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On Splittable Linear Lie Algebras

By

Shigeaki TÔGÔ

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Introduction

Let $\mathfrak{gl}(V)$ be the Lie algebra of all linear endomorphisms of a finite-dimensional vector space V over a field K of characteristic 0. An element X of $\mathfrak{gl}(V)$ is uniqely expressed as $X=S+N$ in such a way that S is a semi-simple matrix, N is a nilpotent matrix and $[S, N]=0$. These S, N are called the semi-simple and nilpotent components of X respectively [5]. A Lie subalgebra $\mathfrak{g}$ of $\mathfrak{gl}(V)$ is called *splittable* [14] provided the components of every element of $\mathfrak{g}$ also belong to $\mathfrak{g}$. E. g., completely reducible linear Lie algebras are splittable [13]. The main purpose of this paper is to study the properties of splittable linear Lie algebras.

Necessary and sufficient conditions for $\mathfrak{g}$ to be splittable are given (Proposition 1). $\mathfrak{g}$ is called algebraic [8, p. 171] if it is the Lie algebra of an algebraic group. $Y \in \mathfrak{gl}(V)$ is called a replica of $X \in \mathfrak{gl}(V)$ [8, p. 180] if Y is contained in the Lie algebra of the smallest algebraic group whose Lie algebra contains X. Then $\mathfrak{g}$ is algebraic if and only if every replica of any element of $\mathfrak{g}$ also belongs to $\mathfrak{g}$ [8, p. 181]. Since the components of any element $X \in \mathfrak{gl}(V)$ are replicas of X. [8, p. 181], every algebraic Lie algebra is splittable. It is known [7] that if $\mathfrak{g}$ is algebraic, $\mathfrak{h}$ is its radical and $\mathfrak{n}$ is the ideal of all nilpotent matrices $\in \mathfrak{h}$, then for any Levi decomposition $\mathfrak{g}=\mathfrak{s}+\mathfrak{h}$ of $\mathfrak{g}$ there exists an abelian subalgebra $\mathfrak{a}$ of semi-simple matrices $\in \mathfrak{h}$ such that $\mathfrak{h}=\mathfrak{n}+\mathfrak{a}$, $\mathfrak{n} \cap \mathfrak{a}=0$, $[\mathfrak{s}, \mathfrak{a}]=0$. An analogue to this for splittable Lie algebras will be proved (Theorem 1). The smallest algebraic Lie algebra containing $\mathfrak{g}$ is called the algebraic hull $\mathfrak{g}^*$ of $\mathfrak{g}$ [7]. In like manner we define the splittable hull $^*\mathfrak{g}$ of $\mathfrak{g}$. It is the smallest splittable Lie algebra containing $\mathfrak{g}$. We shall show that $^*\mathfrak{g}$ is the smallest Lie algebra containing $\mathfrak{g}$ which has the same nilpotent matrices as $\mathfrak{g}^*$ (Theorem 2), and that if $\mathfrak{k}$ is a Cartan subalgebra of $\mathfrak{g}$ then $^*\mathfrak{k}$ ($\mathfrak{k}^*$) is a Cartan subalgebra of $^*\mathfrak{g}$ ($\mathfrak{g}^*$) (Proposition 5). Owing to a Cartan decomposition of $\mathfrak{g}$, M. Gotô [10] showed that $\mathfrak{g}^*$ is the direct sum of $\mathfrak{g}$ and an abelian subalgebra composed of semi-simple matrices. Without making use of a Cartan decomposition a simple proof of this result and its analogue for splittable case will be established

(Proposition 3 and its corollary). Let ${}_*\mathfrak{g}$ and $\mathfrak{g}_*$, respectively, be the largest splittable and the largest algebraic Lie algebras contained in $\mathfrak{g}$. We shall show that ${}_*\mathfrak{g}$ ($\mathfrak{g}_*$) is composed of all elements of $\mathfrak{g}$ whose components (all replicas) are contained in $\mathfrak{g}$, and the radical of ${}_*\mathfrak{g}$ ($\mathfrak{g}_*$) is ${}_*\mathfrak{h}$ ($\mathfrak{h}_*$), where $\mathfrak{h}$ is the radical of $\mathfrak{g}$ (Propositions 7, 8). And we also show that a not necessarily linear Lie algebra over K, whose adjoint representation is splittable, has a faithful splittable representation (Proposition 10). We study certain properties of splittable representations of not necessarily linear Lie algebras whose adjoint representations are splittable.

Let $GL(V)$ be the group of all automorphisms of V. We shall call a subgroup $\mathfrak{G}$ of $GL(V)$ to be *splittable* provided $\mathfrak{G}$ contains the semi-simple component of every element of $\mathfrak{G}$. Then every algebraic group is splittable. We shall show that if K is the real or complex field, any splittable Lie algebra is the Lie algebra of a splittable group and vice versa (Theorem 3). By making use of a result of C. Chevalley [8, p. 157] concerning the enveloping algebras of algebraic groups, we shall show that any subgroup $\mathfrak{G}$ of $GL(V)$ and the smallest algebraic group containing $\mathfrak{G}$ have the same enveloping algebra (Theorem 5). If, in particular, K is the real or complex field and if $\mathfrak{G}$ is connected, then its enveloping algebra is the associative algebra generated by the unit matrix I and elements of the Lie algebra of $\mathfrak{G}$.

Finally we shall give examples to show the interrelations between various types of Lie algebras considered in this paper. The relations are given by the following diagram:

$$\text{(D)}\qquad \begin{array}{ccc} \mathfrak{g} \text{ is algebraic} & \rightarrow & \mathfrak{g} \text{ has the algebraic adjoint representation} \\ \downarrow & & \downarrow \\ \mathfrak{g} \text{ is splittable} & \rightarrow & \mathfrak{g} \text{ has the splittable adjoint representation.} \end{array}$$

The inverse implications do not hold generally.

§ 1. Splittable Lie algebras

Let K be a field of characteristic 0. Let V be a finite-dimensional vector space over K and $\mathfrak{gl}(V)$ be the Lie algebra of all linear endomorphisms of V with the commutator product $[X, Y]=XY-YX$. The elements of $\mathfrak{gl}(V)$ are considered as the square matrices of finite degree with coefficients in K. An element X of $\mathfrak{gl}(V)$ is uniquely represented in the form $X=S+N$, where S is a semi-simple matrix, N is a nilpotent matrix and $[S, N]=0$. Furthermore S and N are polynomials in X. These S, N are called the semi-simple and nilpotent components of X, which we shall denote by S_X and N_X respectively.

DEFINITION 1. *A Lie subalgebra* $\mathfrak{g}$ *of* $\mathfrak{gl}(V)$ *is called splittable provided the components of every element of* $\mathfrak{g}$ *also belong to* $\mathfrak{g}$.

DEFINITION 2. *A basis* $\{X_1, \cdots\cdots, X_n\}$ *of a Lie subalgebra* $\mathfrak{g}$ *is called splittable provided the components of each* X_i *belong to* $\mathfrak{g}$.

A Lie subalgebra $\mathfrak{g}$ is called *algebraic* if it is the Lie algebra of an algebraic group. For any $X \in \mathfrak{gl}(V)$ let $\mathfrak{g}(X)$ be the Lie algebra of the smallest algebraic group whose Lie algebra contains X. Every element of $\mathfrak{g}(X)$ is called a replica of X. Then $\mathfrak{g}$ is algebraic if and only if every replica of any element of $\mathfrak{g}$ belongs to $\mathfrak{g}$. Since the components of X are replicas of X, every algebraic Lie algebra is splittable. It is known [8, p. 159] that $\mathfrak{g}(X) = KX$ if X is a nilpotent matrix. The Lie algebra generated by any family of algebraic Lie algebras is also algebraic [8, p. 175]. A basis of $\mathfrak{g}$ is called algebraic [15] provided all replicas of each element of the basis are contained in $\mathfrak{g}$. Then any Lie algebra with an algebraic basis is algebraic. The derived algebra $\mathfrak{g}'$ of $\mathfrak{g}$ is algebraic [8, p. 177] and therefore any semi-simple linear Lie algebra is also algebraic. The smallest algebraic Lie algebra containing $\mathfrak{g}$ is called the algebraic hull of $\mathfrak{g}$ and will be denoted by $\mathfrak{g}^*$. The derived algebra of $\mathfrak{g}^*$ is also the derived algebra of $\mathfrak{g}$ [8, p. 173].

LEMMA 1. *If* $[X, Y] = 0$ *for* $X, Y \in \mathfrak{gl}(V)$, *then* $S_{X+Y} = S_X + S_Y$, $N_{X+Y} = N_X + N_Y$.

PROOF. Since the components of any matrix Z of $\mathfrak{gl}(V)$ are polynomials in Z [8, p. 181], it follows from $[X, Y] = 0$ that the components of X and Y commute each other. Hence $S_X + S_Y$ is semi-simple, $N_X + N_Y$ is niloptent and $[S_X + S_Y, N_X + N_Y] = 0$. Thus $S_X + S_Y$, $N_X + N_Y$ are the semi-simple and nilpotent components of $X + Y$.

Let V^* denote the dual space of V. Let V_q^p denote the space of p times contravariant and q times covariant tensors on V i.e.

$$V_q^p = \underbrace{V \otimes \cdots\cdots \otimes V}_{p} \otimes \underbrace{V^* \otimes \cdots\cdots \otimes V^*}_{q},$$

where $\otimes$ denotes Kronecker product. For a matrix $X \in \mathfrak{gl}(V)$ we denote by ${}^tX$ the transpose of X and by X^* the matrix $-{}^tX$. If we put

$$\begin{aligned} X_q^p = {} & X \otimes I \otimes \cdots \otimes I \otimes I \otimes \cdots\cdots \otimes I + \cdots\cdots + I \otimes \cdots\cdots \otimes X \otimes I \otimes \cdots\cdots \otimes I \\ & + I \otimes \cdots\cdots \otimes I \otimes X^* \otimes \cdots\cdots \otimes I + \cdots\cdots + I \otimes \cdots\cdots \otimes I \otimes I \otimes \cdots\cdots \otimes X^*, \end{aligned}$$

where I is the unit matrix in $\mathfrak{gl}(V)$, then $X \rightarrow X_q^p$ is a representation of $\mathfrak{gl}(V)$ in $\mathfrak{gl}(V_q^p)$. $\mathrm{e} \in V_q^p$ is said to be a tensor invariant of X if we have $X_q^p e = 0$.

