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On the Complex Orthogonal Transformations

By

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§ 1. Introduction

In this paper it is our main purpose to obtain the exponential representations and the factorization of the complex orthogonal transformations. For this purpose we shall first consider the exponential representations of the orthogonal transformations, by means of which some properties (Theorem 2) of the special orthogonal group is obtained. By starting from this representation we shall obtain the factorization of a complex rotation into the plane rotations, and then we shall show the number of these plane rotations and the relations among these plane rotations.

As for the orthogonal canonical form of complex orthogonal matrices J. Wellstein[1)] has already obtained the elaborate results, and by K. Schröder[2)] a general theory for the exponential representation of semi-simple Lie group has been developped, and also recently J. L. Brenner[3)] has obtained an excellent result for the factorization of an orthogonal transformation into the plane rotations.

§ 2. The canonical representations of an orthogonal transformation

Let V be a complex n-dimensional vector space, and let $g=\|g_{ij}\|$ be a non-degenerate symmetric covariant tensor (i. e., $g_{ij}=g_{ji}$ and det $\|g_{ij}\|\neq 0$), then the complex orthogonal transformation in V with respect to g is represented by a matrix M of order n such that $M^*gM=g$, M^* being the transposed matrix of M. If we write $M^{(g)}=g^{-1}M^*g$ and call $M^{(g)}$ the transposed matrix of M with respect to g, then this operation has the same properties as the ordinary transposition ($*$-operation), and $M^*gM=g$ is written as $M^{(g)}M=1$ (the unit matrix). In particular,

1) J. Wellstein, *Uber symmetrische, alternierende und orthogonale Normalformen von Matrizen*, J. Reine Angew. Math., Vol. 163 (1930), pp. 166-182.

2) K. Schröder, *Einige Sätze aus der Theorie der kontinuirlichen Gruppen linear Transformationen*, Schriften des math. seminars der Univ. Berlin, Vol. 2 (1934), pp. 111-149.

3) J. L. Brenner, *The factorization of orthogonal matrices*, (abstract), Bull. Amer. Math. Soc., Vol. 60 (1954), p. 359.

if $g=\begin{pmatrix} & & 1 \\ & 1 & \\ \cdot^{\cdot^{\cdot}} & & \\ 1 & & \end{pmatrix}$, $M^{(g)}$ means the transposition of M with respect to its second diagonal.

For a coordinate transformation $v=P\,'v$, the covariant tensor $g=\|g_{ij}\|$ and the mixed tensor $k=\|k^i_j\|$ are transformed as

$$'g = P^* g P \quad \text{and} \quad 'k = P^{-1} k P$$

respectively. By a suitable coordinate transformation $'v=Qv$, g is transformed to $\delta=\|\delta_{ij}\|$ (Kronecker's delta), and then $g=Q^*Q$, and $M^*gM=g$ is equivalent to $'M^*\,'M=I$ where $'M=QMQ^{-1}$. In the following we shall consider the orthogonal transformation in a suitable coordinate system where g takes a convenient form for our consideration.

LEMMA 1. *M is orthogonal with respect to both g and g_1, if and only if $M^*gM=g$, $g_1=k^*gk$ for $k\in C(M)$, $C(M)$ being the set of all the matrices commutative with M.*

PROOF. By a suitable coordinate transformation $'v=Uv$, g is transformed to δ, then we have $g=U^*U$, $g_1=U^*\,'gU$ and $'M=UMU^{-1}$. Since $'M^*\,'M=I$, $'M^*\,'g\,'M='g_1$ is equivalent to $'g_1\,'M='M\,'g_1$, that is, $'g_1\in C('M)$. If we write $'k=\exp(\frac{1}{2}\operatorname{Log}\,'g_1)$ where $\operatorname{Log}\,'g_1$ indicates a principal branch of the logarithmic function of matrix $'g_1$,[1] then $'g_1{}^*='g_1\in C('M)$ implies $'k^*='k\in C('M)$, hence we have $'g_1='k^*\,'k$ and $'k^*='k\in C('M)$. Conversely, it is easily seen that $'k\in C('M)$ implies $'k^*\in C('M)$, and therefore $'g_1='k^*\,'k$ is symmetric and belongs to $C('M)$. By returning to the original coordinate system we see that M is orthogonal with respect to both g and g_1 if and only if $M^*gM=g$, $g_1=k^*gk$ for $k\in C(M)$.

REMARK 1 Let $\mathfrak{S}(M)$ be the system of metrics g such that M is orthogonal with respect to g, then we can prove that if $g_1\in\mathfrak{S}(M)$, then $\mathfrak{S}(M)$ contains another metric $g(\neq\alpha g_1$, α is any complex number). We have only to prove, in a coordinate syetem where $g_1=\delta$, that there exists a nondegenerate symmetric tensor g such that $g\in C(M)$ and $g\neq\alpha\delta$. If $M+M^*\neq\beta I$, then $g=M+M^*+\eta I$ $(\det(M+M^*\eta I)\neq 0)$ satisfies the above conditions. If $M+M^*=\beta I$ for some complex number β, then, since $M^*M=I$, $M^2-\beta M+I=0$, that is, $M'^2=\kappa^2 I$ where $M'=M-\dfrac{\beta}{2}I$; therefore we have $C(M)=C(M')$, and by a suitable coordinate transformation, M' is transformed to $M''=\kappa(I_{n_1}\dot{+}(-I_{n_2}))$, $(n_1+n_2=n)$, where I_r means the unit matrix of order r. In this coordinate system, if we take $g''=g_{(n_1)}\dot{+}g_{(n_2)}(\neq\alpha\delta)$ for $\kappa\neq 0$, $g''=g_{(n)}(\neq\alpha\delta)$ for $\kappa=0$, where $g_{(r)}$ means any nondegenerate symmetric

1) K. Morinaga and T. Nôno, *On the logarithmic functions of matrices I*, J. Sci. Hiroshima Univ. (A), Vol. 14, pp. 107-114.

matrix of order r, then our requirements are satisfied.

Lemma 2. *If* $M^*gM=g$, $M_1{}^*g_1M_1=g_1$ *and* $M_1=U^{-1}MU$, *then there exists a matrix* V *such that* $g_1=V^*gV$ *and* $M_1=V^{-1}MV$.

Proof. If we put $'g=U^*gU$, then we have $M_1^*{}'gM_1={}'g$, and hence, by Lemma 1 we have $g_1=k^*{}'gk$, $k\in C(M_1)$, and consequently $g_1=k^*U^*gUk=(Uk)^*g(Uk)$. If we write $V=Uk$, then $g_1=V^*gV$ and $V^{-1}MV=(Uk)^{-1}M(Uk)=k^{-1}U^{-1}MUk=k^{-1}M_1k=M_1$.

By O_g we shall denote the group of orthogonal transformations with respect to g, and then, by considering the case where $g_1=g$ we have

Lemma 3. *If* M, $M_1\in O_g$ *and* $M_1=U^{-1}MU$, *then there exists a matrix* V *such that* $M_1=V^{-1}MV$ *and* $V\in O_g$.

Next let M be an orthogonal matrix with respect to g, i. e., $M^{(g)}M=I$; then as shown in our previous paper,[1] there exists a unique matrix K such that $M=\exp K$ and $K\in\mathfrak{A}_{(0)}$, where $\mathfrak{A}_{(0)}$ is the set of all the matrices satisfying the condition: the imaginary parts of characteristic roots lie in the half-closed interval $(-\pi,\pi]$.

From $M^{(g)}M=I$ it follows that $KK^{(g)}=K^{(g)}K$[1] and $\exp(K+K^{(g)})=I$. If we write $\mathfrak{p}=K+K^{(g)}$, we have $K\mathfrak{p}=\mathfrak{p}K$ and $\exp\mathfrak{p}=I$; and hence $M=\exp K=\exp(K+\mathfrak{p}]$, that is, $\mathfrak{p}$ is a period for K. Therefore K and $\mathfrak{p}$ are simultaneously transformed into Jordan's canonical forms,[1] denoted by K_0 and $\mathfrak{p}_0$ respectively; and then we have

$$K_0+(K^{(g)})_0=\mathfrak{p}_0=\sum\dotplus\eta\begin{pmatrix}2\pi i&&&\\&2\pi i&&\\&&\ddots&\\&&&2\pi i\end{pmatrix},\quad(\eta=0,1),\tag{2.1}$$

where $A\dotplus B$ means the direct sum of A and B, and $\eta=1$ only for the blocks of K_0 belonging to its characteristic roots $a+\pi i$ (a is real). Therefore, $(K^{(g)})_0$ is also Jordan's canonical form of $K^{(g)}$; Since K and $K^{(g)}$ are similar, that is, they have the same Jordan's canonical forms, it follows from (2.1) that there exists a one-to-one correspondence among the blocks of the same order of K_0, corresponding the characteristic root σ to the characteristic root τ such that $\sigma+\tau=0$ or $2\pi i$. Since $K\in\mathfrak{A}_{(0)}$, except for the blocks where $\sigma=0$, πi, the blocks of K_0 are in pairs by the above correspondence. If we put $M_0=\exp K_0$, since $K_0=U^{-1}KU$ by means of some U, we have $M_0=U^{-1}MU$ and $M_0{}^*g_0M_0=g_0$ where $g_0=U^*gU$. And then we have

$$M_0=\exp K_0=\sum_{\sigma\neq0,\pi i}\dotplus\exp K_{(\sigma)}\dotplus\exp\tilde{K}_{(0)}\dotplus\exp\tilde{K}_{(\pi i)}\tag{2.2}$$

1) K.Morinaga and T.Nôno, *ibid.*

where $K_{(\sigma)}=\begin{pmatrix} \sigma & 1 & & \\ & \sigma & \ddots & \\ & & \ddots & 1 \\ & & & \sigma \end{pmatrix}$, and $\tilde{K}_{(0)}$ and $\tilde{K}_{(\pi i)}$ mean the total blocks belonging to the characteristic roots 0 and πi of K_0 respectively; therefore it follows from $M_0^* g_0 = g_0 M_0^{-1}$ that

$$(2.3) \qquad g_0 = \sum_{\sigma \neq 0,\, \pi i} \dot{+}\, g_{(\sigma)} \dot{+} \tilde{g}_{(0)} \dot{+} \tilde{g}_{(\pi i)}$$

corresponding to (2.2), and that $\exp \tilde{K}_{(0)}$ and $\exp \tilde{K}_{(\pi i)}$ are orthogonal with respect to $\tilde{g}_{(0)}$ and $\tilde{g}_{(\pi i)}$ respectively, i. e., by considering (2.1), $\tilde{K}_{(0)} + \tilde{K}_{(0)}^{(\tilde{g}_{(0)})} = 0$ and $(\tilde{K}_{(\pi i)} - \pi i I) + (\tilde{K}_{(\pi i)} - \pi i I)^{(\tilde{g}_{(\pi i)})} = 0$ where I means the unit matrix of the same other as $\tilde{K}_{(\pi i)}$. That is, $\tilde{K}_{(0)}$ and $(\tilde{K}_{(\pi i)} - \pi i I)$ are skewsymmetric with respect to $\tilde{g}_{(0)}$ and $\tilde{g}_{(\pi i)}$ respectively, and consequently $\tilde{K}_{(0)}^{2m+1}$ and $(\tilde{K}_{(\pi i)} - \pi i I)^{2m+1}$ are also skew-symmetric with respect to $\tilde{g}_{(0)}$ and $\tilde{g}_{(\pi i)}$ respectively, and hence $\tilde{K}_{(0)}^{2m+1}$ and $(\tilde{K}_{(\pi i)} - \pi i I)^{2m+1}$ are of even ranks. Therefore the blocks of even order of $\tilde{K}_{(0)}$ are in pairs with the blocks of the same order of $\tilde{K}_{(0)}$, and as for the blocks of even order of $\tilde{K}_{(\pi i)}$ the same is said. Now we shall denote by K' the matrix obtained from K by replacing, in $K = U K_0 U^{-1}$, by $-(a+\pi i)$ the characteristic root $-a+\pi i$ corresponding to the characteristic root $a+\pi i$ in the above pairs (a is real), that is, by replacing in the pairs of blocks $K_{(a+\pi i)}$ and $K_{(-a+\pi i)}$ the blocks $K_{(-a+\pi i)}$ by the blocks $K_{(-a-\pi i)}$, since $K_{(-a+\pi i)} = 2\pi i I + K_{(-a-\pi i)}$ where I means the unit matrix of the same order as $K_{(-a+\pi i)}$, we have $M = \exp K = \exp K'$, and K' has Jordan's canonical form satisfying the conditions:[1)]

1°. *The blocks belonging to the characteristic roots* σ $(\neq 0, \pi i)$ *are in pairs with the blocks of the same order belonging to the characteristic roots* $-\sigma$.

