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The Least Upper Bound of a Damping Coefficient Ensuring the Existence of a Periodic Motion of a Pendulum under Constant Torque

By

Minoru URABE

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§ 1. Introduction.

In the previous paper[(1)], by means of infinitesimal deformation of the solutions, we have studied on the periodic solution of the second kind of the equation of a pendulum as follows:

$$\frac{d^2\theta}{dt^2} + \alpha f(\theta)\frac{d\theta}{dt} + g(\theta) = 0, \qquad (\alpha \geqq 0) \tag{1.1}$$

where $f(\theta)$ and $g(\theta)$ are the periodic integral functions with the common period and $f(\theta)>0$. In that paper, we have also suggested the method of determining the least upper bound of α ensuring the periodic solution of the second kind of (1.1). In this paper, following the idea suggested in that paper, we shall actually calculate the least upper bound of α for the following equation of the special form of (1.1)

$$\frac{d^2\theta}{dt^2} + \alpha\frac{d\theta}{dt} + \sin\theta = \beta. \qquad (\alpha, \beta \geqq 0) \tag{E}$$

For the equation (E), the least upper bound of α ensuring the periodic solution of the second kind was already estimated by F. Tricomi[(2)], L. Amerio[(3)], G. Seifert[(4)]

1) M. Urabe, *Infinitesimal deformation of the periodic solution of the second kind and its application to the equation of a pendulum*, J. Sci. Hiroshima Univ., Ser. A, 18 (1954), 183-219. In the following, we denote this paper by [P].

2) F. Tricomi, *Integrazione di un'equazione differenziale presentatasi in elettrotecnica*, Ann. R. Sci. Norm. Sup. di Pisa, (1933), 1-20.

3) L. Amerio, *Determinazione delle condizione di stabilità per gli di un'equazione interessante l'elettro tecnica*, Ann. Math. pura appl., [4] 30 (1949), 75-90.

4) G. Seifert, *On the existence of certain solution of a nonlinear differential equation*, Z. angew. Math. Phys., 3 (1952), 468-471.

do., *On certain solutions of a pendulum-type equation*, Quart. Appl. Math., 11 (1953), 127-131.

and W. D. Hayes[1], but its true value has not yet been given by anyone up to the present. Our results will give the true values and make clear the accuracy of the estimation found by the above writers.

§ 2. The behavior of the function $\alpha(\beta)$.

Let the least upper bound of α ensuring the periodic solution of the second kind of (E) be $\alpha(\beta)$, considered as the function of β. Then, by §16 of [P], the value of $\alpha(\beta)$ is determined as the value of α for which, β being given, there exists a separatrix-curve of the equations as follows:

$$\left\{\begin{aligned} \frac{d\theta}{dt} &= z, \\ \frac{dz}{dt} &= -\alpha z + \beta - \sin\theta. \end{aligned}\right. \tag{2.1}$$

For $\beta > 1$, there exists no critical point of (2.1), consequently, by §12 of [P], $\alpha(\beta) = \infty$ for $\beta > 1$.

For $\beta < 1$, the critical points of (2.1) are $(\theta_0 + 2k\pi, 0)$ and $(\theta_1 + 2k\pi, 0)$, where $\theta_0 = \mathrm{Sin}^{-1}\beta$, $\theta_1 = \pi - \theta_0$ and k is an arbitrary integer. Put $\theta = \theta_i + \xi$ $(i = 0, 1)$, then, in the neighborhood of the critical points, the equations (2.1) can be written as follows:

$$\frac{d\xi}{dt} = z,$$

$$\frac{dz}{dt} = -\xi\cos\theta_i - \alpha z + \cdots,$$

where the unwritten terms are those of the second and higher orders with regard to ξ and z. Consequently, the characteristic roots λ of the matrix of the coefficients of the linear parts are determined by the equation as follows:

$$\lambda^2 + \alpha\lambda + \cos\theta_i = 0. \tag{2.2}$$

From this, it is easily seen that the points $(\vartheta_0 + 2k\pi, 0)$ are the stable nodes or the foci inclusive of the centers and the points $(\theta_1 + 2k\pi, 0)$ are the saddles. Consequently, the condition of §13 of [P] that all the saddle-like points are the saddles is fulfilled, therefore, by the general results of §15 of [P], the function $\alpha(\beta)$ becomes a one-valued continuous monotone increasing function.