LEMMA 2. *Let* $X \in \mathfrak{gl}(V)$. *Let* $\sum V_q^p$ *be the direct sum of some tensor spaces on* V *and let* W *be a subspace of* $\sum V_q^p$ *which is stable under* X. *If* $\bar{X}$ *denotes the matrix induced* *by X on W, then* $\bar{S}_X=S_{\bar{X}}$, $\bar{N}_X=N_{\bar{X}}$.

Proof. It is easily seen that $(S_X)^p_q$, $(N_X)^p_q$ are the semi-simple and nilpotent components of X^p_q [5]. Lemma 2 follows immediately from this fact.

Lemma 3. *Let $\mathfrak{g}$ be a Lie subalgebra of $\mathfrak{gl}(V)$ whose elements are nilpotent matrices. Then there exists a space W of tensor invariants of $\mathfrak{g}$ such that any matrix of $\mathfrak{gl}(V)$ which induces 0 matrix on W is contained in $\mathfrak{g}$.*

We omit the proof (see [1] and [9]).

Making use of these lemmas we show

Proposition 1. *The following statements on a Lie subalgebra $\mathfrak{g}$ of $\mathfrak{gl}(V)$ are equivalent:*

(1) *$\mathfrak{g}$ is splittable.*

(2) *The radical of $\mathfrak{g}$ is splittable.*

(3) *$\mathfrak{g}$ has a splittable basis.*

Proof. (1)$\rightarrow$(2). The radical $\mathfrak{h}_1$ of the derived algebra $\mathfrak{g}'$ consists of nilpotent matrices [2]. Let W be a sufficiently large space of tensor invariants of $\mathfrak{h}_1$. Then W is stable under $\mathfrak{g}$, since $\mathfrak{h}_1$ is an ideal of $\mathfrak{g}$. Let $\bar{X}$ denote the matrix induced by $X\in\mathfrak{g}$ on W. Given any subset $\mathfrak{m}$ of $\mathfrak{g}$ let $\bar{\mathfrak{m}}$ denote the set of $\bar{X}$ for all $X\in\mathfrak{m}$. If $\mathfrak{h}$ denotes the radical of $\mathfrak{g}$, then $\bar{\mathfrak{h}}$ is the center of $\bar{\mathfrak{g}}$. Suppose now that $\mathfrak{g}$ is splittable. Then by Lemma 2 we see that $\bar{\mathfrak{g}}$ is splittable, and therefore that the center $\bar{\mathfrak{h}}$ is splittable. Hence for any $X\in\mathfrak{h}$ we have $\bar{S}_X=S_{\bar{X}}\in\bar{\mathfrak{h}}$ and therefore there exists an element $X_1\in\mathfrak{h}$ such that $\bar{S}_X=\bar{X}_1$. It follows from Lemma 3 that $S_X-X_1\in\mathfrak{h}_1$, whence $S_X\in\mathfrak{h}$, which shows that $\mathfrak{h}$ is splittable.

(2) $\rightarrow$ (3) is evident, since semi-simple linear Lie algebras are splittable.

(3) $\rightarrow$ (1). Suppose that W and $\bar{X}$ have the same meanings as above. Let $\{X_1,\cdots\cdots\cdots,X_n\}$ be a splittable basis of $\mathfrak{g}$. Let $\mathfrak{g}=\mathfrak{s}+\mathfrak{h}$ be a Levi decomposition of $\mathfrak{g}$, where $\mathfrak{s}$ is a semi-simple subalgebra. Let $X_i=Y_i+Z_i$, $Y_i\in\mathfrak{s}$, $Z_i\in\mathfrak{h}$. Then it is clear that $[\bar{Y}_i,\bar{Z}_i]=0$, whence $S_{\bar{X}_i}=S_{\bar{Y}_i}+S_{\bar{Z}_i}$. Since $S_{X_i}\in\mathfrak{g}$ and $S_{Y_i}\in\mathfrak{s}$, it follows from Lemma 2 that $S_{\bar{Z}_i}\in\bar{\mathfrak{g}}$. Let X be any element of $\mathfrak{g}$. Then $X=Y+\sum_{i=1}^n k_iZ_i$, where $Y\in\mathfrak{s}$ and $k_i\in K$. Hence $S_{\bar{X}}=S_{\bar{Y}}+\sum_{i=1}^n k_iS_{\bar{Z}_i}\in\bar{\mathfrak{g}}$. Therefore $\bar{\mathfrak{g}}$ is splittable. Using Lemma 3 we see easily that $\mathfrak{g}$ is splittable. Thus the proof is completed.

Remark. In the proof of Proposition 1 we used the splittability of semi-simple linear Lie algebras deduced from the algebraicity of the derived algebra of any linear Lie algebra. But an independent proof of this fact was given by N. Jacobson [13]. Using Lemma 3 we can deduce the splittability of the derived algebra of any linear Lie algebra from that of semi-simple linear Lie algebras, since its radical consists of nilpotent matrices.

It is known [13] that $\mathfrak{g}$ is a completely reducible Lie algebra if and only if $\mathfrak{g}=\mathfrak{g}'+\mathfrak{z}$ where $\mathfrak{g}'$ is semi-simple and $\mathfrak{z}$ is the center and consists of semi-simple matrices. Therefore we have

Corollary 1. *A Lie subalgebra $\mathfrak{g}$ of $\mathfrak{gl}(V)$ is completely reducible if and only if $\mathfrak{g}$ is splittable and its radical contains no nilpotent matrices.*

Let L be an extension field of K. We denote by $\mathfrak{g}^L$ the Lie algebra obtained from $\mathfrak{g}$ by extending the basic field K to L. Then we have

Corollary 2. *$\mathfrak{g}$ is splittable if and only if $\mathfrak{g}^L$ is splittable.*

Proof. Suppose that $\mathfrak{g}^L$ is splittable. Then the components of any $X\in\mathfrak{g}$ are contained in $\mathfrak{g}^L$ and also in $\mathfrak{gl}(V)$. Hence they are contained in $\mathfrak{g}$, which shows that $\mathfrak{g}$ is splittable. The converse follows from Proposition 1.

The basic field K is characteristic 0 so that its prime subfield K_0 can be identified with the rational field. Now we have

Proposition 2. *Let $\mathfrak{g}$ be an abelian subalgebra of $\mathfrak{gl}(V)$ composed of semi-simple matrices. Then $\mathfrak{g}$ is algebraic if and only if there exists a Lie subalgebra $\mathfrak{g}_0$ over K_0 such that all eigenvalues of any element of $\mathfrak{g}_0$ are rational numbers and that $\mathfrak{g}_0^K=\mathfrak{g}$. In particular, every replica of any semi-simple matrix $X\in\mathfrak{gl}(V)$ whose eigenvalues are all rational numbers is a scalar multiple of X.*

Proof. Let $\bar{K}$ denote the algebraic closure of K. If $\mathfrak{g}$ is algebraic, then $\mathfrak{g}^{\bar{K}}$ is algebraic [8, p. 181] and may be considered as consisting of diagonal matrices. Hence there exists a Lie algebra $\mathfrak{g}_0$ composed of diagonal matrices with coefficients in K_0 such that $\mathfrak{g}^{\bar{K}}=\mathfrak{g}_0{}^{\bar{K}}$ [8, p. 169], whence $\mathfrak{g}=\mathfrak{g}_0^K$. Conversely assume the existence of such a Lie algebra $\mathfrak{g}_0$. Then $\mathfrak{g}_0{}^{\bar{K}}$ is algebraic [8, p. 169]. It follows [8, p. 181] that $\mathfrak{g}$ is also algebraic. This completes the proof.

Corollary. *A Lie subalgebra $\mathfrak{g}$ of $\mathfrak{gl}(V)$ is algebraic if and only if $\mathfrak{g}$ has a splittable basis such that all eigenvalues of each element of the basis are rational numbers.*

Proof. Suppose that $\{X_1,\cdots\cdots,X_n\}$ is such a basis of $\mathfrak{g}$. It follows from Proposition 2 that $\mathfrak{g}(S_{X_i})=KS_{X_i}$. Since $\mathfrak{g}(X_i)=\mathfrak{g}(S_{X_i})+\mathfrak{g}(N_{X_i})$ [8, p. 165], we have $\mathfrak{g}(X_i)\subseteqq\mathfrak{g}$, which shows that $\{X_1,\cdots\cdots,X_n\}$ is an algebraic basis and therefore $\mathfrak{g}$ is algebraic. Conversely let $\mathfrak{g}$ be algebraic. Then we can take a basis $\{X_1,\cdots\cdots,X_n\}$ composed of nilpotent or semi-simple matrices. Proposition 2 shows that for each semi-simple matrix X_i there exists a Lie algebra $\mathfrak{g}_i$ over K_0 such that all eigenvalues of any element of $\mathfrak{g}_i$ are rational numbers and $\mathfrak{g}_i{}^K=\mathfrak{g}(X_i)$. By replacing all semi-simple matrices X_i by suitable elements of such $\mathfrak{g}_i$'s, we obtain a splittable basis of $\mathfrak{g}$ such that all eigenvalues of any element of the basis are rational numbers, as desired.