2°. *The blocks of even order belonging to the characteristic root* 0 *are in pairs with the blocks of the same order belonging to the characteristic root* 0.

3°. *The blocks of even order belonging to the characteristic root* πi *are in pairs with the blocks of the same order belonging to the characteristic root* $-\pi i$.

Thus K' is similar to the following matrix $\mathring{K}$:

$$(2.4) \qquad \mathring{K} = \sum_{\sigma \neq 0,\, \pi i} \dot{+}\, K_{\mathrm{I}}(\sigma, 2r) \dot{+} \sum \dot{+}\, K_{\mathrm{I}}(\pi i, 4r') \dot{+} \sum \dot{+}\, K_{\mathrm{II}}(2s+1)$$
$$\dot{+} \sum \dot{+}\, K_{\mathrm{III}}(4t) \dot{+} \sum \dot{+}\, K_{\mathrm{IV}}(2p+1) \dot{+} O_q,$$

where

1) This result is essentially the same as the result in J. Wellstein, *ibid.*

$$
(2.5)\quad
\left\{
\begin{array}{ll}
K_{\mathrm{I}}(\sigma, 2r)=\overbrace{\begin{pmatrix}
\begin{pmatrix}
\sigma & 1 & & \\
 & \sigma & \ddots & \\
 & & \ddots & 1 \\
 & & & \sigma
\end{pmatrix} & \\
 & \begin{pmatrix}
-\sigma & -1 & & \\
 & -\sigma & \ddots & \\
 & & \ddots & -1 \\
 & & & -\sigma
\end{pmatrix}
\end{pmatrix}}^{2r}, &
K_{\mathrm{II}}(2s+1)=\overbrace{\begin{pmatrix}
0 & 1 & & & & & & \\
 & 0 & \ddots & & & & & \\
 & & \ddots & \ddots & & & & \\
 & & & 0 & 1 & & & \\
 & & & & 0 & -1 & & \\
 & & & & & 0 & \ddots & \\
 & & & & & & \ddots & \ddots \\
 & & & & & & & 0 & -1 \\
 & & & & & & & & 0
\end{pmatrix}}^{2s+1}, \\
\text{type I. } (\sigma \neq 0,\ r \geqq 1) & \text{type II. } (s \geqq 1) \\
K_{\mathrm{III}}(4t)=\overbrace{\begin{pmatrix}
\begin{pmatrix}
0 & 1 & & \\
 & 0 & \ddots & \\
 & & \ddots & 1 \\
 & & & 0
\end{pmatrix} & \\
 & \begin{pmatrix}
0 & -1 & & \\
 & 0 & \ddots & \\
 & & \ddots & -1 \\
 & & & 0
\end{pmatrix}
\end{pmatrix}}^{4t}, &
K_{\mathrm{IV}}(2p+1)=\overbrace{\begin{pmatrix}
\pi i & 1 & & & & & & \\
 & \pi i & \ddots & & & & & \\
 & & \ddots & \ddots & & & & \\
 & & & \pi i & 1 & & & \\
 & & & & \pi i & -1 & & \\
 & & & & & \pi i & \ddots & \\
 & & & & & & \ddots & -1 \\
 & & & & & & & \pi i
\end{pmatrix}}^{2p+1}, \\
\text{type III. } (t \geqq 1) & \text{type IV. } (p \geqq 0)
\end{array}
\right.
$$

and O_q means the zero matrix of order q. Here if we take

$$
(2.6)\qquad \mathring{g}=\sum \dotplus \mathring{g}(2r) \dotplus \sum \dotplus \mathring{g}(4r') \dotplus \sum \dotplus \mathring{g}(2s+1) \dotplus \sum \dotplus \mathring{g}(4t) \dotplus \sum \dotplus \mathring{g}(2p+1) \dotplus \mathring{g}(q),
$$

where $\mathring{g}(l)=\overbrace{\begin{pmatrix} & & 1 \\ & 1 & \\ \cdot^{\cdot^{\cdot}} & & \\ 1 & & \end{pmatrix}}^{l}$, then we can easily verify that $\mathring{M}=\exp\mathring{K}$ is orthogonal with respect to $\mathring{g}$, i. e., $\mathring{M}^*\mathring{g}\mathring{M}=\mathring{g}$. Therefore, by means of Lemma 2 we have by a suitable coordinate transformation $v=V\mathring{v}$: $\mathring{M}=V^{-1}MV$ and $\mathring{g}=V^*gV$.

In the following we shall often consider the orthogonal transformation in the coordinate system where g takes the form $\mathring{g}$, and we shall write $M^+=\mathring{g}^{-1}M^*\mathring{g}$.

§ 3. The exponential representations of an orthogonal transformation

In this section we shall consider the exponential representations of an orthogonal transformation by means of the skew-symmetric matrices.

For a linear transformation T in V, let V_1 be the minimal subspace of V such that $V=V_1+V_2$, $V_1 \perp V_2$, $TV_1 \subset V_1$ and $Tv=v$ for all $v \in V_2$; if necessary, V_1 will be precisely denoted by $V_1(T)$. Then we shall indicate by $T_1 \perp T_2$ that $V_1(T_1) \perp V_1(T_2)$

and $V_1(T_1) \cap V_1(T_2) = (0)$. Moreover we shall define the dimension of T as follows:

$$\dim T = \dim V_1(T),$$

i. e.,

$$= n - \dim V_2(T)$$

$$= n - q, \qquad (\text{see } (2.4)).$$

It is easily seen that

$$M = \exp K = \exp\left(\frac{1}{2}\mathfrak{p} + \left(K - \frac{1}{2}\mathfrak{p}\right)\right) = \exp\frac{1}{2}\mathfrak{p}\cdot\exp\left(K - \frac{1}{2}\mathfrak{p}\right), \tag{3.1}$$

where $\frac{1}{2}\mathfrak{p}$ is the symmtric part of K, i. e., $\frac{1}{2}\mathfrak{p} = \frac{1}{2}(K + K^{(g)})$ and $K - \frac{1}{2}\mathfrak{p}$ is the skew-symmetric part of K, i. e., $K - \frac{1}{2}\mathfrak{p} = \frac{1}{2}(K - K^{(g)})$. From (2.1) we have

$$\frac{1}{2}\mathfrak{p}_0 = \sum \dotplus \eta \begin{pmatrix} \pi i & & & \\ & \pi i & & \\ & & \ddots & \\ & & & \pi i \end{pmatrix} \sim O_{n_1} \dotplus \pi i I_{n_2}, \qquad (\eta = 0, 1\,;\ n_1 + n_2 = n), \tag{3.2}$$

and consequently

$$\exp\frac{1}{2}\mathfrak{p}_0 \sim I_{n_1} \dotplus (-I_{n_2}),$$

where $A \sim B$ means that A is similar to B. It is clear that $\exp\frac{1}{2}\mathfrak{p}$ is similar to $\exp\frac{1}{2}\mathfrak{p}_0$, i. e., to $I_{n_1} \dotplus (-I_{n_2})$, and $I_{n_1} \dotplus (-I_{n_2})$ is orthogonal with respect to $I_{n_1} \dotplus \mathring{g}(n_2)$, and that $\exp\frac{1}{2}\mathfrak{p}$ is orthogonal with respect to g; therefore by means of Lemma 2, $\exp\frac{1}{2}\mathfrak{p}$ and g are transformed to $I_{n_1} \dotplus (-I_{n_2})$ and $I_{n_1} \dotplus \mathring{g}(n_2)$ simultaneously by a coordinate transformation. Furthermore we have

$$-I_{n_2} = \exp \overbrace{\begin{pmatrix} \pi i & & & & & & \\ & \ddots & & & & & \\ & & \pi i & & & & \\ & & & (\pi i) & & & \\ & & & & -\pi i & & \\ & & & & & \ddots & \\ & & & & & & -\pi i \end{pmatrix}}^{n_2}$$

$((\pi i)$ occurs only for $\det M = -1)$) and

$$\overbrace{\begin{pmatrix} \pi i & & & & & & \\ & \ddots & & & & & \\ & & \pi i & & & & \\ & & & (0) & & & \\ & & & & -\pi i & & \\ & & & & & \ddots & \\ & & & & & & -\pi i \end{pmatrix}}^{n_2}$$

((0) occurs only for $\det M = -1$) is skew-symmetric with respect to $\mathring{g}(n_2)$. By returnnig to the original coordinate system we have [1)]

1) K. Schröder proved that any element of the component of semisimple Lie group $\mathfrak{G}$ is expressed by $\exp U \exp V$ where U and V are the elements of the infinitesimal group of $\mathfrak{G}$. K. Schröder, *ibid.*

$$\text{(3.3)}\qquad \exp\frac{1}{2}\mathfrak{p} = \Gamma \exp K_2, \text{ and consequently } M = \Gamma \exp K_2 \cdot \exp K_1,$$

where Γ is a symmetry which occurs only for the case where $\det M=-1$, and K_2 and K_1 are skew-symmetric with respect to g. Furthermore we can prove

THEOREM 1. *A rotation R is in one-to-one correspondence with a pair of skew-symmetric matrices $\overset{1}{F}$ and $\overset{2}{F}$ as follows:*

$$R = J \exp \overset{1}{F} \cdot \exp \overset{2}{F},\ (\overset{1}{F}\overset{2}{F} = \overset{2}{F}\overset{1}{F}),$$

where $\exp \overset{1}{F} \perp \exp \overset{2}{F}$, $V_1(\exp \overset{2}{F}) = V_1(J)$, *($V_1(J)$ is of even dimension),* $|I(\mu^{(1)})| < \pi$ *for the charactristic roots $\mu^{(1)}$ of $\overset{1}{F}$, $I(\mu^{(2)})=0$ for the characteristic roots $\mu^{(2)}$ of $\overset{2}{F}$, and $Jv=-v$ for all $v \in V_1(J)$.*

PROOF. From (3.1) a rotation R is expressible as

$$\text{(3.4)}\qquad R = J \exp F$$

where $J=\exp\frac{1}{2}\mathfrak{p}$, $\det J=1$ and $F=K-\frac{1}{2}\mathfrak{p}$. By remembering the process from (3.1) to (3.3), we see that

$$\text{(3.5)}\qquad R = J \exp(\overset{1}{F}+\overset{2}{F}) = J \exp \overset{1}{F} \cdot \exp \overset{2}{F},\ (\overset{1}{F}\overset{2}{F} = \overset{2}{F}\overset{1}{F}),$$

where $\overset{1}{F}$ and $\overset{2}{F}$ are the skew-symmetric matrices with respect to g such that

$$\text{(3.6)}\qquad \begin{cases} \exp \overset{1}{F} \perp \exp \overset{2}{F},\ V_{,}(\exp \overset{2}{F}) = V_1(J),\ (V_1(J) \text{ is of even dimension}). \\ |I(\mu^{(1)})| < \pi \text{ for the characteristic roots } \mu^{(1)} \text{ of } \overset{1}{F}, \\ I(\mu^{(2)}) = 0 \text{ for the characteristic roots } \mu^{(2)} \text{ of } \overset{2}{F}, \\ \text{and } Jv=-v \text{ for all } v \in V_1(J). \end{cases}$$

Conversely, corresponding to the skew-symmetric matrices $\overset{1}{F}$ and $\overset{2}{F}$ satisfying the conditions for only $\overset{1}{F}$ and $\overset{2}{F}$ in (3.6), there is uniquely determined J such that $V_1(J)=V_1(\exp \overset{2}{F})$ and $Jv=-v$ for all $v \in V_1(J)$, and therefore a rotation R is uniquely determined by means of (3.5). Thus a rotation R is in one-to-one correspondece with the pair $\overset{1}{F}$ and $\overset{2}{F}$ satisfying the conditions for only $\overset{1}{F}$ and $\overset{2}{F}$ in (3.6).