1) W. D. Hayes, *On the equation for a damped pendulum under constant torque*, Z. angew. Math. Phys., 4 (1953), 398-401.

Increasing β from zero, let the least upper bound of β for which a separatrix-curve exists be β', then evidently $\beta' \leqq 1$. When $\beta' < 1$, by §17 of [P], it must be that $\alpha(\beta') = \infty$, namely that, for any finite α, there exists a periodic solution of the second kind of (E) for β'. This contradicts the well-known fact[1] that, for sufficiently large α, there does not exist any periodic solution of the second kind of (E) for $\beta < 1$. Thus it must be that $\beta' = 1$.

For $\beta = 1$, the critical points of (2.1) are $(\frac{\pi}{2} + 2k\pi,\ 0)$ $(k = 0,\ \pm 1,\ \pm 2,\ \cdots)$. By §11 of [P], these points are the saddle-like points, but not the saddles, and they are of the character shown in Fig. 7 of [P].

Let $\lim_{\beta \to 1-0} \alpha(\beta) = \alpha' < \infty$. If $\alpha(1) > \alpha'$, then, for α_0 such that $\alpha(1) \geqq \alpha_0 > \alpha'$, there exists a periodic solution of the second kind of (E) for $\beta = 1$. Then by the condition of continuation with regard to β[2], we see that, for $\alpha = \alpha_0$ and β such that $0 < 1 - \beta \ll 1$, there exists a periodic solution. Consequently it must be that $\alpha_0 < \alpha(\beta)$. Since $\alpha(\beta)$ is monotone increasing for $0 \leqq \beta < 1$, it must be that $\alpha_0 < \alpha(\beta) \leqq \alpha'$. This contradicts our assumption that $\alpha_0 > \alpha'$. Thus it must be that $\alpha(1) \leqq \alpha'$.

Suppose that $\alpha(1) < \alpha'$. Then, for any α_0 such that $\alpha(1) < \alpha_0 < \alpha'$, there exists a positive number $\beta_0 < 1$ such that, for $\beta_0 < \beta < 1$, it always holds that $\alpha_0 < \alpha(\beta) < \alpha'$. Then, for any such (β, α_0), there exists a periodic solution of the second kind of (E). Now, by §17 of [P], in the phase (θ, z)-plane, the periodic solution moves monotonely upwards when β is continuonsly increased α being fixed. Now, consider the curve $\Delta(\beta)$: $z = (\beta - \sin\theta)/\alpha_0$, then, above $\Delta(\beta)$, the solution $z = z(\theta\,;\beta)$ of the equation

$$(2.3) \qquad \frac{dz}{d\theta} = \frac{-\alpha_0 z + \beta - \sin\theta}{z} = -\frac{\alpha_0}{z}\left(z - \frac{\beta - \sin\theta}{\alpha_0}\right)$$

which corresponds to (2.1), is monotone decreasing, consequently, for the solution $z = z(\theta\,;\beta)$ such that $z\left(\frac{\pi}{2} + 2\pi\,;\beta\right) = h > 2/\alpha_0$, it is valid that

$$z\left(\frac{\pi}{2}\,;\beta\right) > h = z\left(\frac{\pi}{2} + 2\pi\,;\beta\right)$$

for any $\beta \leqq 1$[3]. Consequently, from the uniqueness and stability of the periodic solution of (2.3), the periodic solution corresponding to (β, α_0) lies below the solution

1) For example, cf.: A. A. Andronow and C. E. Chaikin, Theory of oscillations, (1949), 293-300.

2) Cf. §17 of [P].

3) By §6 of [P], the solution $z = z(\theta;\beta)$ such that $z\left(\frac{\pi}{2} + 2\pi;\ \beta\right) = h$ necessarily attains $\theta = \pi/2$.