§ 2. Structure of splittable Lie algebras

Let $\mathfrak{g}$ be a not necessarily linear Lie algebra over K. For any $x \in \mathfrak{g}$ we denote by $\mathrm{ad}_{\mathfrak{g}} x$ the adjoint representation $y \rightarrow [x, y]$. If for a subalgebra $\mathfrak{g}_1$ of $\mathfrak{g}$ there exists a subalgebra $\mathfrak{g}_2$ such that $\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_2$, $\mathfrak{g}_1 \cap \mathfrak{g}_2 = 0$, we say that $\mathfrak{g}$ splits over $\mathfrak{g}_1$ and that $\mathfrak{g}_2$ is a complement of $\mathfrak{g}_1$ in $\mathfrak{g}$.

Now we show the following

Theorem 1. *A Lie subalgebra $\mathfrak{g}$ of $\mathfrak{gl}(V)$ is splittable if and only if its radical $\mathfrak{h}$ splits over the ideal $\mathfrak{n}$ of all nilpotent matrices of $\mathfrak{h}$ and any maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{h}$ composed of semi-simple matrices is a complement of $\mathfrak{n}$. Then for any given maximal semi-simple subalgebra $\mathfrak{S}$ of $\mathfrak{g}$ there exists a maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{h}$ composed of semi-simple matrices such that $[\mathfrak{S}, \mathfrak{a}] = 0$ and conversely. Furthermore if one of $\mathfrak{a}$'s is splittable, then so are $\mathfrak{g}$ and any other $\mathfrak{a}$.*

Proof. Suppose that $\mathfrak{g}$ is splittable. It is clear that if $\mathfrak{h} \neq \mathfrak{n}$ then there exists an abelian subalgebra of $\mathfrak{h}$ composed of semi-simple matrices. Let $\mathfrak{a}$ be such a maximal subalgebra. Then it is clear that $\mathfrak{n} \cap \mathfrak{a} = 0$. Assume that $\mathfrak{h} \neq \mathfrak{n} + \mathfrak{a}$. Since $\mathfrak{h}$, $\mathfrak{n} + \mathfrak{a}$ are stable under $\mathrm{ad}_{\mathfrak{g}} \mathfrak{a}$ which is completely reducible, there exists a subspace $\mathfrak{m}$ such that $\mathfrak{h} = (\mathfrak{n} + \mathfrak{a}) + \mathfrak{m}$, $(\mathfrak{n} + \mathfrak{a}) \cap \mathfrak{m} = 0$, $(\mathrm{ad}_{\mathfrak{g}} \mathfrak{a}) \mathfrak{m} \subseteqq \mathfrak{m}$. Since $[\mathfrak{a}, \mathfrak{m}] \subseteqq \mathfrak{n}$, it follows that $[\mathfrak{a}, \mathfrak{m}] = 0$. Since $\mathfrak{h} \neq \mathfrak{n} + \mathfrak{a}$, we can find an element $X \neq 0$ in $\mathfrak{m}$. By Proposition 1 $\mathfrak{h}$ is splittable, whence $S_X \in \mathfrak{h}$. But the nilpotent matrix N_X belongs to $\mathfrak{n}$, which shows that $S_X \notin \mathfrak{n} + \mathfrak{a}$. It follows from $[\mathfrak{a}, X] = 0$ that $[\mathfrak{a}, S_X] = 0$. Hence the linear space spanned by $\mathfrak{a}$ and S_X is an abelian subalgebra of $\mathfrak{h}$ composed of semi-simple matrices and whose dimension exceeds the dimension of $\mathfrak{a}$ by 1. This contradicts the maximality of $\mathfrak{a}$. Thus we have $\mathfrak{h} = \mathfrak{n} + \mathfrak{a}$. The converse is evident from Proposition 1.

Let $\mathfrak{S}$ be any maximal semi-simple subalgebra of $\mathfrak{g}$. Then, since $\mathrm{ad}_{\mathfrak{g}} \mathfrak{S}$ is completely reducible, it is clear that there exists an abelian subalgebra of $\mathfrak{h}$ whose elements are semi-simple matrices and commute with the elements of $\mathfrak{S}$. If $\mathfrak{a}$ is such a maximal subalgebra, then $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{S} + \mathfrak{a})$ is completely reducible and therefore we can show in the same manner as above that such $\mathfrak{a}$ is also a complement of $\mathfrak{n}$ in $\mathfrak{h}$. Conversely let $\mathfrak{a}$ be any maximal abelian subalgebra of $\mathfrak{h}$ composed of semi-simple matrices. For a matrix $X_1 \in \mathfrak{a}$ let $f(\xi)$ be the characteristic polynomial of $\mathrm{ad}_{\mathfrak{g}} X_1$. If $f(\xi) = \xi^m f_0(\xi)$, $f_0(0) \neq 0$, then there exist polynomials $\phi_0(\xi)$, $\phi_1(\xi)$ such that $f_0(\xi)\phi_0(\xi) + \xi^m \phi_1(\xi) = 1$. If we put $\mathfrak{g}_1 = f_0(\mathrm{ad}_{\mathfrak{g}} X_1)\phi_0(\mathrm{ad}_{\mathfrak{g}} X_1)\mathfrak{g}$, $\mathfrak{m}_1 = (\mathrm{ad}_{\mathfrak{g}} X_1)^m \phi_1(\mathrm{ad}_{\mathfrak{g}} X_1)\mathfrak{g}$, then we have $\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{m}_1$, $\mathfrak{g}_1 \cap \mathfrak{m}_1 = 0$, $\mathfrak{m}_1 \subseteqq \mathfrak{n}$. Since $\mathrm{ad}_{\mathfrak{g}} X_1$ is semi-simple, it is easily seen that the restriction of $\mathrm{ad}_{\mathfrak{g}} X_1$ to $\mathfrak{m}_1$ is an automorphism. It follows that $\mathfrak{g}_1$ is the set of all elements annihilated by $\mathrm{ad}_{\mathfrak{g}} X_1$. Next let $X_2 (\neq X_1)$ be an element of $\mathfrak{a}$ such that $(\mathrm{ad}_{\mathfrak{g}}X_2)\mathfrak{g}_1 \neq 0$. Since $\mathfrak{g}_1$ is also stable under $\mathrm{ad}_{\mathfrak{g}}X_2$, we have $\mathfrak{g}_1 = \mathfrak{g}_2 + \mathfrak{m}_2$, $\mathfrak{g}_2 \cap \mathfrak{m}_2 = 0$, $\mathfrak{m}_2 \subseteqq \mathfrak{n}$, where $\mathfrak{g}_2$ is the set of all elements of $\mathfrak{g}_1$ annihilated by $\mathrm{ad}_{\mathfrak{g}}X_2$. Since $\mathfrak{g}$ is of finite dimension, by repeated application of this construction we finally obtain $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{m}$, $\mathfrak{g}_0 \cap \mathfrak{m} = 0$, where $\mathfrak{g}_0$ is the subalgebra formed by all elements annihilated by $\mathrm{ad}_{\mathfrak{g}}\mathfrak{a}$ and $\mathfrak{m}$ is a subspace contained in $\mathfrak{n}$. If $\mathfrak{h}_0$ denotes the radical of $\mathfrak{g}_0$, then it is easy to see that $\mathfrak{h} = \mathfrak{h}_0 + \mathfrak{n}$. It follows from this fact that any maximal semi-simple subalgebra $\mathfrak{s}$ of $\mathfrak{g}_0$ is a maximal semi-simple subalgebra of $\mathfrak{g}$, which is such that $[\mathfrak{s}, \mathfrak{a}] = 0$.

Suppose now that $\mathfrak{a}$ is algebraic. Since $\mathfrak{h} = \mathfrak{n} + \mathfrak{a}$ and $\mathfrak{g} = \mathfrak{s} + \mathfrak{n} + \mathfrak{a}$, it is evident that $\mathfrak{h}$ and $\mathfrak{g}$ are algebraic. Let $\mathfrak{a}_1$ be any other maximal abelian subalgebra of $\mathfrak{h}$ composed of semi-simple matrices. Then we have $\mathfrak{h} = \mathfrak{n} + \mathfrak{a}_1^*$, $\mathfrak{n} \cap \mathfrak{a}_1^* = 0$, whence dim $\mathfrak{a}_1$ = dim $\mathfrak{a}_1^*$. It follows that $\mathfrak{a}_1 = \mathfrak{a}_1^*$ i.e., $\mathfrak{a}_1$ is algebraic. Thus the theorem is completely proved.

Remark. We know [12] that any maximal semi-simple subalgebra of $\mathfrak{g}$ is a complement of $\mathfrak{h}$ in $\mathfrak{g}$. Hence it follows from Theorem 1 that if $\mathfrak{g}$ is splittable,

$$\mathfrak{g} = \mathfrak{s} + \mathfrak{h},\ \mathfrak{h} = \mathfrak{n} + \mathfrak{a},\ \mathfrak{s} \cap \mathfrak{h} = 0,\ \mathfrak{n} \cap \mathfrak{a} = 0,\ [\mathfrak{s}, \mathfrak{a}] = 0,$$

where $\mathfrak{s}$ is a maximal semi-simple subalgebra of $\mathfrak{g}$ and $\mathfrak{a}$ is an abelian subalgebra of $\mathfrak{h}$ composed of semi-simple matrices.