J can not be continuously determined from $\overset{2}{F}$ by the condition: $V_1(J)=V_1(\exp \overset{2}{F})$. For, if we take $g \equiv \mathring{g}(n) = \begin{pmatrix} & & 1 \\ & 1 & \\ 1^{\cdot^{\cdot^{\cdot}}} & & \end{pmatrix}$ and $\mathring{F}_\eta = \begin{pmatrix} 0 & & & & & & \\ & \ddots & & & & & \\ & & 0 & & & & \\ & & & \eta \cdots & & & \\ & & & \begin{pmatrix} \ddots & \# \\ 0 & \ddots \end{pmatrix} & & & \\ & & & & -\eta & & \\ & & & & & 0 & \\ & & & & & & \ddots \\ & & & & & & & 0 \end{pmatrix}$ where $\overset{2}{F}{}^{+} = -\overset{2}{F}$

and $\begin{pmatrix} \ddots & \# \\ 0 & \ddots \end{pmatrix}$ is of order $2r$, then we have $J_\eta = \begin{pmatrix} 1 &&&&&& \\ & \ddots &&&&& \\ && 1 &&&& \\ &&& \underbrace{-1 \ \ddots \ -1}_{2(r+1)} &&& \\ &&&& 1 && \\ &&&&& \ddots & \\ &&&&&& 1 \end{pmatrix}$; here if η tends to zero, then $\overset{2}{F}_\eta$ tends to $\overset{2}{F} = \begin{pmatrix} 0 &&&&&& \\ & \ddots &&&&& \\ && 0 &&&& \\ &&& \begin{pmatrix} \ddots & \# \\ 0 & \ddots \end{pmatrix} &&& \\ &&&& 0 && \\ &&&&& \ddots & \\ &&&&&& 0 \end{pmatrix}$, and corresponding to this $\overset{2}{F}$ we have $J = \begin{pmatrix} 1 &&&&&& \\ & \ddots &&&&& \\ && 1 &&&& \\ &&& \underbrace{-1 \ \ddots \ -1}_{2r} &&& \\ &&&& 1 && \\ &&&&& \ddots & \\ &&&&&& 1 \end{pmatrix}$; therefore, for $\eta \to 0$, $\overset{2}{F}_\eta$ tends to $\overset{2}{F}$ but J_η does not tend to J. Thus our theorem is proved.

Moreover J is expressed by a skew-symmetric matrx $\overset{3}{F}$ as follows:

$$J = \exp \overset{3}{F}, \tag{3.7}$$

where $\overset{3}{F} \sim \begin{pmatrix} \eta_1 \pi i &&& \\ & \eta_2 \pi i && \\ && \ddots & \\ &&& \eta_{2r} \pi i \end{pmatrix}$, $(\dim J = 2r)$, $\eta_a + \eta_{2r+1-a} = 0$ and $\eta_a^2 = 1$ for $a = 1, 2, \cdots, 2r$

That is, there exist 2^r $\overset{3}{F}$ satisfying (3.7) for a J. Thus a rotation R is expressible as $R = \exp \overset{3}{F} \exp \overset{1}{F} \exp \overset{2}{F}$ with these 2^r arbitrarities.

Next if we write

$$\mathring{B}_2 = \sum{}' \dot{+} K_{\mathrm{IV}} \dot{+} 0, \quad \mathring{B}_1 = \mathring{K} - \mathring{B}_2, \tag{3.8}$$

where $\sum{}' \dot{+}$ means the direct sum of the blocks K_{IV} of different order, each of which occurs odd times in K, then we have

$$\mathring{M} = \exp \mathring{K} = \exp \mathring{B}_2 \cdot \exp \mathring{B}_1, \quad \mathring{B}_1 \mathring{B}_2 = \mathring{B}_2 \mathring{B}_1,$$

that is, in the original coordinate system, (3.9) $$M = \exp K = \exp B_2 \cdot \exp B_1, \quad B_2 B_1 = B_1 B_2.$$

This exponential representation will be used in § **5** to obtain the factorization of a complex rotation into plane rotations. Furthermore we shall put as lollows:

(3.10) $$\begin{cases} \mathring{B}_2 = A_2' + \{\textstyle\sum' \dotplus \begin{pmatrix} 0 & 1 & & & & & \\ & \ddots & \ddots & & & & \\ & & 0 & 1 & & & \\ & & & 0 & -1 & & \\ & & & & 0 & \ddots & \\ & & & & & \ddots & -1 \\ & & & & & & 0 \end{pmatrix} \dotplus 0\}, \\ \\ A_2' = \textstyle\sum' \dotplus \begin{pmatrix} \pi i & & & \\ & \pi i & & \\ & & \ddots & \\ & & & \pi i \end{pmatrix} \dotplus 0, \quad \mathring{A}_1 = \mathring{K} - A_2', \end{cases}$$

then we have

$$\mathring{M} = \exp A_2' \exp \mathring{A}_1, \quad \mathring{A}_1 A_2' = A_2' \mathring{A}_1;$$

by the same method which was used to obtain (3.3) we have

(3.11) $$M = \Gamma \exp A_2 \exp A_1$$

where A_1 and A_2 are skew-symmetric with respect to g, and Γ being a symmetry which occurs only for the case where $\det M = -1$.

From (3.11) we have another representation of R by the skew-symmetric matrices $\overset{1}{f}$ and $\overset{2}{f}$:

(3.12) $$R = j \exp(\overset{1}{f} + \overset{2}{f}) = j \exp \overset{1}{f} \exp \overset{2}{f}, \ (\overset{1}{f}\overset{2}{f} = \overset{2}{f}\overset{1}{f}),$$

where $V_1(j) = V_1(\overset{2}{f} + P_0)$, $\exp \overset{1}{f} \perp \exp \overset{2}{f}$, $|I(\mu^{(1)})| \leqq \pi$ for the characteristic roots $\mu^{(1)}$ of $\overset{1}{f}$, and $\mu^{(2)} = 0$ for the characteristic roots $\mu^{(2)}$ of $\overset{2}{f}$; P_0 occurs only for the case where $\dim \exp \overset{2}{f}$ is odd, and then P_0 indicates the position of (0) corresponding to (-1) (in the canonical form $(\mathring{g}, \mathring{K})$) in the correspondence j and $\overset{2}{f}$:

$$j: \begin{pmatrix} \ddots & & & & \\ & (-1) & & & \\ & & \ddots & & \\ & & & -I_p & \\ & & & & \ddots \end{pmatrix} \quad \overset{2}{f}: \begin{pmatrix} \ddots & & & \\ & (0) & & \\ & & \ddots & \\ & & & \overbrace{\begin{pmatrix} 0 & 1 & & & & \\ & 0 & \ddots & & & \\ & & \ddots & 1 & & \\ & & & \ddots & -1 & \\ & & & & \ddots & -1 \\ & & & & & 0 \end{pmatrix}}^{p} \end{pmatrix}, \ (p \text{ is odd and} \neq 1).$$

Since it may happen that $|I(\mu^{(1)})| = \pi$ for some characteristic roots $\mu^{(1)}$ of $\overset{1}{f}$, the corresponce $R \rightleftarrows (\overset{1}{f}, \overset{2}{f}, P_0)$ is not one-to-one in general; and as seen before, the correspondence $\overset{2}{f} \to j$ is not continuous, consequently the correspondence $(\overset{1}{f}, \overset{2}{f}, P_0) \to R$ is not continuous. As will be seen in Proposition 1 (§ 4), in the representation (3.12), $\overset{2}{f}=0$ if and only if $R=\exp f$ (f being skew-symmetric), but in the representation (3.5), this fact is not seen.

§ 4. Some properties of rotation groups

In this section we shall consider the rotation group 0^+, i. e., the special orthogonal group. As it is clear from (2.4), the necessary and sufficient condition for $M=\exp K \in 0^+$ is that the sum of the ranks of the blocks of $\mathring{K}$ belonging to the characteristic root πi be an even number; and then M is expressible as

$$M = \exp A_2 \exp A_1, \tag{4.1}$$

where A_1 and A_2 ares kew-symmetric with respect to g. Moreover, it is easily seen by taking t as an infinitesimal value that $\exp tX \in O_g$ for every real number t if and only if X is skew-symmetric with respect to g, i. e., $X+X^{(g)}=0$. Such a one-parameter sudgroup $\exp tX$ will be called an orthogonal path. By the same consideration as used in § 2 to obtain (2.4) we see[1] that X is skew-symmetric with respect to g, i. e., $X+X^{(g)}=0$, if and only if 1°, 2°, 3° and 4° are satisfied:

4°. *The blocks of odd order belonging to the characteristic root πi are in pairs with the blocks of the same order belonging to the characteristic root $-\pi i$.*

If there exist odd blocks of the same odd order belonging to the characteristic root πi in $\mathring{K}$, then, by adding the term of period $\mathfrak{p}(M)$ such that $\log M=K+\mathfrak{p}(M)$, the property 4° can not be obtained. That is, there exists the case which can not be expressed by a skew-symmetric matrix.

Let O_{I}^+ be the set of the elements M of O_g^+ which have the orthogonal path, i. e., $M=\exp A$, $A+A^{(g)}=0$, and let $O_{\mathrm{II}}^+=O_g^+-O_{\mathrm{I}}^+$, then we have a decomposition of O_g^+: $O_g^+=O_{\mathrm{I}}^+\cup O_{\mathrm{II}}^+$, $O_{\mathrm{I}}^+\cup O_{\mathrm{II}}^+=\phi$.

From the above consideration we have

PROPOSITION 1. *$M\in O_{\mathrm{I}}^+$ if and only if $A_2=0$ in the representation $M=\exp A_2 \exp A_1$, in other words, $B_2=0$ in the representation $M=\exp B_2 \exp B_1$.*

As for the decomposition of O_g^+: $O_g^+=O_{\mathrm{I}}^+\cup O_{\mathrm{II}}^+$ we have

THEOREM 2. *(i) $O_{\mathrm{II}}^+\neq\phi$ for $n\geqq 4$ and $O_{\mathrm{II}}^+=\phi$ for $n\leqq 3$, (ii) this decomposition is orthogonal invariant: $Q^{-1}O_{\mathrm{I}}^+Q=O_{\mathrm{I}}^+$ and $Q^{-1}O_{\mathrm{II}}^+Q=O_{\mathrm{II}}^+$ for every $Q\in O_g$, (iii) $O_{\mathrm{I}}^+=\{h^2\,;h\in O_g^+\}=\{h^2\,;h\in O_g\}$, (iv) $O_{\mathrm{II}}^+=\{h_1h_2\,;h_1h_2=h_2h_1,\ h_1^2=I,\ h_1\neq I,\ h_1,h_2\in O_{\mathrm{I}}^+\}$.*

1) This result is essentially contained in J. Wellstlin, *ibid.*

PROOF. (i) For $n \geqq 4$, if we take $\mathring{g}=\begin{pmatrix} & & 1 & \\ & 1 & & \\ 1 & & & \\ & & & 1 \end{pmatrix}$ and $\mathring{K}=\begin{pmatrix} \pi i & 1 & & \\ & \pi i & -1 & \\ & & \pi i & \\ & & & \pi i \end{pmatrix}$, then $\mathring{M}=\exp \mathring{K}$ is orthogonal with respect to $\mathring{g}$, but by Proposition 1, $\mathring{M}$ has no orthogonal path, i. e., $\mathring{M}\in O_{\mathrm{II}}^{+}$; it is clear from Proposition 1 that such an element can not exist for $n \leqq 3$.

