theta, \dots, \theta) = 0.$$

Now we shall assume the existence of $f_i(1\leq i<s-2)$ such that $f_i(\theta, \dots, \theta, \theta^h)=0$ for $1\leq h\leq i$. We can take a g_i such that $f_i(\theta, \dots, \theta, \theta^{i+1})=g_i(\theta, \dots, \theta, \theta^{i+2})$, then we have:

$$f_i(\theta, \dots, \theta, \theta^h)=g_i(\theta, \dots, \theta, \theta^{h+1})=0 \quad \text{for} \quad 1\leq h\leq i.$$

Then for the relative n-cocycle $f_{i+1}=f_i+\delta g_i$ we have:

$$\begin{aligned} f_{i+1}(\theta, \dots, \theta, \theta^h) &= f_i(\theta, \dots, \theta, \theta^h)+\theta g_i(\theta, \dots, \theta, \theta^h) \\ &- g_i(\theta^2, \theta, \dots, \theta, \theta^h)+\cdots+g_i(\theta, \dots, \theta^2, \theta^h)-g_i(\theta, \dots, \theta, \theta^{h+1}) \\ &+ \theta^h g_i(\theta, \dots, \theta)=0 \quad \text{for} \quad 1\leq h\leq i, \end{aligned}$$

and

$$\begin{aligned} f_{i+1}(\theta, \dots, \theta, \theta^{i+1}) &= f_i(\theta, \dots, \theta, \theta^{i+1})+\theta g_i(\theta, \dots, \theta, \theta^{i+1}) \\ &- g_i(\theta^2, \theta, \dots, \theta, \theta^{i+1})+\cdots+g_i(\theta, \dots, \theta^2, \theta^{i+1})-g_i(\theta, \dots, \theta, \theta^{i+2}) \\ &+ \theta^{i+1} g_i(\theta, \dots, \theta)=0 \quad \text{for} \quad 1\leq i<s-2. \end{aligned}$$

Thus, by mathematical induction, we have our theorem.

Therefore, when n is an even number, if we denote $Z_1(\mathfrak{R}, \mathfrak{R}_0; \mathfrak{M})$ as the totality of all the relative n-cocycles $f\in Z(\mathfrak{R}, \mathfrak{R}_0; \mathfrak{M})$ with $f(\theta, \dots, \theta, \theta^i)=0$ for $1\leq i<s-1$ and denote $B_1(\mathfrak{R}, \mathfrak{R}_0; \mathfrak{M})=Z_1(\mathfrak{R}, \mathfrak{R}_0; \mathfrak{M})\cap B(\mathfrak{R}, \mathfrak{R}_0; \mathfrak{M})$, then we have:

$$H(\mathfrak{R}, \mathfrak{R}_0; \mathfrak{M})\cong Z_1(\mathfrak{R}, \mathfrak{R}_0; \mathfrak{M})/B_1(\mathfrak{R}, \mathfrak{R}_0; \mathfrak{M}).$$

THEOREM 7. *The condition* $\delta g\in B_1(\mathfrak{R}, \mathfrak{R}_0; \mathfrak{M})$ *is the following:*

$$g(\theta, \dots, \theta, \theta^i)=i\theta^{i-1}\lambda \quad \text{for} \quad 1\leq i\leq s-1$$

and

$$g(\theta, \dots, \theta, \theta^s)=-\sum_{i=1}^{s-1}(s-i)\,a_i\theta^{s-i-1}\lambda,$$

where $g(\theta, \dots, \theta)=\lambda$ *is an arbitrary element of* $\mathfrak{M}$.

PROOF. By Theorem 6, the condition $\delta g\in B_1(\mathfrak{R}, \mathfrak{R}_0; \mathfrak{M})$ is $\delta g(\theta, \dots, \theta, \theta^i)=0$ for $1\leq i<s-1$. Then from

$$\begin{aligned} \delta g(\theta, \dots, \theta, \theta^i) &= \theta g(\theta, \dots, \theta, \theta^i)-g(\theta^2, \dots, \theta, \theta^i)+\cdots \\ &+ \theta g(\theta, \dots, \theta^2, \theta^i,)-g(\theta, \dots, \theta, \theta^{i+1})+\theta^i g(\theta, \dots, \theta)=0, \end{aligned}$$

we have:

(1) $\quad g(\theta, \dots, \theta, \theta^{i+1})=\theta g(\theta, \dots, \theta, \theta^i)+\theta^i g(\theta, \dots, \theta) \quad$ for $\quad 1\leq i<s-1.$

Now, putting $g(\theta, \dots, \theta) = \lambda$, λ being an arbitrary element of $\mathfrak{M}$, (1) is equivalent to

$$g(\theta, \dots, \theta, \theta^i) = i\theta^{i-1}\lambda \quad \text{for} \quad 1 \leq i \leq s-1.$$

By using $\theta^s = -\sum_{i=1}^{s} a_i \theta^{s-1}$, we have:

$$g(\theta, \dots, \theta, \theta^s) = -\sum_{i=1}^{s-1} (s-i)\, a_i \theta^{s-i-1}\lambda, \quad \text{q. e. d.}$$

§ 4. On the structure of n-cocycle and n-cohomology group in integral domain of $\mathfrak{p}$-adic number fields.