$z=z(\theta;\beta)$. Now, from (2.3), for $z>0$, $dz/d\theta$ increases as β increases. Therefore, for $1\geqq\beta>\beta_0$, the solution $z=z(\theta;\beta)$ lies below the solution $z=z(\theta;\beta_0)$. Thus the periodic solution corresponding to (β,α_0) such that $\beta_0<\beta<1$ lies below the solution $z=z(\theta;\beta_0)$. Then, when β is increased monotonely from β_0 and tends to 1, the periodic solution moves monetonely upwards and tends to a certain set C lying below or on the solution $z=z(\theta;\beta_0)$. Because of continuity of the solution, C becomes a periodic solution of (2.3) for $\beta=1$. That is to say that there exists a periodic solution of the second kind of (E) for $\beta=1$ and $\alpha=\alpha_0$. This contradicts our assumption that $\alpha(1)<\alpha_0$.

Thus we see that, when $\lim_{\beta\to1-0}\alpha(\beta)=\alpha'<\infty$, it must be that $\alpha(1)=\alpha'$.

When $\alpha(1)<\infty$ and $\alpha'=\infty$, it is evident that $\alpha(1)<\alpha'$. Then, by the above reasonings, we get again a contradiction. Therefore, if $\alpha(1)<\infty$, it must be that $\alpha'<\infty$, consequently by the above result, it must be that $\alpha(1)=\alpha'$.

When $\alpha(1)$ and $\lim_{\beta\to1-0}\alpha(\beta)$ are both infinity, it is evident that $\alpha(1)=\lim_{\beta\to1-0}\alpha(\beta)$.

Thus we see that $\alpha(1)$ *and* $\lim_{\beta\to1-0}\alpha(\beta)$ *are always equal to each other whether either of them is finite or not.*

§ 3. The equation of a separatrix.

First, for $\beta<1$, we shall seek for the equations of the separatrices passing through the saddle $(\theta_1, 0)$. Put $\theta=\theta_1+\theta'$, then, writing θ instead of θ', (2.1) can be written as follows:

$$\begin{cases} \dfrac{d\theta}{dt} = z, \\ \dfrac{dz}{dt} = -\alpha z + \sin\theta_0 + \sin(\theta-\theta_0), \end{cases} \tag{3.1}$$

where

$$\theta_0 = \sin^{-1}\beta. \qquad (0\leqq\theta_0<\pi/2) \tag{3.2}$$

For the equations (3.1), the saddles are the points where $\theta=2k\pi$ $(k=0, \pm1, \pm2, \cdots)$. By §13 of [P], the equation of a separatrix passing through the saddle $(0, 0)$ is written as follows:

$$z = z(\theta;\alpha,\theta_0), \tag{3.3}$$

where $z(\theta;\alpha,\theta_0)$ is analytic with regard to the arguments. Consequently, put

$$z = c_1\theta + c_2\theta^2 + \cdots + c_n\theta^n + \cdots, \tag{3.4}$$

and substitute this into the equation

$$z\frac{dz}{d\theta} = -\alpha z + \sin\theta_0 + \sin(\theta - \theta_0), \tag{3.5}$$

which follows from (3.1), then, comparing the coefficients of the powers of θ, we have:

$$\begin{cases} c_1^2 = -\alpha c_1 + \cos\theta_0, \\ 3c_1c_2 = -\alpha c_2 + \dfrac{\sin\theta_0}{2\,!}, \\ 4c_1c_3 + 2c_2^2 = -\alpha c_3 - \dfrac{\cos\theta_0}{3\,!}, \\ 5c_1c_4 + 5c_2c_3 = -\alpha c_4 - \dfrac{\sin\theta_0}{4\,!}, \\ 6c_1c_5 + 6c_2c_4 + 3c_3^2 = -\alpha c_5 + \dfrac{\cos\theta_0}{5\,!}, \\ 7c_1c_6 + 7c_2c_5 + 7c_3c_4 = -\alpha c_6 + \dfrac{\sin\theta_0}{6\,!}, \\ \cdots\cdots. \end{cases}$$