(ii) If $M\in O_{\mathrm{I}}^{+}$, then $M=\exp A$, $A+A^{(g)}=0$; for every $Q\in O_g$, $Q^{-1}MQ=\exp Q^{-1}AQ$ and $(Q^{-1}AQ)^{(g)}=Q^{(g)}A^{(g)}(Q^{-1})^{(g)}=-Q^{-1}AQ$. That is, $Q^{-1}MQ\in O_{\mathrm{I}}^{+}$. Since it is obvious that $Q^{-1}O_g^{+}Q=O_g^{+}$ for every $Q\in O_g$, we have $Q^{-1}O_{\mathrm{II}}^{+}Q=O_{\mathrm{II}}^{+}$.

(iii) If $M\in O_{\mathrm{I}}^{+}$, then $M=\exp A$, $A+A^{(g)}=0$; putting $M_1=\exp\frac{1}{2}A$, we have $M=M_1^2$, $M_1\in O_{\mathrm{I}}^{+}\subset O_g^{+}\subset O_g$, that is, $O_{\mathrm{I}}^{+}\subset\{h^2\,;\,h\in O_g^{+}\}\subset\{h^2\,;\,h\in O_g\}$. Conversely, if $M=M_1^2$ and $M_1\in O_g$, then $M\in O_{\mathrm{I}}^{+}$. In order to prove this we write $M_1=\exp K_1$, $K_1\in\mathfrak{A}_{(0)}$ and $\mathfrak{p}=K_1+K_1^{(g)}$, and then from $M_1\in O_g$ it follows that $\exp\mathfrak{p}=\exp(K_1+K_1^{(g)})=I$. Moreover $M_1=\exp K_1=\exp\{(K_1-\frac{1}{2}\mathfrak{p})+\frac{1}{2}\mathfrak{p}\}$ where $K_1-\frac{1}{2}\mathfrak{p}$ is skew-symmetric with respect to g, therefore we have $M=M_1^2=\exp\{2(K_1-\frac{1}{2}\mathfrak{p})+\mathfrak{p}\}=\exp 2(K-\frac{1}{2}\mathfrak{p})\in O_{\mathrm{I}}^{+}$, that is, $\{h^2\,;h\in O_g\}\subset O_{\mathrm{I}}^{+}$. Thus we obtain $O_{\mathrm{I}}^{+}=\{h^2\,;h\in O_g^{+}\}=\{h^2\,;h\in O_g\}$. This fact means that if $M=M_1^2$ for $M_1\in O_g$, then $M=M_2^2$ for $M_2\in O_{\mathrm{I}}^{+}$.

(iv) This is clear from the expression: $M=\exp A_2\exp A_1$ and Proposition 1.

REMARK 2. Let $\mathring{\mathfrak{A}}$ be the set of all the matrices satisfying the condition: the imaginary parts of the characteristic roots lie in the open interval $(-\pi,\pi)$, then from Proposition 1 it follows that if $M\in O_g$ has no characteristic root -1, then $M\in O_{\mathrm{I}}^{+}$; and that if $M=\exp K\in O_g$ and $K\in\mathring{\mathfrak{A}}$, then $M\in O_{\mathrm{I}}^{+}$, that is, $O_0^{+}\equiv\exp\mathring{\mathfrak{A}}\cup O_g\subset O_{\mathrm{I}}^{+}$. From (iv) of Proposition 2 we have $O_{\mathrm{II}}^{+}\subset(O_{\mathrm{I}}^{+})^2$ and hence $O_g^{+}\subset(O_{\mathrm{I}}^{+})^2$. But we can prove that $O_g^{+}\subset(O_0^{+})^2$. For $M\in O_g^{+}$ we have from

$$M\sim\mathring{M}=\exp\ (\hat{B}_2\dotplus\hat{B}_1)=\exp\hat{B}_2\dotplus\exp\hat{B}_1, \tag{3.9}$$

where $\hat{B}_1$ is skew-symmetric with respect to $\mathring{g}$, $\hat{B}_2=\sum'\dotplus K_{\mathrm{IV}}$ and $\mathring{K}=\hat{B}_1\dotplus\hat{B}_2$. Therefere $\exp\frac{1}{2}\hat{B}_1\in O_0^{+}$, consepuently $\exp\hat{B}_1=(\exp\frac{1}{2}\hat{B}_1)^2\in(O_0^{+})^2$. Since $\det M=1$, $\hat{B}_2=\sum\dotplus(K_{\mathrm{IV}}(p_1)\dotplus K_{\mathrm{IV}}(p_2))$ and $\exp(K_{\mathrm{IV}}(p_1)\dotplus K_{\mathrm{IV}}(p_2))$ is a rotation with respect to $\mathring{g}(p_1)\dotplus\mathring{g}(p_2)$. ($p_1$ and p_2 are odd and $p_1<p_2$). And we see that $K_{\mathrm{IV}}(p_1)\dotplus K_{\mathrm{IV}}(p_2)$ is similar to $\pi iI_{2p}+D$, where

$$D=\left(\begin{array}{c|c} \overbrace{\begin{matrix}0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0\end{matrix}}^{p} & \overbrace{\begin{matrix}a_{\frac{1}{2}(p_2-p_1),\,p+1} & & \\ & \ddots & \\ & & a_{p,\,p+p_1+1}\end{matrix}}^{p} \\ \hline & \begin{matrix}0 & -1 & & \\ & 0 & \ddots & \\ & & \ddots & -1 \\ & & & 0\end{matrix}\end{array}\right), \quad (D^+=-D, \quad p_1+p_2=2p),$$

$a_{\frac{1}{2}(p_2-p_1),\ p+1}, \cdots, a_{p,\ p+p_1+1}$ being taken suitably. Because, if we put

$$D_0=K_{\mathrm{IV}}(p_1)\dotplus K_{\mathrm{IV}}(p_2)-\pi i I_{2p}=\overbrace{\begin{pmatrix}0 & 1 & & & & \\ & 0 & \ddots & & & \\ & & \ddots & 1 & & \\ & & & \ddots & -1 & \\ & & & & \ddots & -1 \\ & & & & & 0\end{pmatrix}}^{p_1}\dotplus\overbrace{\begin{pmatrix}0 & 1 & & & & \\ & 0 & \ddots & & & \\ & & \ddots & 1 & & \\ & & & \ddots & -1 & \\ & & & & \ddots & -1 \\ & & & & & 0\end{pmatrix}}^{p_2},$$

then it is verified that the ranks $\rho(D)$ and $\rho(D_0)$ of D and D_0 satisfy

$$(4.2)\qquad \begin{cases} \rho(D^r)=\rho(D_0^r)=2p-2r, & (1\leqq r\leqq p_1), \\ \rho(D^{p_1+s})=\rho(D_0^{p_1+s})=(p_2-p_1)-s, & (1\leqq s\leqq p_2-p_1). \end{cases}$$

Therefore, $\exp\,(K_{\mathrm{IV}}(p_1)+K_{\mathrm{IV}}(p_2))$ is similar to $\exp\,(\pi i I_{2p}+D)$, and also it is clear that $\exp\,(K_{\mathrm{IV}}(p_1)\dotplus K_{\mathrm{IV}}(p_2))$ is orthogonal with respect to $\mathring{g}(p_1)\dotplus\mathring{g}(p_2)$ and $\exp(\pi i I_{2p}+D)$ is orthogonal with respect to $\mathring{g}(p_1+p_2)$. Hence, by Lemma 2 we see that by a suitable coordinate transformation, $\mathring{g}(p_1)+\mathring{g}(p_2)$ and $\exp\ (K_{\mathrm{IV}}(p_1)\dotplus K_{\mathrm{IV}}(p_2))$ are simultaneously transformed to $\mathring{g}(p_1+p_2)$ and $\exp\,(\pi i I_{2p}+D)$ respectively. And then we have

$$(4.3)\qquad \begin{cases} \exp\,(\pi i I_{2p}+D)=\exp\ \pi i I_{2p}\cdot\exp D=\exp\,(\pi i I_p\dotplus(-\pi i)I_p)\cdot\exp D \\ \quad =\exp\frac{1}{2}\,(\pi i I_p\dotplus(-\pi i)I_p)\cdot\exp\frac{1}{2}\,(\pi i I_p\dotplus(-\pi i I_p))\cdot\begin{pmatrix}1 & & \# \\ & 1 & \\ 0 & & \ddots \\ & & & 1\end{pmatrix}. \end{cases}$$

It is clear that $\exp\frac{1}{2}(\pi i I_p\dotplus(-\pi i)I_p)\in O_0^+$ and $\exp\frac{1}{2}(\pi i I_p\dotplus(-\pi i)I_p)\cdot\exp D\in O_0^+$, (with respect to $\mathring{g}(p_1+p_2)$). Therefore we have $\exp(\pi i I_{2p}+D)\in(O_0^+)^2$ (with respect to $\mathring{g}(p_1+p_2)$), that is, $\exp(K_{\mathrm{IV}}(p_1)\dotplus K_{\mathrm{IV}}(p_2))\in(O_0^+)^2$ (with respect to $\mathring{g}(p_1)+\mathring{g}(p_2)$); consequently, $\exp\mathring{B}_2\in(O_0^+)^2$ (with respect to $\sum\dotplus(\mathring{g}(p_1)\dotplus\mathring{g}(p_2))$. Thus we conclude that $\mathring{M} \in (O_0^+)^2$ (with respect to $\mathring{g}$), i. e., $M \in (O_0^+)^2$ (with respect to g).

It is noticed that $\mathring{\mathfrak{A}} \cap \mathfrak{M}^s$ is mapped topologically onto O_0^+ by the mapping $M = \exp A$ where $\mathfrak{M}^s$ means the set of all the skew-symmetric matrices with respect to g. And moreover O_0^+ is the largest subset of O_g^+ which can be mapped topologically onto the subset of the Lie algebra $\mathfrak{M}^s$ of the rotation group O_g^+.[1)]

§ 5. The factorization of a complex rotation

In this section we shall consider the factorization of a complex rotation into the plane rotations.

LEMMA 4. *If M_1, M_2 and $SM_1M_2S^{-1} \in O_g$, then $SM_1M_2S^{-1} = \bar{M}^1\bar{M}^2$, where $\bar{M}_1 = S_1M_1S_1^{-1}$, $\bar{M}_2 = S_1M_2S_1^{-1}$ and $\bar{M}_1$, $\bar{M}_2 \in O_g$.*

PROOF. If M_1, M_2 and $SM_1M_2S^{-1} \in O_g$, then by Lemma 3

$$SM_1M_2S^{-1} = S_1M_1M_2S_1^{-1} = S_1M_1S_1^{-1} \cdot S_1M_2S_1^{-1} = \bar{M}_1\bar{M}_2$$

where $S_1 \in O_g$, $\bar{M}_1 = S_1M_1S_1^{-1}$ and $\bar{M}_2 = S_1M_2S_1^{-1}$, i. e., $\bar{M}_1$, $\bar{M}_2 \in O_g$.

REMARK 3. Lemma 4 is written as follows: If M_1, M_2 and $M_3 \in O_g$, and $M_1M_2 \sim M_3$, then $M_3 = \bar{M}_1\bar{M}_2$ where $\bar{M}_1 \sim M_1$, $\bar{M}_2 \sim M_2$ (simultaneously) and $\bar{M}_1$, $\bar{M}_2 \in O_g$.

