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Infinitesimal Deformation of the Periodic Solution of the Second Kind and its Application to the Equation of a Pendulum

By

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Introduction.

Given the system of the differential equations

$$(E_1) \qquad \frac{dx}{dt} = P(x, \theta), \quad \frac{d\theta}{dt} = Q(x, \theta),$$

where $P(x, \theta)$ and $Q(x, \theta)$ are periodic with regard to θ with a common period. As is well known, for the equations of the above form, there may appear the periodic solutions of the second kind [(1)], namely the solutions corresponding to the periodic solutions of the equation as follows:

$$(E_2) \qquad \frac{dx}{d\theta} = \frac{P(x, \theta)}{Q(x, \theta)} \equiv X(x, \theta).$$

For the equations (E_1), we assume:

1° *$P(x, \theta)$ and $Q(x, \theta)$ are continuous with regard to (x, θ) for $|x|$, $|\theta| < \infty$;*

2° *the conditions of the uniqueness of the solutions are fulfilled;*

3° *$P(x, \theta)$ and $Q(x, \theta)$ are analytic with regard to x for $|x| < \infty$.*

Then the method of infinitesimal deformation which has been previously applied to the cycles by me [(2)] is also applicable to the periodic solutions of the second kind of the equations (E_1).

In this paper, first, we establish the general theory of infinitesimal deformation of the periodic solutions of the second kind of (E_1). Next, we apply the general results thus obtained to the equation of a pendulum as follows;

$$(P) \qquad \frac{d^2\theta}{dt^2} + \alpha f(\theta) \frac{d\theta}{dt} + g(\theta) = 0,$$

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1) N. Minorsky, Introduction to Non-Linear Mechanics, (1947), p.116.

2) M. Urabe, *Infinitesimal Deformation of Cycles*, J. Sci. Hiroshima Univ., Ser. A, 18 (1954), 37–53. In the following, we denote this paper by [P].

where $f(\theta)$ and $g(\theta)$ are the periodic integral functions and $f(\theta)>0$. Then it is shown that, for a certain range of α, there exists a unique periodic solution of the second kind having the fixed absolute stability[1]. Lastly, again making use of the method of infinitesimal deformation, we determine the boundary values of the range of α for which a periodic solution of the second kind exists.

Chapter I. Infinitesimal deformation of the periodic solution of the second kind.

§ 1. Deformation of the solution.

By our assumption, in any point except for the points where $Q(x, \theta)=0$, the function $X(x, \theta)$ is continuous with regard to (x, θ) and is analytic with regard to x for $|x|$, $|\theta|<\infty$. Consequently, for (E_2), the Lipschitz condition is locally fulfilled in any point except for the points where $Q(x, \theta)=0$. Without loss of generality, we may assume that the period of $P(x, \theta)$ and $Q(x, \theta)$ be 2π.

We consider the deformed equations of (E_1), which can be written as follows;

$$(1.1)\qquad \begin{cases} \dfrac{dx}{dt}=P_1(x, \theta, \varepsilon)=P(x, \theta)+\varepsilon H(x, \theta, \varepsilon), \\[2ex] \dfrac{d\theta}{dt}=Q_1(x, \theta, \varepsilon)=Q(x, \theta)+\varepsilon K(x, \theta, \varepsilon). \end{cases}$$

Here we assume that, for sufficiently small $|\varepsilon|$, $H(x, \theta, \varepsilon)$ and $K(x, \theta, \varepsilon)$ are continuous and periodic (with the period 2π) with regard to θ, and are analytic with regard to (x, ε) for $|x|<\infty$. Moreover we assume that, for (1.1), the conditions of uniqueness of the solutions are valid. Then, from (1.1), corresponding to (E_2), the following equation is deduced:

$$(1.2)\qquad \frac{dx}{d\theta}=X_1(x, \theta, \varepsilon)=X(x, \theta)+\varepsilon L(x, \theta, \varepsilon).$$

By our assumption, as for (E_2), for (1.2), the Lipschitz condition is locally fulfilled in any point except for the points where $Q_1(x, \theta, \varepsilon)=0$.