Consequently, it follows that

$$\begin{cases} c_1 = -\dfrac{\alpha}{2} \pm \sqrt{\dfrac{\alpha^2}{4} + \cos\theta_0}\,, \\ c_2 = \dfrac{1}{\alpha + 3c_1} \cdot \dfrac{\sin\theta_0}{2}\,, \\ c_3 = \dfrac{1}{\alpha + 4c_1}\left(-2c_2^2 - \dfrac{\cos\theta_0}{6}\right), \\ c_4 = \dfrac{1}{\alpha + 5c_1}\left(-5c_2c_3 - \dfrac{\sin\theta_0}{24}\right), \\ c_5 = \dfrac{1}{\alpha + 6c_1}\left(-6c_2c_4 - 3c_3^2 + \dfrac{\cos\theta_0}{120}\right), \\ c_6 = \dfrac{1}{\alpha + 7c_1}\left(-7c_2c_5 - 7c_3c_4 + \dfrac{\sin\theta_0}{720}\right), \\ \cdots\cdots. \end{cases} \tag{3.6}$$

For determination of a separatrix, first we calculate the starting values by means of (3.4) where the coefficients c_i's are given by (3.6), and next, we proceed to calculate successively the values of $z = z(\theta\,; \alpha, \theta_0)$ integrating (3.5) numerically by means of the difference formulas. The difference formulas used here are as follows: for extrapolation:

$$z_{r+1} = z_r + \frac{h}{720}\left(720 f_r + 360 \nabla f_r + 300 \nabla^2 f_r + 270 \nabla^3 f_r + 251 \nabla^4 f_r\right);$$

for interpolation:

$$z_{r+1} = z_r + \frac{h}{720}\left(720 f_{r+1} - 360 \nabla f_{r+1} - 60 \nabla^2 f_{r+1} - 30 \nabla^3 f_{r+1} - 19 \nabla^4 f_{r+1}\right),$$

where $h = \pm 0.1$ for starting the integration and $h = \pm 0.2$ for continuing the integration.

For $\beta = 1$, the critical points of (2.1) correspond to $\theta = 2k\pi$ $(k = 0, \pm 1, \pm 2, \cdots)$ in (3.1) where $\theta_0 = \pi/2$. By (2.2), it is easily seen that the characteristic roots of the matrix of the coefficients of the linear parts of (3.1) are $-\alpha$ and 0. Therefore, for $\alpha > 0$, as in §13 of [P], we see that (3.1) has the solution

$$z = z(\theta\,;\,\alpha), \tag{3.7}$$

where $z(\theta\,;\,\alpha)$ is analytic with regard to the arguments and is expanded as follows:

$$z(\theta\,;\,\alpha) = -\alpha\theta + c_2\theta^2 + \cdots\cdots. \tag{3.8}$$

Comparing this with $z(\theta\,;\,\alpha, \theta_0)$, we see that $z(\theta\,;\,\alpha)$ is equal to $z_l(\theta\,;\,\alpha, \pi/2)$, where $z_l(\theta\,;\,\alpha, \theta_0)$ is $z(\theta\,;\,\alpha, \theta_0)$ of (3.3) for which $c_1 = -(\alpha/2) - \sqrt{(\alpha/2)^2 + \cos\theta_0}$, namely the function corresponding to a left separatrix. This result means that $z_l(\theta\,;\,\alpha, \theta_0)$ is continuous with regard to α and θ_0 for $\alpha > 0$ and $0 \leqq \theta_0 \leqq \pi/2$.

We shall show that, *if* $\lim_{\beta \to 1-0} \alpha(\beta) = \alpha' < \infty$, $z = z_l(\theta\,;\,\alpha', \pi/2)$ *represents a separatrix-curve*. If $z_l(\theta\,;\,\alpha', \pi/2) > 0$ for $-2\pi \leqq \theta < 0$, from continuity of the function $z_l(\theta\,;\,\alpha, \theta_0)$, we see that, for θ_0 sufficiently near $\pi/2$, $z_l(-2\pi\,;\,\alpha(\sin\theta_0), \theta_0) > 0$. This contradicts the fact that $z = z_l(\theta\,;\,\alpha(\sin\theta_0), \theta_0)$ represents a separatix-curve. In the same manner, from continuity of the function $z_l(\theta\,;\,\alpha, \theta_0)$, we see that the solution $z = z_l(\theta\,;\,\alpha', \pi/2)$ cannot cross the θ-axis for $-2\pi < \theta < 0$. Thus we see that $z_l(-2\pi\,;\,\alpha', \pi/2) = z_l(0\,;\,\alpha', \pi/2) = 0$ and $z_l(\theta\,;\,\alpha', \pi/2) > 0$ for $-2\pi < \theta < 0$, namely that $z = z_l(\theta\,;\,\alpha', \pi/2) = z(\theta\,;\,\alpha')$ represents a separatrix-curve for $\beta = \sin\theta_0 = 1$. From this, it is readily seen that, *if, for certain* α_1, $z_l(\theta\,;\,\alpha_1, \pi/2) = 0$ *for certain* θ *such that* $-2\pi < \theta < 0$, *then* $\alpha(1) > \alpha_1$. Besides, as in §15 of [P], it can be readily seen that, *if, for certain* α_2, $z_l(\theta\,;\,\alpha_2, \pi/2) > 0$ *for* $-2\pi \leqq \theta < 0$, *then* $\alpha_2 > \alpha(1)$. These results are of use for the determination of $\alpha(1) = \alpha'$.