REMARK 4. Uuless $\sigma = 2l\pi i$ where l is a non-zero integer, we have

$$\exp\begin{pmatrix} \sigma & 1 & & & \\ & \sigma & 0 & & \\ & & 0 & \ddots & \\ & & & \ddots & 0 \\ & & & & 0 \end{pmatrix} \exp\begin{pmatrix} 0 & 0 & & & \\ & 0 & 1 & & \\ & & \sigma & \ddots & \\ & & & \ddots & 1 \\ & & & & \sigma \end{pmatrix} = S\exp\begin{pmatrix} \sigma & 1 & & \\ & \sigma & \ddots & \\ & & \ddots & 1 \\ & & & \sigma \end{pmatrix} S^{-1}.$$

For, $$\exp\begin{pmatrix} \sigma & 1 & & & \\ & \sigma & 0 & & \\ & & 0 & \ddots & \\ & & & \ddots & 0 \\ & & & & 0 \end{pmatrix} \exp\begin{pmatrix} 0 & 0 & & & \\ & 0 & 1 & & \\ & & \sigma & \ddots & \\ & & & \ddots & 1 \\ & & & & \sigma \end{pmatrix} = \begin{pmatrix} e^\sigma & e^\sigma & & & \\ & e^\sigma & 0 & & \\ & & 1 & \ddots & \\ & & & \ddots & 0 \\ & & & & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & & & & \\ & 1 & \dfrac{e^\sigma - 1}{\sigma} & \# & & \\ & & e^\sigma & e^\sigma & & \\ & & & \ddots & \ddots & \\ & & & & \ddots & e^\sigma \\ & & & & & e^\sigma \end{pmatrix}$$

$$= \begin{pmatrix} e^\sigma & e^\sigma & & & & \\ & e^\sigma & \dfrac{e^\sigma - 1}{\sigma}e^\sigma & \# & & \\ & & \ddots & e^\sigma & & \\ & & & \ddots & \ddots & \\ & & & & \ddots & e^\sigma \\ & & & & & e^\sigma \end{pmatrix} = \bar{S}\begin{pmatrix} e^\sigma & 1 & & \\ & e^\sigma & 1 & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & e^\sigma \end{pmatrix}\bar{S}^{-1}$$

(since $\dfrac{e^\sigma - 1}{\sigma}e^\sigma \neq 0$ from the hypothesis, these two matrices have the same elementary

1) K. Morinaga and T. Nono, *ibid.* Concerning the results mentioned in this section we will consider in more detail in the forthcoming paper in this Journal: *On the paths in the matrix space.*

divisors),

$$=\bar{S}\exp\begin{pmatrix}\sigma & \tau & & \# \\ & \sigma & \ddots & \\ & & \ddots & \tau \\ & & & \sigma\end{pmatrix}\bar{S}^{-1}=\exp\bar{S}\begin{pmatrix}\sigma & \tau & & \# \\ & \sigma & \ddots & \\ & & \ddots & \tau \\ & & & \sigma\end{pmatrix}\bar{S}^{-1}\qquad(\tau\neq 0)$$

$$=\exp S\begin{pmatrix}\sigma & 1 & & \\ & \sigma & \ddots & \\ & & \ddots & 1 \\ & & & \sigma\end{pmatrix}S^{-1}=S\exp\begin{pmatrix}\sigma & 1 & & \\ & \sigma & \ddots & \\ & & \ddots & 1 \\ & & & \sigma\end{pmatrix}S^{-1}.$$

Moreover, similarly, we have for $\sigma\neq 2l\pi i$ (l is a non-zero integer)

$$\exp\begin{pmatrix}\sigma & 1 & & & & & \\ & \sigma & \ddots & & & & \\ & & \ddots & 1 & & & \\ & & & \sigma & 0 & & \\ & & & & 0 & \ddots & \\ & & & & & \ddots & 0 \\ & & & & & & 0\end{pmatrix}\exp\begin{pmatrix}0 & 0 & & & & & \\ & \ddots & \ddots & & & & \\ & & \ddots & 0 & & & \\ & & & 0 & 1 & & \\ & & & & \sigma & \ddots & \\ & & & & & \ddots & 1 \\ & & & & & & \sigma\end{pmatrix}=S\exp\begin{pmatrix}\sigma & 1 & & \\ & \sigma & \ddots & \\ & & \ddots & 1 \\ & & & \sigma\end{pmatrix}S^{-1}.$$

We shall call a rotation $R=\exp K'$ as the rotations of type I, II, III or IV according to $K'\sim\mathring{K}=K_{\mathrm{I}}(\sigma, 2r)$ $(\sigma\neq 0)$, $K_{\mathrm{II}}(2s+1)$, $K_{\mathrm{III}}(4t)$ or $K_{\mathrm{IV}}(2p_1+1)\dot{+}K_{\mathrm{IV}}(2p_2+1)$ respectively (see (2.5)). The rank of a rotation is defined as the rank of $\mathring{K}$, and the rotation of rank two will be called a plane rotation. The plane rotations are classified into types I, II and III according to the form of K as follows:

$$R=\exp K',\ K'\sim\mathring{K},\ g\sim\mathring{g}\qquad\text{(simultaneously)}$$

	Type I	Type II	Type III
$\mathring{g}$:	$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$	$\begin{pmatrix}0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{pmatrix}$	$\begin{pmatrix}0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\end{pmatrix}$
$\mathring{K}$:	$\begin{pmatrix}\sigma & 0\\ 0 & -\sigma\end{pmatrix}$ $(\sigma\neq 0)$	$\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & -1\\ 0 & 0 & 0\end{pmatrix}$	$\begin{pmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -1\\ 0 & 0 & 0 & 0\end{pmatrix}$
Characteristic property:	$\mathring{K}^m\neq 0$, $(R-\mathrm{I})^m\neq 0$ (m is any integer)	$\mathring{K}^3=0$, $(R-I)^3=0$	$\mathring{K}^2=0$, $(R-I)^2=0$

$$\mathring{f} = k^{-1}\mathring{K}: \begin{cases} \underset{1}{u} \times \underset{2}{u} & \underset{1}{v} \times \underset{2}{v} & \underset{1}{w} \times \underset{2}{w} \\ (\underset{1}{u}, \underset{2}{u}) = 0 & (\underset{1}{v}, \underset{2}{v}) = 0 & (\underset{1}{w}, \underset{2}{w}) = 0 \\ \|\underset{1}{u}\| \cdot \|\underset{2}{u}\| \neq 0 & \|\underset{1}{v}\| \neq 0,\ \|\underset{2}{v}\| = 0 & \|\underset{1}{w}\| = 0,\ \|\underset{2}{w}\| = 0 \\ \text{in other forms,} & & \\ k(\underset{1}{e} \times \underset{2}{e}) & k(\underset{1}{e} \times (\underset{2}{e} + i\underset{3}{e})) & k((\underset{1}{e} + i\underset{2}{e}) \times (\underset{3}{e} \times i\underset{4}{e})) \\ & (\underset{a}{e}, \underset{b}{e}) = \delta_{ab}\,(a, b = 1, 2, 3, 4), & \end{cases}$$

wher $\underset{1}{u} \times \underset{2}{u} = \underset{12}{uu} - \underset{21}{uu}$, $(\underset{1}{u}, \underset{2}{u})$ means the inner product of $\underset{1}{u}$ and $\underset{2}{u}$, and $\|u\| = (u, u)^{\frac{1}{2}}$. It is worthy of notice that an ordinary plane rotation is of type I.

THEOREM 3. A rotation of type I and of rank $\rho = 2r$ is a product of a plane rotation of type I and a rotation of type I and of rank $\rho - 2$. A rotation of type I and of rank $\rho = 2r$ is a porduct of $\kappa = \frac{1}{2}\rho = r$ plane rotations of type I:

$$R = R_1 R_2 \cdots R_\kappa, \quad \kappa = \tfrac{1}{2}\rho,$$

where

$$R_i \perp R_j, \quad R_i R_j = R_j R_i, \quad \text{for } |i - j| \neq 1, \quad i, j = 1, 2, \cdots, \kappa.$$

PROOF. By means of Remark 4 we have

$$\exp k_1 \exp k_2 = S \exp k \cdot S^{-1},$$

wher $k_1 = \begin{pmatrix} \sigma & & & & \\ & 0 & & & \\ & & \ddots & & \\ & & & 0 & \\ & & & & -\sigma \end{pmatrix}$, $k_2 = \begin{pmatrix} 0 & 1 & & & & & \\ \sigma & \ddots & \ddots & & & & \\ & \ddots & \ddots & 1 & & & \\ & & \sigma & -\sigma & -1 & & \\ & & & \ddots & \ddots & & \\ & & & & -\sigma & -1 & \\ & & & & & 0 & \end{pmatrix}$ and $k = \begin{pmatrix} \sigma & 1 & & & & \\ \sigma & \ddots & \ddots & & & \\ & \ddots & \ddots & 1 & & \\ & & \sigma & -\sigma & -1 & \\ & & & \ddots & \ddots & \ddots \\ & & & & \ddots & -1 \\ & & & & & -\sigma \end{pmatrix}$.

Since $\exp k_1$, $\exp k_2$, $\exp k \in O_{\mathring{g}}$, form Lemma 3 it follows that

$$\exp k = S_1^{-1} \exp k_1 \exp k_2 \cdot S_1 = S_1^{-1} \exp k_1 S_1 \cdot S_1^{-1} \exp k_2 \cdot S_1, \ S_1 \in O_{\mathring{g}}. \tag{5.1}$$

And $\exp k_1$ is a plane rotation of type I, and $\exp k_2$ is a rotation of type I and of rank $\rho - 2$; since $S_1 \in O_{\mathring{g}}$, it is clear that $S_1^{-1} \exp k_1 \cdot S_1$ and $S_1^{-1} \exp k_2 \cdot S_1$ are the rotations of the same properties as $\exp k_1$ and $\exp k_2$ respectively. Hence, by the mathematical induction, this theorem is proved, remaining the relation among the plane rotations R_i. In order to see the relation among R_i, we must consider more in detail the factorization of R. If we put

$$h_1=\begin{pmatrix}\sigma &&&& \\ & 0 &&& \\ && \ddots && \\ &&& 0 & \\ &&&& -\sigma\end{pmatrix},\quad h_2=\begin{pmatrix}0 & 1 &&&&& \\ & \sigma &&&&& \\ && 0 &&&& \\ &&& \ddots &&& \\ &&&& 0 && \\ &&&&& -\sigma & -1 \\ &&&&&& 0\end{pmatrix}\ \text{and}\ h_3=\begin{pmatrix}0 &&&&&&&&&& \\ & 0 & 1 &&&&&&&& \\ && \sigma & 1 &&&&&&& \\ &&& \ddots & \ddots &&&&&& \\ &&&& \ddots & 1 &&&&& \\ &&&&& \sigma &&&&& \\ &&&&&& -\sigma & -1 &&& \\ &&&&&&& -\sigma & \ddots && \\ &&&&&&&& \ddots & -1 && \\ &&&&&&&&& -\sigma & -1 & \\ &&&&&&&&&& 0 & \\ &&&&&&&&&&& 0\end{pmatrix},$$

then we have

$$\exp h_1 \exp h_2 \exp h_3 = \begin{pmatrix} e^{\sigma} & e^{\sigma}\dfrac{e^{\sigma}-1}{\sigma} &&&&&&&&&& \\ & e^{\sigma} & \dfrac{e^{\sigma}-1}{\sigma} &&&&&&&& \# & \\ && e^{\sigma} & e^{\sigma} &&&&&&&& \\ &&& \ddots & \ddots &&&&&&& \\ &&&& \ddots & e^{\sigma} &&&&&& \\ &&&&& e^{\sigma} &&&&&& \\ &&&&&& e^{-\sigma} & e^{-\sigma} &&&& \\ &&&&&&& \ddots & \ddots &&& \\ &&&&&&&& \ddots & e^{-\sigma} && \\ &&&&&&&&& e^{-\sigma} & \dfrac{e^{-\sigma}-1}{\sigma} & \\ &&&&&&&&&& e^{-\sigma} & \dfrac{e^{-\sigma}-1}{\sigma} \\ &&&&&&&&&&& e^{-\sigma} \end{pmatrix}$$

$$=T^{-1}\begin{pmatrix} e^{\sigma} & 1 &&&&& \\ & \ddots & \ddots &&&& \\ && \ddots & 1 &&& \\ &&& e^{\sigma} &&& \\ &&&& e^{-\sigma} & -1 & \\ &&&&& \ddots & \ddots \\ &&&&&& \ddots & -1 \\ &&&&&&& e^{-\sigma}\end{pmatrix}T=S^{-1}\exp\begin{pmatrix} \sigma & 1 &&&&& \\ & \sigma & \ddots &&&& \\ && \ddots & 1 &&& \\ &&& \sigma &&& \\ &&&& -\sigma & -1 & \\ &&&&& -\sigma & \ddots \\ &&&&&& \ddots & -1 \\ &&&&&&& -\sigma\end{pmatrix}S=S^{-1}\exp k\cdot S,$$

and $\exp h_1$, $\exp h_2$, $\exp h_3$ and $\exp k \in O_{\mathfrak{g}}^{\circ}$. Hence, by means of Lemma 3, we have

$$\exp k = S_1 \exp h_1 \exp h_2 \exp h_3\, S_1^{-1} = \exp \bar{h}_1 \exp \bar{h}_2 \exp \bar{h}_3,$$

where S_1, $\exp h_1$, $\exp h_2$, $\exp h_3 \in O_{\mathfrak{g}}^{\circ}$. And then it is easily seen that $\exp \bar{h}_1$ and $\exp \bar{h}_2$ are the plane rotations of type I, and $\exp \bar{h}_3$ is a rotation of type I and of rank $\rho-4$. From the above construction of h_1, h_2 and h_3, by means of the mathematical induction, the following relation among R_i is seen:

$$R_i \perp R_j,\quad R_iR_j = R_jR_i \ \text{for}\ |i-j| \neq 1,\quad i,j=1,2,\cdots,\kappa.$$

Thus, this theorem is completely proved.