Let k, K ($k \subset K$) be $\mathfrak{p}$-adic number fields, and $\mathfrak{o}$, $\mathfrak{O}$ its valuation ring, respectively, and $\mathfrak{P}$ ($\neq (0), (1)$) prime ideal in $\mathfrak{O}$. Let D be the relative different of K/k. Then, there exists $\theta \in \mathfrak{O}$ such that $\mathfrak{O} = \mathfrak{o}[\theta]$. Now, let $F(x) = x^s + a_1 x^{s-1} + \dots + a_s = 0$ be the irreducible defining equation of θ in $\mathfrak{o}[x]$, then $(F'(\theta)) = D$, where $F'(x)$ denotes the derivative of $F(x)$ by x. Now, we shall define the group $Z(\mathfrak{O}, \mathfrak{o}; \mathfrak{O}/\mathfrak{P}^r)$ of the relative n-cocycle and the relative n-cohomology group $H(\mathfrak{O}, \mathfrak{o}; \mathfrak{O}/\mathfrak{P}^r)$ (for natural number r) as in § 3. Then we have:

THEOREM 8. *For odd numbers of n*

$$Z(\mathfrak{O}, \mathfrak{o}; \mathfrak{O}/\mathfrak{P}^r) \cong \mathfrak{O}/(\mathfrak{P}^r, D) \qquad (r = 1, 2, \dots)$$

and for even numbers of n (in addition, $n = 1, 3$)

$$H(\mathfrak{O}, \mathfrak{o}; \mathfrak{O}/\mathfrak{P}^r) \cong \mathfrak{O}/(\mathfrak{P}^r, D) \qquad (r = 1, 2, \dots),$$

where $(\mathfrak{P}^r, D)$ denotes the greatest common divisor of $\mathfrak{P}^r$ and D.

PROOF. If n is an odd number, then, from Theorem 5, $f \in Z(\mathfrak{O}, \mathfrak{o}; \mathfrak{O}/\mathfrak{P}^r)$ is uniquely determined by the value of $f(\theta, \theta, \dots, \theta) \equiv \lambda \bmod \mathfrak{P}^r$. Then, also by Theorem 5, we have:

$$f(\theta, \theta, \dots, \theta, F(\theta)) \equiv F'(\theta)\lambda \equiv 0 \qquad (\text{mod } \mathfrak{P}^r).$$

Thus, we have easily:

$$Z(\mathfrak{O}, \mathfrak{o}; \mathfrak{O}/\mathfrak{P}^r) \cong \mathfrak{O}/(\mathfrak{P}^r, D) \qquad (r = 1, 2, \dots).$$

If n is an even number, then, from Theorem 6, $f \in Z_1(\mathfrak{O}, \mathfrak{o}; \mathfrak{O}/\mathfrak{P}^r)$ is uniquely determined by the value of $f(\theta, \dots, \theta, \theta^{s-1})$. But the condition $f = \delta g \in B_1(\mathfrak{O}, \mathfrak{o}; \mathfrak{O}/\mathfrak{P}^r)$ is the following:

$$f(\theta, \dots, \theta, \theta^{s-1}) \equiv \delta g(\theta, \dots, \theta, \theta^{s-1}) \equiv \theta g(\theta, \dots, \theta, \theta^{s-1})$$
$$- g(\theta^2, \theta, \dots, \theta^{s-1}) + g(\theta, \theta^2, \dots, \theta^{s-1}) - \cdots + g(\theta, \dots, \theta^2, \theta^{s-1})$$
$$- g(\theta, \dots, \theta, \theta^s) + \theta^{s-1} g(\theta, \dots, \theta) \qquad (\mathrm{mod}\ \mathfrak{P}^r).$$

Now, putting $g(\theta, \dots, \theta) \equiv \lambda$ mod $\mathfrak{P}^r$, by Theorem 7 we have:

$$f(\theta, \dots, \theta, \theta^{s-1}) \equiv \theta g(\theta, \dots, \theta, \theta^{s-1}) + \theta^{s-1} g(\theta, \dots, \theta) - g(\theta, \dots, \theta, \theta^s)$$
$$\equiv s\theta^{s-1}\lambda + \sum_{i=1}^{s-1}(s-i)\, a_i \theta^{s-i-1}\lambda \equiv F'(\theta)\lambda \qquad (\mathrm{mod}\ \mathfrak{P}^r).$$

Thus, we have:

$$H(\mathfrak{O}, \mathfrak{o}; \mathfrak{O}/\mathfrak{P}^r) \cong \mathfrak{O}/(\mathfrak{P}^r, D) \quad (r = 1, 2, \dots), \quad \text{q. e. d.}$$

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