§ 4. Calculation of $\alpha(\beta)$.

By (3.2), $\beta=\sin\theta_0$, consequently we consider $\alpha(\beta)$ as the function of θ_0 and denote it by $\alpha(\theta_0)$.

We assume that, for certain $\theta_0=\theta_0^*$ $(<\pi/2)$, the value α^* of $\alpha(\theta_0^*)$ and the separatrix-curve $z=z^*(\theta)$ are known. Then, by (15.4) of [P], we have:

$$\alpha(\theta_0^* + \delta\theta_0^*) = \alpha^* + \delta\alpha^*, \tag{4.1}$$

where

$$\delta\alpha^* \doteqdot \cos\theta_0^* \cdot \frac{\displaystyle\int_0^{2\pi} e^{\alpha^*\int_\pi^\theta \frac{d\theta}{z^*}}\, d\theta}{\displaystyle\int_0^{2\pi} z^* e^{\alpha^*\int_\pi^\theta \frac{d\theta}{z^*}}\, d\theta} \cdot \delta\theta_0^*. \tag{4.2}$$

By this formula, for $\theta_0=\theta_0^*+\delta\theta_0^*$, we can find the approximate value of $\alpha(\theta_0)=\alpha(\theta_0^*+\delta\theta_0^*)$. In actual calculation, in order to avoid computing the improper integral, in place of (4.2), we may use the approximate value as follows:

$$\delta\alpha^* = \cos\theta_0^* \cdot \frac{\displaystyle\int_\varepsilon^{2\pi-\varepsilon} e^{\alpha^*\int_\pi^\theta \frac{d\theta}{z^*}}\, d\theta}{\displaystyle\int_\varepsilon^{2\pi-\varepsilon} z^* e^{\alpha^*\int_\pi^\theta \frac{d\theta}{z^*}}\, d\theta} \cdot \delta\theta_0^*, \tag{4.3}$$

where ε is a sufficiently small positive number. The integration can be carried on numerically. In this paper, we have put $\varepsilon=0.2$ and used Simpson's rule for integration.

Next we assume that, for $\theta_0=\theta_0^*$ $(<\pi/2)$, the approximate value α^* of $\alpha(\theta_0^*)$ is known. Let the true value of $\alpha(\theta_0^*)$ be $\alpha=\alpha^*+\delta\alpha^*$. For $\theta_0=\theta_0^*$, let the right separatrix passing through (0,0) and the left separatrix passing through $(2\pi, 0)$ be $z=z_r(\theta;\alpha)$ and $z=z_l(\theta;\alpha)$ respectively. Then, for $\alpha=\alpha^*+\delta\alpha^*$, there must arise a separatrix-curve, namely the above both separatices must coincide with each other. For this, it is necessary and sufficient that

$$\varDelta(\alpha) \equiv z_r(c;\alpha) - z_l(c;\alpha) = 0, \tag{4.4}$$

where c is an arbitrary number such that $0<c<2\pi$. Thus it is seen that, in order to find the true value α of $\alpha(\theta_0^*)$, it needs only to solve the equation (4.4). Since $z_r(c;\alpha)$ and $z_l(c;\alpha)$ are analytic with regard to α, $\Delta(\alpha)$ is evidently analytic with regard to α. Moreover, by our assumption, the approximate root of (4.4) is already known. Therefore the equation (4.4) can be solved numerically by Newton-Raphson's method, if we know $d\Delta/d\alpha$ for the approximate root α^*.