Corollary. *Let $\mathfrak{g}$ be a Lie subalgebra of $\mathfrak{gl}(V)$, $\mathfrak{h}$ be its radical and $\mathfrak{n}$ be the ideal of all nilpotent matrices of $\mathfrak{h}$. Then the following statements are equivalent:*

(1) *$\mathfrak{g}$ is algebraic.*

(2) *$\mathfrak{h}$ is algebraic.*

(3) *$\mathfrak{h}$ splits over $\mathfrak{n}$ and any maximal abelian subalgebra of $\mathfrak{h}$ composed of semi-simple matrices is algebraic and is a complement of $\mathfrak{n}$.*

Proof. (1) $\rightarrow$ (2), (3) $\rightarrow$ (1) are evident. Therefore we shall show only (2) $\rightarrow$ (3). Suppose that $\mathfrak{h}$ is algebraic. If $\mathfrak{a}$ is a maximal abelian subalgebra of $\mathfrak{h}$ composed of semi-simple matrices, then it follows from Theorem 1 that $\mathfrak{h} = \mathfrak{n} + \mathfrak{a}$, $\mathfrak{n} \cap \mathfrak{a} = 0$. For any $X \in \mathfrak{a}$ every element Y of $\mathfrak{g}(X)$ is a semi-simple matrix. Since $\mathfrak{h}$ is algebraic, Y is contained in $\mathfrak{h}$ and therefore $Y = N + A$, $N \in \mathfrak{n}$, $A \in \mathfrak{a}$. It follows from $[Y, A] = 0$ that $Y - A$ is semi-simple, whence $Y - A = 0$ i.e. $Y \in \mathfrak{a}$. Thus $\mathfrak{a}$ is algebraic, completing the proof.

§ 3. Splittable and algebraic hulls

We begin with the definition of splittable hull of a Lie subalgebra $\mathfrak{g}$ of $\mathfrak{gl}(V)$, whose existence is clear from the definition of splittability.

Definition 3. *The smallest splittable Lie algebra $^*\mathfrak{g}$ containing a Lie subalgebra $\mathfrak{g}$*

is called the splittable hull of $\mathfrak{g}$.

We shall first prove the following

PROPOSITION 3. *Let* $\mathfrak{g}$ *be a Lie subalgebra of* $\mathfrak{gl}(V)$ *and* $\mathfrak{h}$ *be its radical. Then* $^*\mathfrak{g}$ *is the direct sum of* $\mathfrak{g}$ *and an abelian subalgebra* $\mathfrak{b}$ *composed of semi-simple matrices; every element of* $\mathfrak{b}$ *can be represented as a finite sum of components of elements of* $\mathfrak{h}$. *The radical of* $^*\mathfrak{g}$ *is* $^*\mathfrak{h}$.

PROOF. Let $\mathfrak{g}=\mathfrak{s}+\mathfrak{h}$ be a Levi decomposition of $\mathfrak{g}$, where $\mathfrak{s}$ is a semi-simple subalgebra. If we put $\mathfrak{g}_1=\mathfrak{s}+{}^*\mathfrak{h}$, then it is obvious that $\mathfrak{g}_1$ is a Lie algebra contained in $^*\mathfrak{g}$ and $^*\mathfrak{h}$ is its radical. It follows from Proposition 1 that $\mathfrak{g}_1$ is splittable, whence $\mathfrak{g}_1={}^*\mathfrak{g}$, which shows that $^*\mathfrak{h}$ is the radical of $^*\mathfrak{g}$. Let $\mathfrak{n}$ be the linear space spanned by N_X for all $X\in\mathfrak{h}$. Since $\mathfrak{h}$ is solvable, it follows from Lie's theorem that $\mathfrak{n}$ is a Lie algebra composed of nilpotent matrices. We assert that $\mathfrak{n}$ is the set of all nilpotent matrices of $^*\mathfrak{h}$. Let $\mathfrak{a}_1$ be the linear space spanned by S_X for all $X\in\mathfrak{h}$. Then $\mathfrak{n}+\mathfrak{a}_1$ is a Lie algebra contained in $^*\mathfrak{h}$ and has a splittable basis. Hence by Proposition 1 we see that $^*\mathfrak{h}=\mathfrak{n}+\mathfrak{a}_1$. Let W be a sufficiently large space of tensor invariants of $\mathfrak{n}$. Then the space W is stable under $^*\mathfrak{h}$, since $\mathfrak{n}$ is an ideal of $^*\mathfrak{h}$. If X is a nilpotent matrix of $^*\mathfrak{h}$, then $X=N+A$, $N\in\mathfrak{n}$, $A\in\mathfrak{a}_1$, whence $\bar{X}=\bar{A}$. Since $\bar{\mathfrak{a}}_1$ is abelian, $\bar{A}$ is a semi-simple matrix, which shows that $\bar{X}=0$. It follows from Lemma 3 that $X\in\mathfrak{n}$. Therefore the assertion is proved. Now by Theorem 1 we see that $^*\mathfrak{h}=\mathfrak{n}+\mathfrak{a}$ where $\mathfrak{a}$ is an abelian subalgebra composed of semi-simple matrices. By considering a sufficiently large space of tensor invariants of $\mathfrak{h}'$, it is easily seen that $\mathfrak{a}_1\subseteqq\mathfrak{h}'+\mathfrak{a}$, whence $\mathfrak{n}\subseteqq\mathfrak{h}+\mathfrak{a}$ and therefore $^*\mathfrak{h}=\mathfrak{h}+\mathfrak{a}$. Putting $\mathfrak{a}=\mathfrak{h}\cap\mathfrak{a}+\mathfrak{b}$, $\mathfrak{h}\cap\mathfrak{b}=0$, we have $^*\mathfrak{g}=\mathfrak{g}+\mathfrak{b}$, $\mathfrak{g}\cap\mathfrak{b}=0$. Since $^*\mathfrak{h}$ is the linear space spanned by components of all elements of $\mathfrak{h}$, it is clear that $\mathfrak{b}$ has the desired properties. The proof is completed.

COROLLARY. *Let* $\mathfrak{g}$ *be a Lie subalgebra of* $\mathfrak{gl}(V)$ *and* $\mathfrak{h}$ *be its radical. Then* $\mathfrak{g}^*$ *is the direct sum of* $\mathfrak{g}$ *and an abelian subalgebra* $\mathfrak{b}$ *composed of semi-simple matrices; every element of* $\mathfrak{b}$ *can be represented as a finite sum of replicas of elements of* $\mathfrak{h}$. *The radical of* $\mathfrak{g}^*$ *is* $\mathfrak{h}^*$.

PROOF. It follows from Proposition 3 that $^*\mathfrak{g}=\mathfrak{g}+\mathfrak{b}_1$ where $\mathfrak{b}_1$ is an abelian subalgebra of semi-simple matrices of $^*\mathfrak{h}$. Let $\mathfrak{a}$ be a maximal abelian subalgebra of $^*\mathfrak{h}$ composed of semi-simple matrices which contains $\mathfrak{b}_1$. Then by Theorem 1 we see that $^*\mathfrak{g}=\mathfrak{s}+\mathfrak{n}+\mathfrak{a}$ where $\mathfrak{s}$ is a maximal semi-simple subalgebra and $\mathfrak{n}$ is the set of all nilpotent matrices of $^*\mathfrak{h}$. Since $^*\mathfrak{g}=\mathfrak{g}+\mathfrak{a}$ and $\mathfrak{g}^*=\mathfrak{s}+\mathfrak{n}+\mathfrak{a}^*$, it is clear that $\mathfrak{g}^*=\mathfrak{g}+\mathfrak{a}^*$. Putting $\mathfrak{a}^*=\mathfrak{g}\cap\mathfrak{a}^*+\mathfrak{b}$, $\mathfrak{g}\cap\mathfrak{b}=0$, we have $\mathfrak{g}^*=\mathfrak{g}+\mathfrak{b}$, $\mathfrak{g}\cap\mathfrak{b}=0$. It follows immediately from Corollary of Theorem 1 that $\mathfrak{g}^*=\mathfrak{s}+\mathfrak{h}^*$, which shows that $\mathfrak{h}^*$ is the radical of $\mathfrak{g}^*$. The second statement of the corollary is contained in the following proposition.

PROPOSITION 4. *$\mathfrak{g}^*$ is the linear space spanned by all replicas of elements of $\mathfrak{g}$.*

PROOF. The linear space $\mathfrak{g}_1$ spanned by all replicas of elements of $\mathfrak{g}$ forms a Lie algebra contained in $\mathfrak{g}^*$ and has an algebraic basis. Therefore $\mathfrak{g}_1$ is itself algebraic, whence $\mathfrak{g}_1=\mathfrak{g}^*$, completing the proof.

For any Lie subalgebra $\mathfrak{g}$ of $\mathfrak{gl}(V)$ it is evident that $\mathfrak{g}\subseteqq{}^*\mathfrak{g}\subseteqq\mathfrak{g}^*$. We shall show that there exists the following intimate connection between ${}^*\mathfrak{g}$ and $\mathfrak{g}^*$.

THEOREM 2. *Let $\mathfrak{g}$ be a Lie subalgebra of $\mathfrak{gl}(V)$. Then ${}^*\mathfrak{g}$ is the smallest Lie algebra containing $\mathfrak{g}$ which has the same nilpotent matrices as $\mathfrak{g}^*$.*

PROOF. Let $\mathfrak{g}_1$ be a Lie algebra containing $\mathfrak{g}$ which has the same nilpotent matrices as $\mathfrak{g}^*$. Then it is easily seen that $\mathfrak{g}_1\cap\mathfrak{g}^*$ is splittable and therefore ${}^*\mathfrak{g}\subseteqq\mathfrak{g}_1$. Let ${}^*\mathfrak{g}=\mathfrak{s}+\mathfrak{n}+\mathfrak{a}$ be a decomposition of ${}^*\mathfrak{g}$ indicated in Theorem 1, where $\mathfrak{s}$ is a maximal semi-simple subalgebra of ${}^*\mathfrak{g}$, $\mathfrak{n}$ is the ideal of all nilpotent matrices of the radical of ${}^*\mathfrak{g}$ and $\mathfrak{a}$ is an abelian subalgebra of semi-simple matrices such that $[\mathfrak{s}, \mathfrak{a}]=0$. It follows that $\mathfrak{g}^*=\mathfrak{s}+\mathfrak{n}+\mathfrak{a}^*$. We take a sufficiently large space W of tensor invariants of $\mathfrak{n}$. Since $\mathfrak{n}$ is an ideal of $\mathfrak{g}^*$ [8, p. 173], W is stable under $\mathfrak{g}^*$. Let X be any nilpotent matrix of $\mathfrak{g}^*$. Then $X=Y+N+A$, $Y\in\mathfrak{s}$, $N\in\mathfrak{n}$, $A\in\mathfrak{a}^*$, and therefore $\bar{X}=(\bar{S}_Y+\bar{A})+\bar{N}_Y$. Since $[\bar{S}_Y, \bar{A}]=0$, $\bar{S}_Y+\bar{A}$ is semi-simple, whence $\bar{X}=\bar{N}_Y$. It follows from Lemma 3 that $X-N_Y\in\mathfrak{n}$, which shows that $X\in{}^*\mathfrak{g}$. The proof is completed.