Lemma 5. *A Plane rotation of type II with respect to g is a product of two plane rotations of type I with respect to g.*

Proof. If we put

$$k_1 = \begin{pmatrix} -\sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sigma \end{pmatrix} \text{ and } k_2 = \begin{pmatrix} \sigma & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & -\sigma \end{pmatrix} \qquad (\sigma \neq 0),$$

then we have

$$\exp k_1 = \begin{pmatrix} e^{-\sigma} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{\sigma} \end{pmatrix} \text{ and } \exp k_2 = \begin{pmatrix} e^{\sigma} & \dfrac{e^{\sigma}-1}{\sigma} & -\dfrac{(\cosh\sigma - 1)}{\sigma^2} \\ 0 & 1 & \dfrac{e^{-\sigma}-1}{\sigma} \\ 0 & 0 & e^{-\sigma} \end{pmatrix},$$

and since σ can be taken such as $\dfrac{e^{\sigma}-1}{\sigma} \neq 0$, we have

$$\exp k_1 \exp k_2 = \begin{pmatrix} 1 & \dfrac{1-e^{-\sigma}}{\sigma} & \# \\ 0 & 1 & \dfrac{1-e^{\sigma}}{\sigma} \\ 0 & 0 & 1 \end{pmatrix} = \bar{S} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \bar{S}^{-1}$$

$$= \bar{\bar{S}} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \bar{\bar{S}}^{-1} = \bar{\bar{S}} \exp \begin{pmatrix} 0 & 1 & \# \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \bar{\bar{S}}^{-1}$$

$$= S^{-1} \exp \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} S.$$

that is, we have

$$\exp \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} = S \exp k_2 \exp k_1 S^{-1},$$

and moreover

$$\exp k_1,\ \exp k_2,\ \exp \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \in O_{\mathring{g}}.$$

Hence, by Lemmas 3 and 4 we have

$$\exp \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} = S_1 \exp k_2 \exp k_1 \cdot S_1^{-1} = S_1 \exp k_2 S_1^{-1} \cdot S_1 \exp k_1 S_1^{-1} = \exp \bar{k}_2 \cdot \exp \bar{k}_1,$$

where

$$S_1,\ \exp \bar{k}_1,\ \exp \bar{k}_2 \in O_g \text{ and } \bar{k}_2^m,\ \bar{k}_1^m \neq 0 \ (m \text{ is any integer}).$$

Here, since $\exp k_1$ and $\exp k_2$ are the plane rotations of type I, $\exp \bar{k}_1$ and $\exp \bar{k}_2$ are also the plane rotations of type I. Thus the lemma is proved.

LEMMA 6. *A plane rotation ol tyve III with respect to g is a product of two plane rotations of type I with respect to g.*

PROOF. We shall first determine σ_1, σ_2, τ_1 and τ_2 such that

$$\begin{pmatrix} 1+\sigma_1 & \tau_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1+\sigma_2 & \tau_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \sigma_1 \sigma_2 \neq 0,$$

that is,

$$(5.2) \qquad (1+\sigma_1)(1+\sigma_2) = 1, \quad (1+\sigma_1)\tau_2 + \tau_1 = 1, \qquad \sigma_1\sigma_2 \neq 0;$$

and we shall take σ_1', τ_1', σ_2' and τ_2' determined by

$$\exp \begin{pmatrix} \sigma_1' & \tau_1' \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1+\sigma_1 & \tau_1 \\ 0 & 1 \end{pmatrix}, \ \exp \begin{pmatrix} \sigma_2' & \tau_2' \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1+\sigma_2 & \tau_2 \\ 0 & 1 \end{pmatrix}.$$

Here if we write

$$(5.3) \qquad k_1 = \begin{pmatrix} P_1 & 0 \\ 0 & -P_1^{+} \end{pmatrix},\ k_2 = \begin{pmatrix} P_2 & 0 \\ 0 & -P_2^{+} \end{pmatrix} \text{ where } P_1 = \begin{pmatrix} \sigma_1' & \tau_1' \\ 0 & 0 \end{pmatrix},\ P_2 = \begin{pmatrix} \sigma_2' & \tau_2' \\ 0 & 0 \end{pmatrix},$$

then we have $\exp k_1$, $\exp k_2 \in O_{\mathring{g}}$ because $k_1^{+} = -k_1$, $k_2^{+} = -k_2$, $(k^{+} = \mathring{g}^{-1} k^* \mathring{g})$. Form (5.2) and (5.3) it follows that

$$\exp k_1 \exp k_2 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \exp \begin{pmatrix} 0 & 1 & & \\ & 0 & & \\ & & 0 & -1 \\ & & & 0 \end{pmatrix}.$$

On the other hand, since $\sigma_1\sigma_2 \neq 0$, i.e., $\sigma_1'\sigma_2' \neq 0$ we have $k_1^m \neq 0$, $k_2^m \neq 0$; and moreover the ranks of k_1 and k_2 are both two. Therefore $\exp k_1$ and $\exp k_2$ are the plane rotations of type II. Thus, a plane rotation of type III is a product of two plane rotations of type I.

REMARK 5. The rotation which is a product of two symmetries is a plane rotation of type I or II, and vice versa.

For, let $\underset{1}{v}$ and $\underset{2}{v}$ be the axes of two symmetries, then the rotation which is a product of these two symmetries is given by $\exp k$ such that $gk = \underset{1}{v} \times \underset{2}{v}$. And since $\|\underset{1}{v}\| \neq 0$, this rotation is a plane rotation of type I or II. The converse is evident from the fact that $\underset{1}{v} \times \underset{2}{v} = \underset{1}{v} \times (\underset{1}{v} + \underset{2}{v})$.

REMARK 6. A plane rotation of type III is expressible as a product of four symmetries but not as a product of two symmetries.

For, from Lemma 6 and Remark 5, a plane rotation of type III is a product of four symmetries. A rotation is a product of an even number of symmetries, but from Remark 5 a plane rotation of type III is not a product of two symmetries. Thus our assertion is proved.

Llmma 7. *A rotation* $\exp\begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & -1 & \\ & & & 0 & -1 \\ & & & & 0 \end{pmatrix}$ *with respect to $\mathring{g}$ is a product of two plane rotations of type I with respect to $\mathring{g}$.*

Proof. If we put

$$k_1 = \begin{pmatrix} 0 & 1 & & & \\ & \sigma & & & \\ & & 0 & & \\ & & & -\sigma & -1 \\ & & & & 0 \end{pmatrix} \text{ and } k_2 = \begin{pmatrix} 0 & 0 & & & \\ & -\sigma & 1 & & \\ & & 0 & -1 & \\ & & & \sigma & 0 \\ & & & & 0 \end{pmatrix}, \qquad (\sigma \neq 0),$$

then we have

$$\exp k_1 = \begin{pmatrix} 1 & \frac{e^{\sigma}-1}{\sigma} & & \# & \\ & e^{\sigma} & 0 & & \\ & & 1 & 0 & \\ & & & e^{-\sigma} & \frac{e^{-\sigma}-1}{\sigma} \\ & & & & 1 \end{pmatrix} \text{ and } \exp k_2 = \begin{pmatrix} 1 & 0 & & & \\ & e^{-\sigma} & \frac{e^{-\sigma}-1}{-\sigma} & & \# \\ & & 1 & -\frac{e^{\sigma}-1}{\sigma} & \\ & & & e^{\sigma} & 0 \\ & & & & 1 \end{pmatrix},$$

and therefore

$$\exp k_1 \exp k_2 = \begin{pmatrix} 1 & \frac{1-e^{-\sigma}}{\sigma} & & & \# \\ & 1 & \frac{1-e^{\sigma}}{-\sigma} & & \\ & & 1 & -\frac{e^{\sigma}-1}{\sigma} & \\ & & & 1 & \frac{e^{-\sigma}-1}{\sigma} \\ & & & & 1 \end{pmatrix} = S^{-1} \exp\begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & -1 & \\ & & & 0 & -1 \\ & & & & 0 \end{pmatrix} S,$$

where σ can be taken such as $\frac{1-e^{\sigma}}{\sigma} \neq 0$. Hence we have

$$S \exp k_1 \exp k_2 \, S^{-1} = \exp\begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & -1 & \\ & & & 0 & -1 \\ & & & & 0 \end{pmatrix},$$

and moreover, since $\exp k_1$, $\exp k_2$, $\exp\begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & -1 & \\ & & & 0 & -1 \\ & & & & 0 \end{pmatrix} \in O_{\mathring{g}}$, $S_1(\in O_{\mathring{g}})$ can be taken in place of S; $\exp k_1$ and $\exp k_2$ are the plane rotations of type I, so that we obtain

$$\exp\begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & -1 & \\ & & & 0 & -1 \\ & & & & 0 \end{pmatrix} = \exp \bar{k}_1 \exp \bar{k}_2,$$

where $\exp \bar{k}_1$ and $\exp \bar{k}_2$ are the plane rotations of type I. Thus, this lemma is proved.

LEMMA 8. *A rotation* $\exp\begin{pmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & 0 & & & \\ & & & 0 & -1 & \\ & & & & 0 & -1 \\ & & & & & 0 \end{pmatrix}$ *with respect to* $\mathring{g}$ *is a product of two plane rotations of type I with respect to* $\mathring{g}$.

PROOF. If we put

$$k_1 = \begin{pmatrix} 0 & 1 & & & & \\ & \sigma & & & & \\ & & 0 & & & \\ & & & 0 & & \\ & & & & -\sigma & -1 \\ & & & & & 0 \end{pmatrix} \text{ and } k_2 = \begin{pmatrix} 0 & 0 & & & & \\ & -\sigma & 1 & & & \\ & & 0 & & & \\ & & & 0 & -1 & \\ & & & & \sigma & \\ & & & & & 0 \end{pmatrix}, \quad (\sigma \neq 0),$$

then we can prove this lemma similarly as in the proof of Lemma 7.

THEOREM 4. *A rotation R of type II and of rank $\rho = 2s$ is expressible as a product of κ plane rotations $R_1, R_2, \cdots, R_\kappa$ of type I:*

$$R = R_1 R_2 \cdots R_\kappa, \quad \kappa = \tfrac{1}{2}(\rho + (1 - i\rho)),$$

where $R_i \perp R_j$ and $R_i R_j = R_j R_i$ for $|i-j| \neq 1$, $i, j = 1, 2, \cdots, \kappa$.