For $\alpha=\alpha^*+\delta\alpha^*$, by (14.13) of [P], $z_r(\theta;\alpha)$ and $z_l(\theta;\alpha)$ are expressed as follows:

$$\text{(4.5)}\qquad \begin{cases} z_r(\theta;\alpha) = z_r^{(0)}(\theta) + z_r^{(1)}(\theta)\,\delta\alpha^* + \cdots, \\ z_l(\theta;\alpha) = z_l^{(0)}(\theta) + z_l^{(1)}(\theta)\,\delta\alpha^* + \cdots, \end{cases}$$

where

$$\text{(4.6)}\qquad z_r^{(0)}(\theta) = z_r(\theta;\alpha^*), \quad z_l^{(0)}(\theta) = z_l(\theta;\alpha^*),$$

and

$$\text{(4.7)}\qquad \begin{cases} z_r^{(1)}(\theta) = -\dfrac{1}{z_r^{(0)}(\theta)}\, e^{-\alpha^*\int_c^\theta \frac{d\theta}{z_r^{(0)}}} \displaystyle\int_0^\theta z_r^{(0)} e^{\alpha^*\int_c^\theta \frac{d\theta}{z_r^{(0)}}}\, d\theta, \\ z_l^{(1)}(\theta) = -\dfrac{1}{z_l^{(0)}(\theta)}\, e^{-\alpha^*\int_c^\theta \frac{d\theta}{z_l^{(0)}}} \displaystyle\int_{2\pi}^\theta z_l^{(0)} e^{\alpha^*\int_c^\theta \frac{d\theta}{z_l^{(0)}}}\, d\theta. \end{cases}$$

Then, for $\alpha=\alpha^*$, from (4.4), (4.5) and (4.7), it follows that

$$\frac{d\Delta}{d\alpha} = \frac{1}{z_l^{(0)}(c)} \int_{2\pi}^c z_l^{(0)} e^{\alpha^*\int_c^\theta \frac{d\theta}{z_l^{(0)}}}\, d\theta - \frac{1}{z_r^{(0)}(c)} \int_0^c z_r^{(0)} e^{\alpha^*\int_c^\theta \frac{d\theta}{z_r^{(0)}}}\, d\theta.$$

Since $|z_r^{(0)}(c) - z_l^{(0)}(c)| = |\Delta(\alpha^*)| \ll 1$, it follows that

$$\text{(4.8)}\qquad \left.\frac{d\Delta}{d\alpha}\right|_{\alpha=\alpha^*} \doteqdot -\frac{1}{z^{(0)}(c)} \int_0^{2\pi} z^{(0)} e^{\alpha^*\int_c^\theta \frac{d\theta}{z^{(0)}}}\, d\theta,$$

where $z^{(0)}(\theta)$ is a function such that

$$\text{(4.9)}\qquad \begin{cases} z^{(0)}(\theta) \doteqdot z_r^{(0)}(\theta) & \text{for } 0 \leqq \theta \leqq c, \\ z^{(0)}(\theta) \doteqdot z_l^{(0)}(\theta) & \text{for } c \leqq \theta \leqq 2\pi. \end{cases}$$

Thus, by Newton-Raphson's method, the first approximate value of $\delta\alpha^*$ is given by

$$\delta\alpha^* = \frac{\varDelta(\alpha^*)}{\dfrac{1}{z^{(0)}(c)}\displaystyle\int_0^{2\pi} z^{(0)} e^{\alpha^*\int_c^\theta \frac{d\theta}{z^{(0)}}} d\theta} . \tag{4.10}$$