In the phase (θ, x)-plane, we take a point (θ_0, k) such that $Q(k, \theta_0)\neq 0$, then there exists a unique solution $x=x(\theta)$ of (E_2) such that $x(\theta_0)=k$. From the continuity of $Q_1(x, \theta, \varepsilon)$, for sufficiently small $|\varepsilon|$, $Q_1(x, \theta, \varepsilon)\neq 0$ in the sufficiently narrow neighborhood of (θ_0, k), consequently, for sufficiently small $|c|$ and $|\varepsilon|$,

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1) We say that, when the solution is stable or unstable, it has the absolute stability and that, when the solution is semi-stable, it has the half stability.

there exists a unique solution $x=x(\theta\,;c,\varepsilon)$ of (1.2) such that $x(\theta_0\,;c,\varepsilon)=k+c$. Put

$$x(\theta\,;\,c,\,\varepsilon)-x(\theta)=u(\theta\,;\,c,\,\varepsilon), \tag{1.3}$$

then, from (E_2) and (1.2), it follows that

$$\frac{du}{d\theta}=X_1(x+u,\theta,\varepsilon)-X(x,\theta)=U(u,\theta,\varepsilon). \tag{1.4}$$

Here $U(u,\theta,\varepsilon)$ is continuous with regard to θ and is analytic with regard to (u,ε). Consequently, for (1.4), the Lipschitz condition is locally fulfilled in any point except for the points where $Q(x,\theta)=0$ or $Q_1(x,\theta,\varepsilon)=0$. From the initial conditions of $x(\theta)$ and $x(\theta\,;c,\varepsilon)$, it follows that

$$u(\theta_0\,;\,c,\,\varepsilon)=c. \tag{1.5}$$

From the analyticity of $U(u,\theta,\varepsilon)$, the solution $u(\theta\,;c,\varepsilon)$ of (1.4) is expanded as follows:

$$u(\theta\,;c,\varepsilon)=u_0(\theta)+u_1(\theta\,;c,\varepsilon)+u_2(\theta\,;c,\varepsilon)+\cdots\cdots+u_n(\theta\,;c,\varepsilon)+\cdots\cdots,$$

where $u_n(\theta\,;c,\varepsilon)$ is a homogeneous polynomial of n-th degree with regard to (c,ε). Then, from (1.5), it follows that

$$u_0(\theta_0)=0,\ u_1(\theta_0;c,\varepsilon)=c,\ u_2(\theta_0;c,\varepsilon)=\cdots\cdots=u_n(\theta_0;c,\varepsilon)=\cdots=0. \tag{1.6}$$

Now, for $c=\varepsilon=0$, from (1.4), it follows that

$$du_0/d\theta=X(x+u_0,\theta)-X(x,\theta).$$

This is satisfied by $u_0=0$, consequently, by the uniqueness of the solution, $u_0(\theta)=0$ because of (1.6). Thus the expansion of $u(\theta\,;c,\varepsilon)$ becomes

$$u(\theta\,;\,c,\,\varepsilon)=u_1(\theta\,;\,c,\,\varepsilon)+u_2(\theta\,;\,c,\,\varepsilon)+\cdots+u_n(\theta\,;\,c,\,\varepsilon)+\cdots. \tag{1.7}$$

From the analyticity of $X_1(x,\theta,\varepsilon)$, it follows that

$$U(u,\theta,\varepsilon)=u\frac{\partial X(x,\theta)}{\partial x}+\varepsilon L_0(x,\theta)+\cdots, \tag{1.8}$$

where the unwritten terms are those of the second and higher orders with regard to (u,ε) and $L_0(x,\theta)=L(x,\theta,0)$. Substituting (1.7) and (1.8) into (1.4), we have:

$$\frac{du_1}{d\theta}+\frac{du_2}{d\theta}+\cdots+\frac{du_n}{d\theta}+\cdots=(u_1+u_2+\cdots+u_n+\cdots)\frac{\partial X(x,\theta)}{\partial x}+\varepsilon L_0(x,\theta)+\cdots.$$

Comparing the terms of the same degrees with regard to c and ε, we have:

$$
(1.9) \quad \begin{cases} \dfrac{du_1}{d\theta} = \dfrac{\partial X}{\partial x} u_1 + \varepsilon L_0, \\[2ex] \dfrac{du_n}{d\theta} = \dfrac{\partial X}{\partial x} u_n + R_n (u_1, u_2, \cdots u_{n-1}), \qquad (n \geqq 2) \end{cases}
$$

where R_n is a polynomial of $u_1, u_2, \cdots\cdots, u_{n-1}$. Put

$$
(1.10) \qquad h(\theta) = \int_{\theta_0}^{\theta} \frac{\partial X}{\partial x} d\theta.
$$