COROLLARY. *A Lie subalgebra $\mathfrak{g}$ of $\mathfrak{gl}(V)$ is splittable if and only if $\mathfrak{g}$ and $\mathfrak{g}^*$ have the same nilpotent matrices. Then any linear space $\mathfrak{g}_1$ such that $\mathfrak{g}\subseteqq\mathfrak{g}_1\subseteqq\mathfrak{g}^*$ is also a splittable Lie algebra.*

Let $\mathfrak{g}$ be a Lie algebra over K. As is well known, a nilpotent subalgebra $\mathfrak{k}$ is called a *Cartan subalgebra* of $\mathfrak{g}$ if the only elements x such that $(\mathrm{ad}_{\mathfrak{g}}z)^m x=0$ for every $z\in\mathfrak{k}$ and a suitable integer m are the elements of $\mathfrak{k}$. Then it is clear that any Cartan subalgebra is a maximal nilpotent subalgebra of $\mathfrak{g}$. Now we have

LEMMA 4. *Let $\mathfrak{g}$ be a Lie subalgebra of $\mathfrak{gl}(V)$ and $\mathfrak{k}$ be a Cartan subalgebra of $\mathfrak{g}$. Then there exists a subspace $\mathfrak{m}$ contained in the derived algebra of $\mathfrak{g}$ such that $\mathfrak{g}=\mathfrak{k}+\mathfrak{m}$, $\mathfrak{k}\cap\mathfrak{m}=0$.*

PROOF. For any $X\in\mathfrak{k}$ let $\mathfrak{g}_X$ denote the set of all elements Y of $\mathfrak{g}$ such that $(\mathrm{ad}_{\mathfrak{g}}X)^m Y=0$ for a suitable integer m. Then it is easily seen that $\mathfrak{g}_X$ is the set of all elements annihilated by $\mathrm{ad}_{\mathfrak{g}}S_X$. Since $\mathfrak{k}$ is nilpotent, it follows from Proposition 3 and Lie's theorem that the set of S_X for all $X\in\mathfrak{k}$ coincides with the set $\mathfrak{a}$ of all semi-simple matrices of ${}^*\mathfrak{k}$. Therefore $\mathfrak{k}$ is the set of all elements annihilated by $\mathrm{ad}_{\mathfrak{g}}\mathfrak{a}$. Since $\mathfrak{a}$ is abelian, we have $\mathfrak{g}=\mathfrak{k}+\mathfrak{m}$, $\mathfrak{k}\cap\mathfrak{m}=0$, $\mathfrak{m}\subseteqq\mathfrak{g}'$, by the same argument as in the proof of the second part of Theorem 1.

Using this lemma we show the following propositions.

PROPOSITION 5. *Let $\mathfrak{g}$ be a Lie subalgebra of $\mathfrak{gl}(V)$. If $\mathfrak{k}$ is a Cartan subalgebra of $\mathfrak{g}$, then ${}^*\mathfrak{k}$, $\mathfrak{k}^*$ are Cartan subalgebras of ${}^*\mathfrak{g}$ and $\mathfrak{g}^*$ respectively. Conversely if K is the complex field, any Cartan subalgebra of ${}^*\mathfrak{g}$ and $\mathfrak{g}^*$ can be represented in such a form.*

PROOF. Let $\mathfrak{k}$ be a Cartan subalgebra of $\mathfrak{g}$ and let $\mathfrak{g}=\mathfrak{k}+\mathfrak{m}$ be a decomposition of $\mathfrak{g}$ indicated in Lemma 4, where $\mathfrak{m}$ is a subspace contained in the derived algebra of $\mathfrak{g}$. It is clear that ${}^*\mathfrak{g}={}^*\mathfrak{k}+\mathfrak{m}$ and $\mathfrak{g}^*=\mathfrak{k}^*+\mathfrak{m}$. Then it is easy to see that ${}^*\mathfrak{k}$ and $\mathfrak{k}^*$ are Cartan subalgebras of ${}^*\mathfrak{g}$ and $\mathfrak{g}^*$ respectively. The converse part follows immediately from a result due to C. Chevalley [4].

PROPOSITION 6. *A Lie subalgebra of $\mathfrak{gl}(V)$ is splittable (algebraic) if and only if any Cartan subalgebra is splittable (algebraic).*

PROOF. Suppose that a Lie subalgebra $\mathfrak{g}$ is splittable (algebraic). If $\mathfrak{k}$ is a Cartan subalgebra of $\mathfrak{g}$, then ${}^*\mathfrak{k}$ ($\mathfrak{k}^*$) is a nilpotent subalgebra contained in $\mathfrak{g}$. Since any Cartan subalgebra is a maximal nilpotent subalgebra, we have ${}^*\mathfrak{k}=\mathfrak{k}$ ($\mathfrak{k}^*=\mathfrak{k}$) i. e., $\mathfrak{k}$ is splittable (algebraic). Conversely let a Cartan subalgebra $\mathfrak{k}$ be splittable (algebraic). Since $\mathfrak{g}=\mathfrak{k}+\mathfrak{m}$ and $\mathfrak{m}\subseteqq\mathfrak{g}'$ by Lemma 4, it follows that $\mathfrak{g}$ has a splittable (algebraic) basis and therefore $\mathfrak{g}$ is splittable (algebraic), completing the proof.

§ 4. The largest splittable and the largest algebraic Lie algebras contained in a linear Lie algebra

Let $\mathfrak{g}$ be a Lie subalgebra of $\mathfrak{gl}(V)$. Then there exists the largest splittable Lie algebra contained in $\mathfrak{g}$, which we shall denote by ${}_*\mathfrak{g}$. We can prove the following result which implies the existence of ${}_*\mathfrak{g}$: The Lie algebra generated by any family of splittable Lie algebras is itself splittable. Let $\{\mathfrak{g}_i : i\in I\}$ be any family of splittable Lie algebras and let $\tilde{\mathfrak{g}}$ be the Lie algebra generated by all $\mathfrak{g}_i$'s. First we consider the case where I is a finite set $\{1, 2, \cdots, h\}$. Then we have $\tilde{\mathfrak{g}}=\mathfrak{g}_1+\mathfrak{g}_2+\cdots+\mathfrak{g}_h+\tilde{\mathfrak{g}}'$, which shows that $\tilde{\mathfrak{g}}$ has a splittable basis and therefore $\tilde{\mathfrak{g}}$ is splittable. In the case where I is an infinite set, let E be any finite subset of I and $\tilde{\mathfrak{g}}_E$ be the Lie algebra generated by $\mathfrak{g}_i$ for all $i\in E$. If we denote by $\tilde{\mathfrak{g}}_{E_0}$ one of maximal dimension among $\tilde{\mathfrak{g}}_E$'s, then we have $\tilde{\mathfrak{g}}_E=\tilde{\mathfrak{g}}_{E_0}$ for any finite subset E of I containing E_0, whence $\tilde{\mathfrak{g}}=\tilde{\mathfrak{g}}_{E_0}$. Therefore by the first consideration $\tilde{\mathfrak{g}}$ is splittable, as desired. Now we shall show

PROPOSITION 7. *Let $\mathfrak{g}$ be a Lie subalgebra of $\mathfrak{gl}(V)$ and $\mathfrak{h}$ be its radical. Then ${}_*\mathfrak{g}$*

is the set of all elements of $\mathfrak{g}$ *whose components are contained in* $\mathfrak{g}$ *and it is also the linear space spanned by all semi-simple and all nilpotent matrices of* $\mathfrak{g}$. ${}_*\mathfrak{g}$ *is an ideal of* $\mathfrak{g}$ *and the radical of* ${}_*\mathfrak{g}$ *is* ${}_*\mathfrak{h}$.

Proof. Let $\mathfrak{g}_1$ be the set of all elements X of $\mathfrak{g}$ such that $S_X \in \mathfrak{g}$ and let $\mathfrak{g}_2$ be the linear space spanned by all semi-simple and all nilpotent matrices of $\mathfrak{g}$. Then it is obvious that ${}_*\mathfrak{g} \subseteqq \mathfrak{g}_1 \subseteqq \mathfrak{g}_2 \subseteqq \mathfrak{g}$. But $\mathfrak{g}_2$ forms a Lie algebra with a splittable basis, which is splittable by Proposition 1. Hence ${}_*\mathfrak{g} = \mathfrak{g}_1 = \mathfrak{g}_2$. The fact that ${}_*\mathfrak{g}$ is an ideal of $\mathfrak{g}$ follows immediately from $\mathfrak{g}' \subseteqq {}_*\mathfrak{g}$. Let $\mathfrak{g} = \mathfrak{s} + \mathfrak{h}$ be a Levi decomposition of $\mathfrak{g}$. Since $\mathfrak{s} \subseteqq {}_*\mathfrak{g}$, it follows that ${}_*\mathfrak{g} \cap \mathfrak{h}$ is the radical of ${}_*\mathfrak{g}$, which is splittable by Proposition 1. Hence we have ${}_*\mathfrak{g} \cap \mathfrak{h} \subseteqq {}_*\mathfrak{h}$ and therefore ${}_*\mathfrak{g} \cap \mathfrak{h} = {}_*\mathfrak{h}$. This completes the proof.