PROOF. If we put

$$k_1 = \begin{pmatrix} 0 & 1 & & & & \\ & \sigma & & & & \\ & & 0 & & & \\ & & & \ddots & & \\ & & & & -\sigma & -1 \\ & & & & & 0 \end{pmatrix}, \quad k_2 = \begin{pmatrix} 0 & 0 & & & & & \\ & -\sigma & 1 & & & & \\ & & 0 & & & & \\ & & & \ddots & & & \\ & & & & 0 & -1 & \\ & & & & & \sigma & \\ & & & & & & 0 \end{pmatrix}$$

and
$$k_3 = \begin{pmatrix} 0 &&&&&&&&&&& \\ & 0 &&&&&&&&&& \\ && 0 & 1 &&&&&&&& \\ &&& \ddots & \ddots &&&&&&& \\ &&&& 0 & 1 &&&&&& \\ &&&&& 0 & -1 &&&&& \\ &&&&&& 0 & \ddots &&&& \\ &&&&&&& \ddots & -1 &&& \\ &&&&&&&& 0 && \\ &&&&&&&&& 0 & \\ &&&&&&&&&& 0 \end{pmatrix}, \qquad (\sigma \neq 0),$$

then we have

$$\exp k_1 \cdot \exp k_2 \cdot \exp k_3 = T \begin{pmatrix} 1 & 1 &&&&&& \\ & 1 & \ddots &&&&& \\ && \ddots & \ddots &&&& \\ &&& 1 & 1 &&& \\ &&&& 1 & -1 && \\ &&&&& 1 & \ddots & \\ &&&&&& \ddots & -1 \\ &&&&&&& 1 \end{pmatrix} T^{-1} = S^{-1} \exp \begin{pmatrix} 0 & 1 &&&&& \\ & \ddots & \ddots &&&& \\ && 0 & 1 &&& \\ &&& 0 & -1 && \\ &&&& 0 & \ddots & \\ &&&&& \ddots & -1 \\ &&&&&& 0 \end{pmatrix} S.$$

In a similar manner as before we have

$$\exp \begin{pmatrix} 0 & 1 &&&&& \\ & \ddots & \ddots &&&& \\ && 0 & 1 &&& \\ &&& 0 & -1 && \\ &&&& 0 & \ddots & \\ &&&&& \ddots & -1 \\ &&&&&& 0 \end{pmatrix} = S_1 \exp k_1 \exp k_2 \exp k_3 \cdot S_1^{-1} = \exp \bar{k}_1 \exp \bar{k}_2 \exp \bar{k}_3,$$

where $S_1, \exp k_1, \exp k_2, \exp k_3 \in O_{\mathfrak{g}}$. And then $\exp \bar{k}_1$ and $\exp \bar{k}_2$ are the plane rotations of type I, and $\exp \bar{k}_3$ is a rotation of type II and of rank $2s-4$.

Therefore, in the case where $2s \equiv 0 \pmod 4$, by means of the mathematical induction, from Lemma 7 it follows that R is expressible as a product of s plane rotations of type I; in the case where $2s \equiv 2 \pmod 4$, by the mathematical induction, from Lemma 5 it follows that R is expressible as a product of $s+1$ plane rotations of type I. Thus, in general, a rotation of type II and of rank $\rho = 2s$ is expressible as a product of $\frac{1}{2}(\rho + (1-i^{\rho}))$ plane rotations of type I. The remaining part of this theorem is evident from the construction of k_1, k_2 and k_3.

THEOREM 5. *A rotation R of type III and of rank $\rho = 4t-2$ is expressible as a product of κ plane rotations $R_1, R_2, \cdots, R_\kappa$ of type I:*

$$R = R_1 R_2 \cdots R_\kappa, \quad \kappa = \tfrac{1}{2}\rho + 1 = 2t,$$

where $R_i \perp R_j$ and $R_i R_j = R_j R_i$ for $|i-j| \neq 1$, $i, j = 1, 2, \cdots, \kappa$.

PROOF. If we put

$$k_1 = \begin{pmatrix} 0 & 1 & & & & & \\ & \sigma & & & & & \\ & & 0 & & & & \\ & & & \ddots & & & \\ & & & & 0 & & \\ & & & & & -\sigma & -1 \\ & & & & & & 0 \end{pmatrix}, \quad k_2 = \begin{pmatrix} 0 & & & & & & & \\ & \sigma & 1 & & & & & \\ & & 0 & & & & & \\ & & & 0 & & & & \\ & & & & \ddots & & & \\ & & & & & 0 & & \\ & & & & & & 0 & -1 \\ & & & & & & & -\sigma & \\ & & & & & & & & 0 \end{pmatrix}$$

and

$$k_3 = \begin{pmatrix} 0 & & & & & & & & & & & \\ & 0 & & & & & & & & & & \\ & & 0 & 1 & & & & & & & & \\ & & & 0 & \ddots & & & & & & & \\ & & & & \ddots & 1 & & & & & & \\ & & & & & 0 & & & & & & \\ & & & & & & 0 & -1 & & & & \\ & & & & & & & \ddots & \ddots & & & \\ & & & & & & & & 0 & -1 & & \\ & & & & & & & & & 0 & & \\ & & & & & & & & & & 0 & \\ & & & & & & & & & & & 0 \end{pmatrix}, \qquad (\sigma \neq 0),$$

then we can prove this theorem by means of Lemmas 6 and 8, in the similar manner as in the proof of Theorem 4.

THEOREM 6. *A rotation R of type IV and of rank ρ is expressible as a product of κ plane rotations $R_1, R_2, \cdots, R_\kappa$ of type I:*

$$R = R_1 R_2 \cdots R_\kappa, \quad \kappa = 2\rho,$$

where $R_i \perp R_j$ and $R_i R_j = R_j R_i$ for $|i-j| \neq 1$, $i, j = 1, 2, \cdots, \kappa$.

PROOF. If we put

$$k_1 = \begin{pmatrix} \pi i & 1 & & & & \\ & 0 & 0 & & & \\ & & \ddots & \ddots & & \\ & & & \ddots & 0 & \\ & & & & 0 & -1 \\ & & & & & -\pi i \end{pmatrix}, \quad k_2 = \begin{pmatrix} 0 & & & & & & \\ & \pi i & 1 & & & & \\ & & 0 & 0 & & & \\ & & & \ddots & \ddots & & \\ & & & & \ddots & 0 & \\ & & & & & 0 & -1 \\ & & & & & & -\pi i & \\ & & & & & & & 0 \end{pmatrix}$$

and
$$k_3=\begin{pmatrix}
0 &&&&&&&&&& \\
& 0 &&&&&&&&& \\
&& \pi i & 1 &&&&&&& \\
&&& \ddots & \ddots &&&&&& \\
&&&& \pi i & 1 &&&&& \\
&&&&& \pi i & -1 &&&& \\
&&&&&& \pi i & \ddots &&& \\
&&&&&&& \ddots & -1 && \\
&&&&&&&& \pi i && \\
&&&&&&&&& 0 & \\
&&&&&&&&&& 0
\end{pmatrix},$$

then we have

$\exp k_1 \exp k_2 \exp k_3$

$$=\begin{pmatrix}
-1 & \frac{2}{\pi}i &&&&& \\
& 1 &&&&& \\
&& 1 &&&& \\
&&& \ddots &&& \\
&&&& 1 && \\
&&&&& 1 & \frac{2}{\pi}i \\
&&&&&& -1
\end{pmatrix}
\begin{pmatrix}
1 &&&&&&& \\
& -1 & \frac{2}{\pi}i &&&&& \\
&& 1 &&&&& \\
&&& 1 &&&& \\
&&&& \ddots &&& \\
&&&&& 1 && \\
&&&&&& 1 & \frac{2}{\pi}i \\
&&&&&&& -1 \\
&&&&&&&& 1
\end{pmatrix}
\begin{pmatrix}
1 &&&&&&&&&& \\
& 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots && \\
&& -1 & -1 &&&&& \vdots && \\
&&& \ddots & \ddots && \# && \vdots && \\
&&&& -1 & -1 &&& \vdots && \\
&&&&& -1 & 1 && \vdots && \\
&&&&&& -1 & \ddots & \vdots && \\
&&&&&&& \ddots & 1 && \\
&&&&&&&& -1 && \\
&&&&&&&&& 1 & \\
&&&&&&&&&& 1
\end{pmatrix}$$

$$=\begin{pmatrix}
-1 & -\frac{2}{\pi}i &&&&&&&&&& \\
& -1 & -\frac{2}{\pi}i &&&&&&&&& \\
&& -1 & -1 &&&&&&&& \\
&&& \ddots & \ddots &&&&&&& \\
&&&& -1 & -1 &&&&&& \\
&&&&& -1 & 1 &&&&& \\
&&&&&& -1 & \ddots &&&& \\
&&&&&&& \ddots & 1 &&& \\
&&&&&&&& -1 & \frac{2}{\pi}i && \\
&&&&&&&&& -1 & \frac{2}{\pi}i & \\
&&&&&&&&&& -1
\end{pmatrix}
=S\exp\begin{pmatrix}
\pi i & 1 &&&&& \\
& \ddots & \ddots &&&& \\
&& \pi i & 1 &&& \\
&&& \pi i & -1 && \\
&&&& \pi i & \ddots & \\
&&&&& \ddots & -1 \\
&&&&&& \pi i
\end{pmatrix}S^{-1},$$

and moreover,

$$\exp\begin{pmatrix}\pi i & 1 & 0\\ & 0 & -1\\ & & -\pi i\end{pmatrix}\exp\begin{pmatrix}0 & & \\ & \pi i & \\ & & 0\end{pmatrix}=\begin{pmatrix}-1 & \frac{2}{\pi}i & \sharp\\ & 1 & \frac{2}{\pi}i\\ & & -1\end{pmatrix}\begin{pmatrix}1 & & \\ & -1 & \\ & & 1\end{pmatrix}=\begin{pmatrix}-1 & \frac{-2}{\pi}i & \sharp\\ & -1 & \frac{2}{\pi}i\\ & & -1\end{pmatrix}$$

$$= T\exp\begin{pmatrix}\pi i & 1 & 0\\ & \pi i & -1\\ & & \pi i\end{pmatrix}T^{-1}.$$

Therefore, by the method used above repeatedly, we have

$$\exp K_{\mathrm{IV}}(2p'+1)\equiv\exp\overbrace{\begin{pmatrix}\pi i & 1 & & & & & \\ & \ddots & \ddots & & & & \\ & & \pi i & 1 & & & \\ & & & \pi i & -1 & & \\ & & & & \pi i & \ddots & \\ & & & & & \ddots & -1\\ & & & & & & \pi i\end{pmatrix}}^{2p'+1}=R'_1\,R'_2\cdots R'_{p'}\,N',$$

where $R'_1, R'_2, \cdots, R'_{p'}$ are the plane rotations of type I, and $N'=\exp\overbrace{\begin{pmatrix}0 & & & & & & \\ & \ddots & & & & & \\ & & 0 & & & & \\ & & & \pi i & & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0\end{pmatrix}}^{2p'+1}$.

As seen easily from this result by taking the transposition with respect to $\mathring{g}$ ($+-$ transposition), we have

$$\exp K_{\mathrm{IV}}(2p''+1)\equiv\exp\overbrace{\begin{pmatrix}\pi i & 1 & & & & & \\ & \ddots & \ddots & & & & \\ & & \pi i & 1 & & & \\ & & & \pi i & -1 & & \\ & & & & \pi i & \ddots & \\ & & & & & \ddots & -1\\ & & & & & & \pi i\end{pmatrix}}^{2p''+1}=N''\,R''_{p''}\cdots R''_1,$$

where $R''_1, R''_2, \cdots, R''_{p''}$ are the plane rotations of type I, and $N''=\exp\overbrace{\begin{pmatrix}0 & & & & & & \\ & \ddots & & & & & \\ & & 0 & & & & \\ & & & -\pi i & & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0\end{pmatrix}}^{2p''+1}$.