Then, by the initial conditions (1.6), the equations (1.9) are integrated as follows:

$$
(1.11) \quad \begin{cases} u_1(\theta\,; c, \varepsilon) = ce^{h(\theta)} + \varepsilon e^{h(\theta)} \displaystyle\int_{\theta_0}^{\theta} e^{-h(\theta)} L_0(x, \theta)\, d\theta, \\[2ex] u_n(\theta\,; c, \varepsilon) = e^{h(\theta)} \displaystyle\int_{\theta_0}^{\theta} e^{-h(\theta)} R_n d\theta. \qquad (n \geqq 2) \end{cases}
$$

Thus (1.7) becomes

$$
(1.12) \qquad u(\theta\,; c, \varepsilon) = \left(ce^h + \varepsilon e^h \int_{\theta_0}^{\theta} e^{-h} L_0 d\theta\right) + u_2(\theta\,; c, \varepsilon) + \cdots.
$$

§ 2. Stability of the periodic solution.

We assume that $Q\{x(\theta), \theta\} \neq 0$ for $\theta_0 \leqq \theta \leqq \theta_0 + 2\pi$, and that $x = x(\theta)$ is a periodic solution of (E_2), namely the solution corresponding to the periodic solution of the second kind of (E_1). Then it is evident that

$$
(2.1) \qquad x(\theta + 2\pi) = x(\theta).
$$

In this paragraph, making use of the results of the preceding paragraph, we shall investigate the stability of such a periodic solution. From the continuity of $Q(x, \theta)$, it is evident that, in the sufficiently narrow neighborhood of $x = x(\theta)$, $Q(x, \theta) \neq 0$. Therefore, for sufficiently small $|c|$, the solution $x = x(\theta\,; c, 0)$ lies in the neighborhood of $x = x(\theta)$ for $\theta_0 \leqq \theta \leqq \theta_0 + 2\pi$.

For $\varepsilon = 0$, from (1.12), it is valid that

$$
(2.2) \qquad u = u(\theta\,; c, 0) = ce^h + c^2 u_2(\theta) + \cdots + c^n u_n(\theta) + \cdots,
$$

where

$$
(2.3) \qquad u_n(\theta) = u_n(\theta\,; c, 0)/c^n.
$$

From (1.6) and (1.10), it is valid that

(2.4) $$u_2(\theta_0) = \cdots = u_n(\theta_0) = \cdots = 0, \quad h(\theta_0) = 0.$$

Put

(2.5) $$h_0 = h(\theta_0 + 2\pi) = \int_{\theta_0}^{\theta_0+2\pi} \frac{\partial X}{\partial x}\, d\theta,$$

then, from the periodicity of $X(x, \theta)$ and $x(\theta)$, it follows that

(2.6) $$h(\theta + 2\pi) = h(\theta) + h_0.$$

Now, so long as $c \neq 0$, from (2.2), it follows that

(2.7) $$u(\theta_0 + 2\pi; c, 0)/u(\theta_0; c, 0) = e^{h_0} + cu_2(\theta_0 + 2\pi) + \cdots + c^{m-1}u_m(\theta_0 + 2\pi) + \cdots.$$

Consequently, if $h_0 \neq 0$, it follows that

$$\begin{cases} \textit{when } h_0 < 0, \quad 0 < \dfrac{u(\theta_0+2\pi; c, 0)}{u(\theta_0; c, 0)} < 1, \\ \qquad \textit{namely } x = x(\theta) \textit{ is stable}; \\ \textit{when } h_0 > 0, \quad 1 < \dfrac{u(\theta_0+2\pi; c, 0)}{u(\theta_0; c, 0)}, \end{cases}$$

namely $x = x(\theta)$ is unstable.[1]

These are the conditions corresponding to those of orbital stability of Poincaré.

When $h_0 = 0$, $u_2(\theta_0 + 2\pi) = \cdots = u_{m-1}(\theta_0 + 2\pi) = 0$ *and* $u_m(\theta_0 + 2\pi) = a_m \neq 0$, (2.7) is written as follows:

$$\frac{u(\theta_0+2\pi; c, 0)}{u(\theta_0; c, 0)} = 1 + c^{m-1}u_m(\theta_0 + 2\pi) + \cdots.$$

Consequently, we have the conditions of stability as follows:

m	*sign of* a_m	*stability of* $x = x(\theta)$		
odd	$-$	*stable*		*absolute stability*
	$+$	*unstable*		
even	$-$	*stable for*	$c > 0$	*half stability*
		unstable for	$c < 0$	
	$+$	*unstable for*	$c > 0$	
		stable for	$c < 0$	

1) For the equation of the form (E_2), we mean the stability with regard to increasing θ. Consequently, for the equations of the forms (E_1) and (E_2), the stability is same or opposite according as $Q(x, \theta) > 0$ or < 0.