Next we consider the largest algebraic Lie algebra contained in a Lie subalgebra $\mathfrak{g}$ and denote it by $\mathfrak{g}_*$. In the same manner as above we can prove the following

Proposition 8. *Let* $\mathfrak{g}$ *be a Lie subalgebra of* $\mathfrak{gl}(V)$ *and* $\mathfrak{h}$ *be its radical. Then* $\mathfrak{g}_*$ *is the set of all elements* X *of* $\mathfrak{g}$ *such that* $\mathfrak{g}(X) \subseteqq \mathfrak{g}$ *and it is also the linear space spanned by all elements of* $\mathfrak{g}$ *whose eigenvalues are all rational numbers.* $\mathfrak{g}_*$ *is an ideal of* $\mathfrak{g}$ *and the radical of* $\mathfrak{g}_*$ *is* $\mathfrak{h}_*$.

From the above two propositions we have

Corollary. *Let* $\mathfrak{g}$ *be a Lie subalgebra of* $\mathfrak{gl}(V)$. *Then* $\mathfrak{g}$, ${}_*\mathfrak{g}$ *and* $\mathfrak{g}_*$ *have the same nilpotent matrices. Any linear space* $\mathfrak{g}_1$ *such that* $\mathfrak{g}_* \subseteqq \mathfrak{g}_1 \subseteqq {}_*\mathfrak{g}$ *is also a splittable Lie algebra.*

§ 5. Splittable groups

Let $GL(V)$ be the group of all automorphisms of V whose unit element is denoted by I. It is known [8, p. 184] that any element s of $GL(V)$ is uniquely represented in the form $s = uv$, where u is the semi-simple component of s, $v - I$ is nilpotent and u, v commute each other. Furthermore u and v are polynomials in s.

Definition 4. *A subgroup* $\mathfrak{G}$ *of* $GL(V)$ *is called splittable provided the semi-simple component of every element of* $\mathfrak{G}$ *also belongs to* $\mathfrak{G}$.

For any $s \in GL(V)$ its semi-simple component is contained in any algebraic group containing s [8, p. 184]. Hence every algebraic group is splittable. Let v be an element of $GL(V)$ such that $v \neq I$ and $v - I$ is nilpotent. Then it is known [8, p. 183] that the smallest algebraic group containing v is irreducible and its Lie algebra is generated by the nilpotent matrix N such that $v = \exp N$.

In this section we assume that K is the real or complex field. Then any subgroup of $GL(V)$ is a Lie group whose Lie algebra is a subalgebra of $\mathfrak{gl}(V)$ [6].

Using Corollary of Theorem 2 we show the following

THEOREM 3. *Let $\mathfrak{G}$ be a subgroup of automorphisms of a vector space V whose basic field is the real or complex field. Then $\mathfrak{G}$ is splittable if and only if its Lie algebra is splittable.*

PROOF. Suppose that the Lie algebra $\mathfrak{g}$ of $\mathfrak{G}$ is splittable. If we denote by $\mathfrak{G}^*$ the smallest algebraic group containing $\mathfrak{G}$, then the Lie algebra of $\mathfrak{G}^*$ is the algebraic hull $\mathfrak{g}^*$ of $\mathfrak{g}$ [8, p. 171]. For any $s\in\mathfrak{G}$ let u be its semi-simple component and put $v=u^{-1}s$. Since $\mathfrak{G}^*$ is splittable, we have $u, v\in\mathfrak{G}^*$. If $v\neq I$, then we consider the smallest algebraic group $\mathfrak{G}_v$ containing v. Since $v-I$ is nilpotent, the Lie algebra of $\mathfrak{G}_v$ is generated by the nilpotent matrix N such that $v=\exp N$. Since $\mathfrak{G}_v\subseteqq\mathfrak{G}^*$, it follows that $N\in\mathfrak{g}^*$. By Corollary of Theorem 2 we see that $N\in\mathfrak{g}$, whence $v\in\mathfrak{G}$ and therefore $u\in\mathfrak{G}$. This shows that $\mathfrak{G}$ is splittable. The converse is evident.

From this theorem and results of §1, §3 we have the following corollaries.

COROLLARY 1. *A subgroup of $GL(V)$ is splittable if and only if its largest solvable normal subgroup is splittable.*

COROLLARY 2. *Every subgroup of $GL(V)$ whose Lie algebra is completely reducible is splittable.*

COROLLARY 3. *The group generated by any family of splittable groups is also splittable.*

COROLLARY 4. *Let $\mathfrak{G}$ be a splittable group whose center is discrete. If $\mathfrak{G}$ is the direct product of subgroups $\mathfrak{G}_i$, $i\in I$, then each $\mathfrak{G}_i$ is splittable.*

PROOF. It is an immediate consequence of Theorem 3 and the corresponding result on Lie algebras which follows from Proposition 3.

COROLLARY 5. *A subgroup $\mathfrak{G}$ of $GL(V)$ is splittable if and only if the set of all elements s of $\mathfrak{G}$ such that $s-I$ is nilpotent coincides with the corresponding set of the smallest algebraic group containing $\mathfrak{G}$.*

On solvable splittable groups we have the following

PROPOSITION 9. *Let $\mathfrak{G}$ be a connected solvable splittable subgroup of $GL(V)$. Then there exist a connected normal subgroup $\mathfrak{N}$ and a connected abelian subgroup $\mathfrak{A}$ such that $\mathfrak{G}=\mathfrak{N}\mathfrak{A}$, $\mathfrak{N}\cap\mathfrak{A}=I$; for any $s\in\mathfrak{N}$ $s-I$ is nilpotent and every element of $\mathfrak{A}$ is semi-simple. Furthermore if $\mathfrak{G}$ is nilpotent, then $\mathfrak{A}$ is a central normal subgroup of $\mathfrak{G}$.*

PROOF. Let $\mathfrak{g}$ be the Lie algebra of $\mathfrak{G}$. Let $\mathfrak{g}=\mathfrak{n}+\mathfrak{a}$ be a decomposition of $\mathfrak{g}$ indicated in Theorem 1 and let $\mathfrak{N}$, $\mathfrak{A}$ be the connected subgroups of $GL(V)$ whose

Lie algebras are $\mathfrak{n}$, $\mathfrak{a}$ respectively. Since $\mathfrak{n}$ is an ideal of $\mathfrak{g}$, $\mathfrak{N}$ is a normal subgroup of $\mathfrak{G}$. Hence we have $\mathfrak{G}=\mathfrak{N}\mathfrak{A}$. Using Engel's theorem on nilpotent Lie algebras it is easily seen that $s-I$ is a nilpotent matrix for any $s\in\mathfrak{N}$. Since $\mathfrak{a}$ is an abelian subalgebra of semi-simple matrices, it follows that every element of $\mathfrak{A}$ is semi-simple. Now it is clear that $\mathfrak{N}\cap\mathfrak{A}=I$. Furthermore if $\mathfrak{G}$ is nilpotent, then $\mathfrak{g}$ is also nilpotent. Therefore for any $X\in\mathfrak{a}$ $\mathrm{ad}_{\mathfrak{g}}X$ is nilpotent and semi-simple, whence $\mathrm{ad}_{\mathfrak{g}}X=0$, which shows that $\mathfrak{a}$ is contained in the center of $\mathfrak{g}$. Therefore $\mathfrak{A}$ is a central normal subgroup of $\mathfrak{G}$.

§ 6. Lie algebras with splittable adjoint representations

In this section we shall study a not necessarily linear Lie algebra whose adjoint representation is splittable. We start with

LEMMA 5. *Let $\mathfrak{g}$ be a Lie subalgebra of $\mathfrak{gl}(V)$. If, for any $X\in\mathfrak{gl}(V)$ such that $[X,\mathfrak{g}]\subseteq\mathfrak{g}$, we denote $(\mathrm{ad}_{\mathfrak{g}}X)Y=[X,Y]$, $Y\in\mathfrak{g}$, then we have $S_{\mathrm{ad}_{\mathfrak{g}}X}=\mathrm{ad}_{\mathfrak{g}}S_X$, $N_{\mathrm{ad}_{\mathfrak{g}}X}=\mathrm{ad}_{\mathfrak{g}}N_X$.*

PROOF. It is sufficient to show the statement for $\mathrm{ad}_{\mathfrak{gl}(V)}X$, since $\mathrm{ad}_{\mathfrak{g}}$ is the restriction of $\mathrm{ad}_{\mathfrak{gl}(V)}$ to $\mathfrak{g}$. But Lemma 5 can be easily verified for $\mathrm{ad}_{\mathfrak{gl}(V)}X$.