In actual calculation, in order to make as small as possible the errors caused by numerical integration of the differential equation, we put $c=\pi$ and, in order to avoid computing the improper integral, we substitute ε and $2\pi-\varepsilon$ for the limit 0 and 2π of the integral in the denominator of (4.10). Thus the first approximate value of $\delta\alpha^*$ is calculated by

$$\delta\alpha^* = \frac{\varDelta(\alpha^*)}{\dfrac{1}{z^{(0)}(\pi)}\displaystyle\int_\varepsilon^{2\pi-\varepsilon} z^{(0)} e^{\alpha^*\int_\pi^\theta \frac{d\theta}{z^{(0)}}} d\theta} , \tag{4.11}$$

where $\varDelta(\alpha^*) = z_r(\pi\,;\alpha^*) - z_l(\pi\,;\alpha^*)$. After having modified the approximate value α^* by $\delta\alpha^*$ calculated by (4.11), we repeat the above process taking $\alpha^*+\delta\alpha^*$ instead of α^* as the approximate value of $\alpha(\theta_0{}^*)$. Iterating this process, we find the true value of $\alpha(\theta_0{}^*)$ as the limit of the approximate values found successively. As is well known, in this process of iteration, we need not compute the denominator of (4.11) in each step of iteration, but, in place of it, we may use the fixed value found in the first step.

Now, by §16 of [P], for $\beta=0$, i. e. $\theta_0=0$, $\alpha(0)=0$, consequently the separatrix-curve for $\theta_0=0$ is easily found as follows:

$$z = z^*(\theta) = 2\sin\frac{\theta}{2} .$$

Then, for $\theta_0=\theta_0^{(1)}=\delta\theta_0{}^*$, by means of (4.3), we can find the approximate value α^* of $\alpha(\theta_0^{(1)})$. For this α^*, by means of the method of §3, we compute $z_r(\theta\,;\alpha^*)$ for $\theta=0, 0.2, 0.4, \cdots, 3.0,\ \pi$ and $z_l(\theta\,;\alpha^*)$ for $\theta=2\pi,\ 2\pi-0.2,\ 2\pi-0.4,\ \cdots,\ 2\pi-3.0,\ \pi$. If $z_r(\pi\,;\alpha^*)=z_l(\pi\,;\alpha^*)$, α^* becomes a true value of $\alpha(\theta_0^{(1)})$ and

$$z = \begin{cases} z_r(\theta\,;\alpha^*) & \text{for } 0\leqq\theta\leqq\pi, \\ z_l(\theta\,;\alpha^*) & \text{for } \pi\leqq\theta\leqq 2\pi \end{cases}$$

becomes an equation of the separatrix-curve for $\theta_0=\theta_0^{(1)}$. If $\varDelta(\alpha^*) = z_r(\pi\,;\alpha^*) - z_l(\pi;\alpha^*) \neq 0$, then, by means of (4.11), we compute the first correction $\delta\alpha^*$ of α^*. At this juncture, as $z^{(0)}(\theta)$, we may use the function $z^*(\theta)$ corresponding to the separatrix-curve for $\theta_0=0$. Taking $\alpha^*+\delta\alpha^*$ instead of α^*, we repeat the above process and continue till we get $\varDelta(\alpha^*)=0$. In actual calculation, this iteration ceases after finite times of repetition since the only finite decimal places are required in actual calculation. Thus, after finite times of repetition of the above process, we get $\Delta(\alpha^*)=0$, namely the true value α^* of $\alpha(\theta_0^{(1)})$ and the equation of the separatix-curve for $\theta_0=\theta_0^{(1)}$. Next, for $\theta_0=\theta_0^{(2)}=\theta_0^{(1)}+\delta\theta_0{}^*$, in the same manner as for $\theta_0=\theta_0^{(1)}$, we find the true value of $\alpha(\theta_0^{(2)})$ and the equation of the separatrix-curve for $\theta_0=\theta_0^{(2)}$. Continuing this process, we compute $\alpha(\theta_0)$ and the equation of the separatrix-curve successively for $\theta_0=0, 0.2, 0.4, \cdots, 1.4$[1].

For $\theta_0=\pi/2$, (4.3) and (4.11) are not applicable since the analytic behavior of $z_r(\theta;\alpha)$ is unknown for $\theta_0=\pi/2$. Therefore the other method must be devised. For this purpose, first, by means of the interpolation formula — for example, Newton's backward formula —, we find the approximate value α^* of $\alpha(\pi/2)=\alpha' = \lim_{\theta_0\to(\pi/2)-0} \alpha(\theta_0)$. For this α^*, by means of the method of §3, we compute $z_l(\theta;\alpha^*,\pi/2)$ for $\theta=2\pi, 2\pi-0.2, 2\pi-0.4, \cdots\cdots$. If $z_l(\theta;\alpha^*,\pi/2)>0$ for $0\leqq\theta<2\pi$, we decrease α^* by small amount and, if $z_l(\theta;\alpha^*,\pi/2)=0$ for certain $0<\theta<2\pi$, we increase α^* by small amount. By this method, ultimately we find:

$$\text{for } \alpha^*=1.1925,\ z_l(\theta;\alpha^*,\pi/2)=0 \text{ for } \theta\doteqdot 1.4;$$

$$\text{for } \alpha^*=1.1930,\ z_l(\theta;\alpha^*,\pi/2)>0 \text{ for } 0\leqq\theta<2\pi.$$

Thus, by §3, *we see that* $\alpha(\pi/2)=1.193$ *correct to three decimal places.*

The true values of $\alpha(\theta_0)$ computed correct to three decimal places in the above way are shown in Table 1, compared with the bounds already found by G. Seifert[2] and W. D. Hayes[3]. These values are shown also graphically in Fig. 1. The separatrix-curves for each θ_0 are shown in Fig. 2.

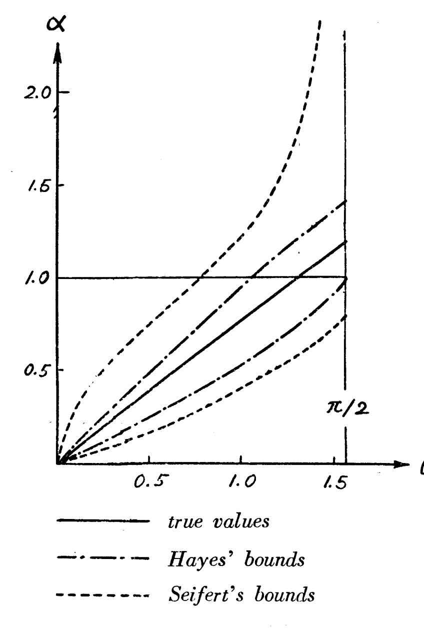

Fig. 1

1) Of course, in computing (4.11), as for $\theta_0=\theta_0{}^{(1)}$, as $z^{(0)}(\theta)$, we may use the function corresponding to the separatix-curve for the preceding θ_0.

2) G. Seifert, ibid.

3) W. D. Hayes, ibid.

Table 1. *The values of* $\alpha(\theta_0)$.

θ_0	lower bounds		true values	upper bounds	
	Seifert's	Hayes'		Hayes'	Seifert's
0	0.000	0.000	0.000	0.000	0.000
0.2	0.071	0.100	0.157	0.200	0.452
0.4	0.138	0.202	0.313	0.398	0.660
0.6	0.217	0.307	0.469	0.592	0.841
0.8	0.307	0.415	0.623	0.778	1.023
1.0	0.406	0.538	0.774	0.958	1.236
1.2	0.522	0.675	0.924	1.130	1.548
1.4	0.659	0.839	1.071	1.288	2.247
$1.571=\pi/2$	0.798	1.000	1.193	1.414	∞

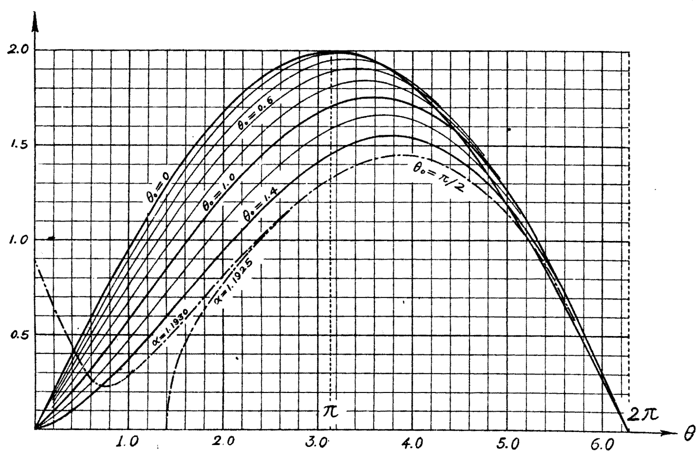

Fig. 2

Department of Mathematics,
Faculty of Science,
Hiroshima University,