Since $N=\exp\left(\begin{pmatrix} \overbrace{\begin{pmatrix} 0 & & & & & \\ & \ddots & & & & \\ & & 0 & & & \\ & & & \pi i & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{pmatrix}}^{2p'+1} & \\ & \overbrace{\begin{pmatrix} 0 & & & & & \\ & \ddots & & & & \\ & & 0 & & & \\ & & & -\pi i & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{pmatrix}}^{2p''+1} \end{pmatrix}\right)$ is orthogonal with respect to both

$\overset{\circ}{g}=\begin{pmatrix} \overbrace{\begin{pmatrix} & & 1 \\ & 1 & \\ ⋰ & & \\ 1 & & \end{pmatrix}}^{2p'+1} & \\ & \overbrace{\begin{pmatrix} & & 1 \\ & 1 & \\ ⋰ & & \\ 1 & & \end{pmatrix}}^{2p''+1} \end{pmatrix}$ and $\overset{1}{g}=\begin{pmatrix} \overbrace{\begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}}^{p'} & & \\ & \begin{pmatrix} & & 1 \\ & 1 & \\ ⋰ & & \\ 1 & & \end{pmatrix} & \\ & & \overbrace{\begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}}^{p''} \end{pmatrix}$, by mean of Lemma 2, we

see that by a suitable coordinate transformation, $\overset{\circ}{g}$ and N are transformed to $\overset{1}{g}$ and

$N=\exp\left(\begin{pmatrix} \begin{pmatrix} 0 & & \\ & \ddots & \\ & & 0 \end{pmatrix} & & \\ & \begin{pmatrix} \pi i & & & & \\ & 0 & & & \\ & & \ddots & & \\ & & & 0 & \\ & & & & -\pi i \end{pmatrix} & \\ & & \begin{pmatrix} 0 & & \\ & \ddots & \\ & & 0 \end{pmatrix} \end{pmatrix}\right)$ simultaneously. Therefore, N_1N_2 is a plane

rotation of type I; consequently, a rotation $R\sim\exp(K_{\mathrm{IV}}(2p'+1)\dot{+}K_{\mathrm{IV}}(2p''+1))$ of type IV and of rank $\rho=2(p_1+p_2+1)$ is expressible as a product of κ plane rotations of type I:

$$R=R_1R_2\cdots R_\kappa,\quad \kappa=\tfrac{1}{2}\rho=(p_1+p_2+1).$$

It follows from the construction of these plane rotations R_i that $R_i \perp R_j$ and $R_iR_j=R_jR_i$ for $|i-j|\neq 1$, $i,j=1,2,\cdots,\kappa$.

By summarizing Theorems 2, 4, 5 and 6 we have

THEOREM 7. *A rotation R is expressible as a product of $\kappa(R)$ plane rotations $R_1, R_2, \cdots, R_\kappa$ of type I:*

$$R = R_1 R_2 \cdots R_{\kappa(R)}, \quad \kappa(R) = m_1 + m_2 + m_3 + m_4 + s_{22} + s_3 = \tfrac{1}{2}\rho(R) + s_{22} + s_3,^{1)}$$

where $R_i \perp R_j$ and $R_i R_j = R_j R_i$ for $|i-j| \neq 1$, $i, j = 1, 2, \cdots, \kappa(R)$, $2m_1$, $2m_2$, $2m_3$ and $2m_4$ are the total sums of the ranks of the blocks of type I, II, III and IV respectively, $\rho(R)$ is the rank of R, s_{22} is the number of the blocks of type II and of rank $\equiv 2 (\mathrm{mod}\ 4)$, and s_3 is the number of the blocks of type III.

In particular, as for the blocks of type II and of rank $\equiv 2(\mathrm{mod}\ 4)$, we have seen that

$$\exp K_{\mathrm{II}}(2s'+1) \equiv \exp \overbrace{\begin{pmatrix} 0 & 1 & & & & & & & \\ & 0 & \ddots & & & & & & \\ & & \ddots & \ddots & & & & & \\ & & & 0 & 1 & & & & \\ & & & & 0 & -1 & & & \\ & & & & & 0 & \ddots & & \\ & & & & & & \ddots & \ddots & \\ & & & & & & & 0 & -1 \\ & & & & & & & & 0 \end{pmatrix}}^{2s'+1} = R'_1 \cdots R'_{s'-1} L'$$

and

$$\exp K_{\mathrm{II}}(2s''+1) \equiv \exp \overbrace{\begin{pmatrix} 0 & 1 & & & & & & & \\ & 0 & \ddots & & & & & & \\ & & \ddots & \ddots & & & & & \\ & & & 0 & 1 & & & & \\ & & & & 0 & -1 & & & \\ & & & & & 0 & \ddots & & \\ & & & & & & \ddots & \ddots & \\ & & & & & & & 0 & -1 \\ & & & & & & & & 0 \end{pmatrix}}^{2s''+1} = L'' R''_{s''-1} \cdots R''_1,$$

where $R'_1, R'_2, \cdots, R'_{s'-1}$ and $R''_1, R''_2, \cdots, R''_{s''-1}$ are the plane rotations of type I, and

$$L' = \exp \begin{pmatrix} 0 & 1 & \\ & 0 & -1 \\ & & 0 \end{pmatrix}, \quad L'' = \exp \begin{pmatrix} 0 & 1 & \\ & 0 & -1 \\ & & 0 \end{pmatrix}.$$

Since $L = \exp \begin{pmatrix} \begin{pmatrix} 0 & 1 & \\ & 0 & -1 \\ & & 0 \end{pmatrix} & \\ & \begin{pmatrix} 0 & 1 & \\ & 0 & -1 \\ & & 0 \end{pmatrix} \end{pmatrix}$ is orthogonal with respect to both

1) As easily seen, $\rho(R) = 2(m_1 + m_2 + m_3 + m_4)$.

$\mathring{g}=\begin{pmatrix} \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix} & \\ & \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix} \end{pmatrix}$ and $\overset{1}{g}=\begin{pmatrix} & & & & & 1 \\ & & & & 1 & \\ & & & 1 & & \\ & & 1 & & & \\ & 1 & & & & \\ 1 & & & & & \end{pmatrix}$, by means of Lemma 2, we see that by a coordinate transformation, L and $\mathring{g}$ are transformed to L and $\overset{1}{g}$ simultaneously, so in the latter coordinatate system, by lemma 8, this rotation L is expressible as a product of two plane rotations. By this procedure, the rotation corresponding to the total blocks of type II is written as a product of $m_2+\frac{1}{2}(1-(-1)^{m_2})$ plane rotations of type I, $2m_2$ being the rank of total blocks of type II. After this process it is not asserted that $R_i \perp R_j$ and $R_iR_j=R_jR_i$ *for* $|i-j|\neq 1$, $i, j=1, 2, \cdots, \kappa$.

Thus, in the notations of Theorem 7 we have

THEOREM 8. *A rotation is expressible as a product of $\omega(R)$ plane rotations, where*

$$\omega(R)=m_1+m_2+m_3+m_4+\tfrac{1}{2}(1-(-1)^{m_2})+s_3=\tfrac{1}{2}\rho(R)+\tfrac{1}{2}(1-(-1)^{m_2})+s_3.$$

Next we shall consider the dimension of a rotation R. The dimension $\dim(R)$ of R is by definition: $\dim(R)=n-q$ (see § 3, p. 312), so that we have

$$\dim(R)=2(m_1+m_2+m_3+m_4)+s_2+2s_3=\rho(R)+s_2+2s_3,$$

in the notations used above, and s_2 being the number of the blocks of type II. By taking account of $m_2 \geqq s_2$ and $m_3 \geqq s_3$, form the above expression of $\dim(R)$ we have

$$\rho(R) \leqq \dim(R) \leqq 2\rho(R)$$

and moreover it is obvious that $\dim(R) \leqq n$, so that we have

$$\rho(R) \leqq \dim(R) \leqq \min(2\rho(R), n).$$

(As for $\kappa(R)$ and $\omega(R)$ also we have $\frac{1}{2}\rho(R) \leqq \omega(R) \leqq \kappa(R) \leqq \rho(R)$).

Next, for any integer λ such that $2m \leqq \lambda \leqq \min(4m, n)$, we shall take the integers l_1, η and l_2 such that

$$4l_1+3\eta+2l_2=\lambda \quad \text{and} \quad 2l_1+2\eta+2l_2=2m,$$

for example, $l_1=\frac{1}{2}(\lambda-\eta-2m)$, $l_2=m-(l_1+\eta)$ and

$$\eta=\begin{cases} 1 & \text{for the case } \lambda \text{ is odd} \\ 0 & \text{for the case } \lambda \text{ is even} \end{cases}, \qquad (l_2 \leqq 0 \text{ since } \lambda \leqq 4m).$$

Here, if we take

$$\mathring{K} = \sum^{l_1} \dotplus \begin{pmatrix} 0 & 1 & & \\ & 0 & & \\ & & 0 & -1 \\ & & 0 & \end{pmatrix} \dotplus \sum^{\eta} \dotplus \begin{pmatrix} 0 & 1 & \\ & 0 & -1 \\ & 0 & \end{pmatrix} \dotplus \sum_{\substack{i=1 \\ (\sigma_i \neq 0)}}^{l_2} \dotplus \begin{pmatrix} \sigma_i & 0 \\ 0 & -\sigma_i \end{pmatrix} \dotplus O_q,$$

and corresponding to this $\mathring{K}$,

$$\mathring{g} = \sum^{l_1} \dotplus \mathring{g}(4) \dotplus \sum^{\eta} \dotplus \mathring{g}(3) \dotplus \sum^{l_2} \dotplus \mathring{g}(2) \dotplus \mathring{g}(q),$$

then the rotation $\exp \mathring{K}$ with respect to $\mathring{g}$ is of rank $2m$ and of dimension λ. Thus we have

THEOREM 9. *The range of the dimensions of the rotations with respect to any various metrics, the ranks of these rotations being equal to* $2m$. *is the closed interval* $[2m, \min(4m, n)]$ *of integers.*

REMARK 8. It is well known [1)] that a rotation R is expressible as a product of μ symmetries such that $\mu \leqq \dim(R)$. From this fact, Remark 5 and Lemma 5 it follows immediately that the rotation is expressible as a product of ν plane rotations of type I such that $\nu \leqq \dim(R)$. Our results give explicitly the number of symmetries: $\mu = 2\omega(R) = \rho(R) + 2s_3 + (1-(-1)^{m_2})$, the number of plane rotations of type I: $\nu = \omega(R) = \frac{1}{2}\rho(R) + s_3 + \frac{1}{2}(1-(-1)^{m_2})$ and moreover the orthogonality relation of these plane rotations. As well known, a real rotation of rank ρ is expressible as a product of $\frac{1}{2}\rho$ real plane rotations:

$$R = R_1 R_2 \cdots R_{\frac{1}{2}\rho}$$

where $R_i \perp R_j$ and $R_i R_j = R_j R_i$ for all $i \neq j$, $i, j = 1, 2, \cdots, \frac{1}{2}\rho$. Our result (Theorem 7) is considered as an extension of this fact for the complex rotation. From our consideration the number μ of the symmetries are calculated more exactly; that is, if we remember that a plane rotation of type I or type II is a product of two symmetries (Remark 5), and that a plane rotation of type III is expressible as a product of four symmetries but not as a product of two symmeties (Remark 6), then we obtain the number μ of symmetries as follows:

Type I (rank ρ): $\mu = 2\kappa = \rho$ (from Theorem 2).

Type II (rank ρ): By tracing the proof of Theorem 3, we see that in the case where $\rho \equiv 0 \pmod 4$ a rotation R is expressible as a product of $\frac{1}{2}\rho$ plane rotations of type I, consepuently, as a product of $\rho (= 2 \times \frac{1}{2}\rho)$ symmetries, and that in the case where $\rho \equiv 2 \pmod 4$ R is expressible as a product of $\frac{1}{2}(\rho - 2)$ plane rotations

1) E. Cartan, Leçons sur la théorie des spineurs I, (1938), pp. 13–17.
J. Dieudonné, Sur les groupes classiques, (1948), pp. 20–17.

of type I and one plane rotation of type II, and hence R is expressible as a product of $\rho(=2\cdot\frac{1}{2}(\rho-2)+2)$ symmetries. That is, $\mu=\rho$.

Type III (rank ρ): $\mu=2\kappa=\rho+2$ (from Theorem 4).

Type IV (rank ρ): $\mu=2\kappa=\rho$ (from Theorem 5).

Thus we have

THEOREM 10. *A rotation of rank ρ is expressible as a product of $\rho+2s_3$ symmetries where s_3 is the number of the blocks of type III.*

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