LEMMA 6. *Let $\mathfrak{g}$ be a Lie algebra over K whose adjoint representation is splittable. Let $\mathfrak{h}$ be its radical and $\mathfrak{n}$ be the largest nilpotent ideal. Then there exist a maximal semi-simple subalgebra $\mathfrak{s}$ and an abelian subalgebra $\mathfrak{a}$ such that $\mathfrak{h}=\mathfrak{n}+\mathfrak{a}$, $\mathfrak{n}\cap\mathfrak{a}=0$, $[\mathfrak{s},\mathfrak{a}]=0$, and in the adjoint representation of $\mathfrak{g}$, $\mathfrak{a}$ is represented faithfully by a Lie algebra composed of semi-simple matrices. If, in particular, $\mathfrak{g}$ is a linear Lie algebra, then $\mathfrak{n}$ is the direct sum of the central ideal of $\mathfrak{g}$ composed of all semi-simple matrices of $\mathfrak{n}$ and an ideal of $\mathfrak{g}$ containing all nilpotent matrices of $\mathfrak{h}$.*

PROOF. Let $\mathfrak{s}$ be a maximal semi-simple subalgebra of $\mathfrak{g}$ and put $\bar{\mathfrak{s}}=\mathrm{ad}_{\mathfrak{g}}\mathfrak{s}$, $\bar{\mathfrak{n}}=\mathrm{ad}_{\mathfrak{g}}\mathfrak{n}$. Then $\bar{\mathfrak{s}}$ is a maximal semi-simple subalgebra of $\mathrm{ad}\,\mathfrak{g}$ and $\bar{\mathfrak{n}}$ is the set of all nilpotent matrices of the radical of $\mathrm{ad}\,\mathfrak{g}$. Since $\mathrm{ad}\,\mathfrak{g}$ is splittable, by Theorem 1 we see that $\mathrm{ad}\,\mathfrak{g}=\bar{\mathfrak{s}}+\bar{\mathfrak{n}}+\bar{\mathfrak{a}}$, $\bar{\mathfrak{n}}\cap\bar{\mathfrak{a}}=0$, $[\bar{\mathfrak{s}},\bar{\mathfrak{a}}]=0$, where $\bar{\mathfrak{a}}$ is an abelian subalgebra of semi-simple matrices. Let $\mathfrak{a}_1$ be the complete inverse image of $\bar{\mathfrak{a}}$. Then for any $x\in\mathfrak{a}_1$ and $y\in\mathfrak{s}+\mathfrak{a}_1$, $[x,y]$ is contained in the center $\mathfrak{z}$ of $\mathfrak{g}$, whence $(\mathrm{ad}_{\mathfrak{g}}x)^2y=0$. Since $\mathrm{ad}_{\mathfrak{g}}x$ is semi-simple, it follows that $(\mathrm{ad}_{\mathfrak{g}}x)y=0$ i. e. $[x,y]=0$, which shows that $[\mathfrak{a}_1,\mathfrak{s}+\mathfrak{a}_1]=0$. If we put $\mathfrak{a}_1=\mathfrak{z}+\mathfrak{a}$, $\mathfrak{z}\cap\mathfrak{a}=0$, then $\mathfrak{a}$ satisfies the desired properties. If, in particular, $\mathfrak{g}$ is a linear Lie algebra, then the set $\mathfrak{b}$ of all semi-simple matrices of $\mathfrak{n}$ is obviously a central ideal of $\mathfrak{g}$. Let $\mathfrak{n}_1$ be a linear space containing all nilpotent matrices of $\mathfrak{n}$ such that $\mathfrak{n}=\mathfrak{b}+\mathfrak{n}_1$, $\mathfrak{b}\cap\mathfrak{n}_1=0$.

Then it is clear that $\mathfrak{n}_1$ is an ideal of $\mathfrak{g}$. The proof is completed.

Using these lemmas we shall show

PROPOSITION 10. *A Lie algebra over K has faithful splittable representation if and only if its adjoint representation is splittable.*

PROOF. Let $\mathfrak{g}$ be a Lie algebra over K whose adjoint representation is splittable. We may suppose that $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{gl}(V)$, since any Lie algebra over K has a faithful representation by finite matrices [1, 3, 11]. Let $\mathfrak{g}=\mathfrak{s}+\mathfrak{b}+\mathfrak{n}_1+\mathfrak{a}$ be a decomposition of $\mathfrak{g}$ indicated in Lemma 6, where $\mathfrak{b}$ is the central ideal of all semi-simple matrices of the largest nilpotent ideal and $\mathfrak{n}_1$ is an ideal containing all nilpotent matrices of the radical. Let $\tilde{\mathfrak{n}}_1$ be the set of all N_X for $X\in\mathfrak{n}_1$ and $\tilde{\mathfrak{a}}$ be the set of all S_X for $X\in\mathfrak{a}$. Then it is clear that $\tilde{\mathfrak{n}}_1$ and $\tilde{\mathfrak{a}}$ are Lie algebras. Let us consider the mapping

$$\mathfrak{s}+\mathfrak{b}\ni X \longrightarrow \tilde{X}=X$$
$$\mathfrak{n}_1\ni X \longrightarrow \tilde{X}=N_X$$
$$\mathfrak{a}\ni X \longrightarrow \tilde{X}=S_X.$$

Then it is obvious that the mapping gives a faithful representation of $\mathfrak{g}$ by a Lie algebra $\tilde{\mathfrak{g}}=\mathfrak{s}+\mathfrak{b}+\tilde{\mathfrak{n}}_1+\tilde{\mathfrak{a}}$, which is splittable by Proposition 1. The converse is evident.

COROLLARY. *If $\mathfrak{g}$ is a Lie algebra over K whose adjoint representation is splittable, then so is the radical of $\mathfrak{g}$.*

Now we shall show the following

THEOREM 4. *Let $\mathfrak{g}$ be a Lie algebra over K whose adjoint representation is splittable and let $\mathfrak{g}_1$ be any subalgebra which contains the center. Then the following statements are equivalent*:

(1) *The adjoint representation of $\mathfrak{g}$ induces a splittable representation of $\mathfrak{g}_1$.*

(2) *There exists a faithful splittable representation of $\mathfrak{g}$ which induces a splittable representation of $\mathfrak{g}_1$.*

(3) *Every splittable representation of $\mathfrak{g}$ induces a splittable representation of $\mathfrak{g}_1$.*

PROOF. (1) $\rightarrow$ (3). We assume (1) satisfied. First let ρ be a faithful splittable representation of $\mathfrak{g}$. Then for any $X\in\mathfrak{g}_1$ there exists $Y\in\mathfrak{g}$ such that $S_{\rho(X)}=\rho(Y)$. Since $\mathrm{ad}_{\mathfrak{g}}\mathfrak{g}_1$ is splittable, $\mathrm{ad}_{\rho(\mathfrak{g})}\rho(\mathfrak{g}_1)$ is also splittable. It follows from Proposition 3 and Lemma 5 that $\mathrm{ad}_{\rho(\mathfrak{g})}{}^*\rho(\mathfrak{g}_1)=\mathrm{ad}_{\rho(\mathfrak{g})}\rho(\mathfrak{g}_1)$. Hence there exists $Y_1\in\mathfrak{g}_1$ such that $\rho(Y)-\rho(Y_1)$ is contained in the center of $\rho(\mathfrak{g})$. Since ρ is faithful, it follows that $Y-Y_1$ is contained in the center of $\mathfrak{g}$. But the center is contained in $\mathfrak{g}_1$ by assumption. Therefore we have $Y\in\mathfrak{g}_1$, whence $\rho(Y)\in\rho(\mathfrak{g}_1)$. Thus $\rho(\mathfrak{g}_1)$ is splittable in this case. Next let ρ be any splittable representation of $\mathfrak{g}$. By Proposition 10 we take a faithful splittable representation σ of $\mathfrak{g}$ and put $\rho'=\rho\dotplus\sigma$, where $\dotplus$ denotes direct sum. Then ρ' is a faithful splittable representation of $\mathfrak{g}$. Hence $\rho'(\mathfrak{g}_1)$ is splittable by the above consideration and therefore for any $X\in\mathfrak{g}_1$ there exists $Y\in\mathfrak{g}_1$ such that $S_{\rho'(X)}=\rho'(Y)$. Then we have $S_{\rho(X)}=\rho(Y)$, which shows that $\rho(\mathfrak{g}_1)$ is splittable.

(3) $\rightarrow$ (2) follows immediately from Proposition 10.

(2) $\rightarrow$ (1). We may assume that $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{gl}(V)$ and $\mathfrak{g}_1$ is splittable. Then Lemma 5 implies that $\mathrm{ad}_{\mathfrak{g}}\mathfrak{g}_1$ is splittable. Thus the theorem is completely proved.

Corollary. *Let $\mathfrak{g}$ be a Lie algebra over K whose adjoint representation is splittable. Then every splittable representation of $\mathfrak{g}$ induces splittable representations of the radical, the largest nilpotent ideal, maximal nilpotent subalgebras and the center.*

Proof. Let $\mathfrak{h}$, $\mathfrak{n}$ and $\mathfrak{k}$ be the radical, the largest nilpotent ideal and a maximal nilpotent subalgebra of $\mathfrak{g}$ respectively. The center $\mathfrak{z}$ is contained in each of them. It is obvious that $\mathrm{ad}_{\mathfrak{g}}\mathfrak{h}$ is the radical of $\mathrm{ad}\,\mathfrak{g}$, $\mathrm{ad}_{\mathfrak{g}}\mathfrak{n}$ is composed of nilpotent matrices, $\mathrm{ad}_{\mathfrak{g}}\mathfrak{k}$ is a maxmal nilpotent subalgebra of $\mathrm{ad}\,\mathfrak{g}$ and $\mathrm{ad}_{\mathfrak{g}}\mathfrak{z}=0$. It follows from the splittability of $\mathrm{ad}\,\mathfrak{g}$ that they are all splittable. The statement then follows immediately from Theorem 4.

We shall show the following lemma in order to prove a result due to M. Gotô.

Lemma 7. *Let $\mathfrak{g}$ be a Lie subalgebra of $\mathfrak{gl}(V)$. For any $X\in\mathfrak{gl}(V)$ such that $[X,\mathfrak{g}]\subseteqq\mathfrak{g}$, we have $\mathrm{ad}_{\mathfrak{g}}\mathfrak{g}(X)=\mathfrak{g}(\mathrm{ad}_{\mathfrak{g}}X)$.*

Proof. If X is a semi-simple matrix, it follows from Proposition 2 that there exists a Lie algebra $\mathfrak{g}_0$ over K_0 such that all eigenvalues of any element of $\mathfrak{g}_0$ are rational numbers and $\mathfrak{g}(X)=\mathfrak{g}_0{}^K$. Then we have $\mathrm{ad}_{\mathfrak{g}}\mathfrak{g}(X)=(\mathrm{ad}_{\mathfrak{g}}\mathfrak{g}_0)^K$. By Proposition 2 we see that $\mathrm{ad}_{\mathfrak{g}}\mathfrak{g}(X)$ is algebraic, whence $\mathfrak{g}(\mathrm{ad}_{\mathfrak{g}}X)\subseteqq\mathrm{ad}_{\mathfrak{g}}\mathfrak{g}(X)$. Let $\bar{K}$ denote the algebraic closure of K and let $\mathfrak{g}^{\bar{K}}(X)$ denote the set of all replicas of X as the endomorphism of $V^{\bar{K}}$. Since $X\in\mathfrak{gl}(V^{\bar{K}})$ may be considered as a diagonal matrix, it is easily seen [8, p. 160] that $\mathrm{ad}_{\mathfrak{g}}\mathfrak{g}^{\bar{K}}(X)\subseteqq\mathfrak{g}^{\bar{K}}(\mathrm{ad}_{\mathfrak{g}}X)$. This, together with $\mathfrak{g}(X)=\mathfrak{g}^{\bar{K}}(X)\cap\mathfrak{gl}(V)$ [8, p. 181], gives $\mathrm{ad}_{\mathfrak{g}}\mathfrak{g}(X)\subseteqq\mathfrak{g}(\mathrm{ad}_{\mathfrak{g}}X)$. Hence we have $\mathrm{ad}_{\mathfrak{g}}\mathfrak{g}(X)=\mathfrak{g}(\mathrm{ad}_{\mathfrak{g}}X)$. Now let X be any element of $\mathfrak{gl}(V)$ such that $[X,\mathfrak{g}]\subseteqq\mathfrak{g}$. Then $\mathrm{ad}_{\mathfrak{g}}\mathfrak{g}(X)=\mathrm{ad}_{\mathfrak{g}}\mathfrak{g}(S_X)+\mathrm{ad}_{\mathfrak{g}}\mathfrak{g}(N_X)=\mathfrak{g}(\mathrm{ad}_{\mathfrak{g}}S_X)+\mathfrak{g}(\mathrm{ad}_{\mathfrak{g}}N_X)=\mathfrak{g}(\mathrm{ad}_{\mathfrak{g}}X)$, completing the proof.

We are now able to prove the following result [10]: *Every Lie algebra over K whose adjoint representation is algebraic has a faithful algebraic representation.* Let $\mathfrak{g}$ be a Lie algebra over K whose adjoint representation is algebraic. Then by Proposition 10 we may assume that $\mathfrak{g}$ is a splittable Lie subalgebra of $\mathfrak{gl}(V)$. Let $\mathfrak{g}=\mathfrak{s}+\mathfrak{n}+\mathfrak{a}$

be a decomposition of $\mathfrak{g}$ indicated in Theorem 1, where $\mathfrak{s}$ is a maximal semi-simple subalgebra, $\mathfrak{n}$ is the ideal of all nilpotent matrices of the radical and $\mathfrak{a}$ is an abelian subalgebra of semi-simple matrices such that $[\mathfrak{s}, \mathfrak{a}]=0$. Corollary of Theorem 1 tells us that $\mathrm{ad}_{\mathfrak{g}}\mathfrak{a}$ is algebraic, since $\mathrm{ad}\,\mathfrak{g}$ is algebraic. It follows from Proposition 4 and Lemma 7 that there exists a central ideal $\mathfrak{b}$ such that $\mathfrak{a}^*=\mathfrak{a}+\mathfrak{b}$, $\mathfrak{a}\cap\mathfrak{b}=0$. Since $\mathfrak{a}^*$ is abelian and consists of semi-simple matrices, by Proposition 2 we see that there exists a Lie algebra $\mathfrak{a}_0$ over K_0 such that all eigenvalues of any element of $\mathfrak{a}_0$ are rational numbers and $\mathfrak{a}^*=\mathfrak{a}_0{}^K$. Let $\{X_1, \cdots\cdots, X_m\}$, $\{X_{m+1}, \cdots\cdots, X_n\}$ and $\{Y_1, \cdots\cdots, Y_n\}$ be the bases of $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{a}_0$ respectively. Then $X_i=\sum_{j=1}^{n} k_{ji}Y_j$ $(i=1, \cdots\cdots, n)$ where $k_{ji}\in K$ $(i, j=1, \cdots\cdots, n)$. Hence $X_i=\sum_{j=1}^{m} k'_{ji}Y_j+Z_i$ $(i=1, \cdots\cdots, m)$ *where* $k'_{ji}\in K$ $(i, j=1, \cdots\cdots, m)$ and $Z_i\in\mathfrak{b}$ $(i=1, \cdots\cdots, m)$. If we put $\tilde{X}_i=X_i-Z_i$ $(i=1, \cdots\cdots, m)$ and $\tilde{\mathfrak{a}}=K\tilde{X}_1+\cdots\cdots+K\tilde{X}_m$, then we have $\tilde{\mathfrak{a}}=KY_1+\cdots\cdots+KY_m$, which shows that $\tilde{\mathfrak{a}}$ is algebraic. Then the mapping

$$\mathfrak{s}+\mathfrak{n}\ni X \longrightarrow \tilde{X}=X$$

$$\mathfrak{a}\ni X_i \longrightarrow \tilde{X}_i\in\tilde{\mathfrak{a}} \qquad (i=1, \cdots\cdots, m)$$

gives a faithful representation of $\mathfrak{g}$ by an algebraic Lie algebra $\tilde{\mathfrak{g}}=\mathfrak{s}+\mathfrak{n}+\tilde{\mathfrak{a}}$.

§ 7. Enveloping algebras of linear groups

The enveloping algebra of a subgroup $\mathfrak{G}$ of $GL(V)$ is the set of all linear combinations of elements of $\mathfrak{G}$. It was proved by C. Chevalley [8, p. 157] that the enveloping algebra of an irreducible algebraic group is the associative algebra generated by the unit matrix I and elements of its Lie algebra. It is known [8, p. 184] that a semi-simple matrix $s\in GL(V)$ is contained in an irreducible algebraic group whose elements are polynomials in s. But this result can be easily established for any element of $GL(V)$.

By making use of these results we can show

THEOREM 5. *A subgroup $\mathfrak{G}$ of $GL(V)$ and the smallest algebraic group containing $\mathfrak{G}$ have the same enveloping algebra.*

PROOF. Let $\mathfrak{G}^*$ be the smallest algebraic group containing $\mathfrak{G}$. Let $\mathfrak{E}$, $\mathfrak{E}^*$ be the enveloping algebras of $\mathfrak{G}$, $\mathfrak{G}^*$ respectively. Then $\mathfrak{E}\subseteqq\mathfrak{E}^*$. For any $s\in\mathfrak{G}$ we denote by $\mathfrak{G}_s$ an irreducible algebraic group containing s whose elements are polynomials in s. Since $\mathfrak{G}_s\subseteqq\mathfrak{E}$, the enveloping algebra of $\mathfrak{G}_s$ is contained in $\mathfrak{E}$ and therefore the Lie algebra $\mathfrak{g}_s$ of $\mathfrak{G}_s$ is also contained in $\mathfrak{E}$. Let $\mathfrak{g}_1$ be the Lie algebra generated by all $\mathfrak{g}_s$ for $s\in\mathfrak{G}$. Then the smallest algebraic group $\mathfrak{G}_1$

containing all $\mathfrak{G}_s$ for $s \in \mathfrak{G}$ is irreducible and its Lie algebra is $\mathfrak{g}_1$ [8, p. 175]. It follows from $\mathfrak{g}_1 \subseteqq \mathfrak{E}$ that $\mathfrak{G}_1 \subseteqq \mathfrak{E}$, whence $\mathfrak{G}^* \subseteqq \mathfrak{E}$ and therefore $\mathfrak{E}^* \subseteqq \mathfrak{E}$. Thus we have $\mathfrak{E}^* = \mathfrak{E}$ and our theorem is proved.

From this theorem and Proposition 4 we have

Corollary. *Let K be the real or complex field and let $\mathfrak{G}$ be a connected subgroup of $GL(V)$. Then the enveloping algebra of $\mathfrak{G}$ is the associative algebra generated by the unit matrix I and elements of its Lie algebra.*

Proof. Let $\mathfrak{G}^*$ be the smallest algebraic group containing $\mathfrak{G}$. Then the Lie algebra of $\mathfrak{G}^*$ is the algebraic hull $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$ of $\mathfrak{G}$. Since $\mathfrak{G}$ is connected, it follows [8, p. 189] that $\mathfrak{G}^*$ is irreducible. By Theorem 5 we see that the enveloping algebra $\mathfrak{E}$ of $\mathfrak{G}$ is identical with the enveloping algebra of $\mathfrak{G}^*$ and therefore it is the associative algebra generated by I and elements of $\mathfrak{g}^*$. Proposition 4 tells us that every element of $\mathfrak{g}^*$ is a finite sum of replicas of elements of $\mathfrak{g}$. Since any replica of $X \in \mathfrak{gl}(V)$ is a polynomial in X, we conclude that $\mathfrak{E}$ is the associative algebra generated by I and elements of $\mathfrak{g}$. The proof is completed.

§ 8. Examples

We have considered various types of Lie algebras. The interrelations between them are given by the diagram (D). The following examples show that, in the diagram, the inverse implications do not hold generally.

Example 1. Let $\mathfrak{g} = KX_1 + KX_2$, $X_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \tau \end{pmatrix}$, $X_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, where K is the real field and τ is an irrational number. Then $\mathfrak{g}$ is a splittable Lie algebra whose adjoint representation is algebraic, but it is not an algebraic Lie algebra.

Example 2. Let $\mathfrak{g} = KX_1 + KX_2 + KX_3$, $X_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \tau \end{pmatrix}$, $X_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $X_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, where K and τ have the same meanings as in Example 1. Then $\mathfrak{g}$ is a splittable Lie algebra whose adjoint representation is not algebraic.

Example 3. Let X be any element of $\mathfrak{gl}(V)$ such that $S_X \neq 0$ and $N_X \neq 0$. Then the Lie algebra generated by X gives an example of a Lie algebra which is not splittable and whose adjoint representation is algebraic.

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Faculty of Science,
Hiroshima University