When $h_0 = 0$, $u_2(\theta_0 + 2\pi) = \cdots = u_m(\theta_0 + 2\pi) = \cdots = 0$, from (2.7), it follows that $u(\theta_0 + 2\pi\,; c, 0) = c$ for any c, namely *there appears a continuum of the periodic solutions.*

§ 3. Deformation of the periodic solution.

As in §2, we assume that $Q\{x(\theta), \theta\} \neq 0$ for $\theta_0 \leqq \theta \leqq \theta_0 + 2\pi$ and that $x = x(\theta)$ is a periodic solution of (E_2). From the continuity of $Q_1(x, \theta, \varepsilon)$, it is evident that, in the sufficiently narrow neighborhood of $x = x(\theta)$, $Q_1(x, \theta, \varepsilon) \neq 0$ for sufficiently small $|\varepsilon|$. Therefore, for sufficiently small $|c|$ and $|\varepsilon|$, the solution $x = x(\theta; c, \varepsilon)$ of (1.2) lies in the neighborhood of $x = x(\theta)$ for $\theta_0 \leqq \theta \leqq \theta_0 + 2\pi$.

We consider the quantity $\Phi(c, \varepsilon) = u(\theta_0 + 2\pi\,; c, \varepsilon) - u(\theta_0\,; c, \varepsilon)$. Then, whether or not, in the neighborhood of $x = x(\theta)$, there exists a periodic solution of (1.2), namely the periodic solution of the second kind of the deformed equations of (E_1), is decided by whether or not there exists a real root c of small absolute value of the equation $\Phi(c, \varepsilon) = 0$ for sufficiently small $|\varepsilon|$. From (1.12), the equation $\Phi(c, \varepsilon) = 0$ is expressed as follows:

$$\Phi(c, \varepsilon) = c\,(e^{h_0} - 1) + \varepsilon e^{h_0} I + u_2(\theta_0 + 2\pi\,; c, \varepsilon) + \cdots = 0, \tag{3.1}$$

where

$$I = \int_{\theta_0}^{\theta_0 + 2\pi} e^{-h} L_0\, d\theta. \tag{3.2}$$

Then, as in §4 of [P], we have the conclusion:

For sufficiently slightly deformed equations of (E_2), *in the neighborhood of* $x = x(\theta)$,

(i) *in the case where* $x = x(\theta)$ *has the absolute stability, there exists at least one periodic solution* $x = x(\theta\,; c, \varepsilon)$ *having the same absolute stability as that of* $x = x(\theta)$; *moreover the periodic solution* $x = x(\theta\,; c, \varepsilon)$ *is unique when* $h_0 \neq 0$ *or* $I \neq 0$, *and specially when* $I \neq 0$ *for any* θ_0, *the solution* $x = x(\theta\,; c, \varepsilon)$ *does not intersect with the solution* $x = x(\theta)$;

(ii) *in the case where* $x = x(\theta)$ *has the half stability,*

(a) *when* $\varepsilon I / a_m < 0$, *there exists one and only one periodic solution lying in each side of* $x = x(\theta)$ *in the phase* (θ, x)*-plane having the same stability as that of the side containing that periodic solution;*

(b) *when* $\varepsilon I / a_m > 0$, *there exists no periodic solution;*

(c) *when* $I = 0$, *the existence of a periodic solution is decided by that of a real root of the equation* (3.1); *when there exist the real roots* $c_1, c_2, \cdots, c_k$, *there exist* k *periodic solutions which are arranged in the phase plane according to the magnitude* *of $c_1, c_2, \cdots, c_k$; when $c_1, c_2, \cdots, c_k$ are all distinct, the corresponding periodic solutions have the alternating absolute stability; when some of the roots coincide with each other, the corresponding periodic solution has the absolute or half stability according as the number of coincident roots is odd or even;*

(iii) *in the case where there exists a continuum of the periodic solutions in the neighborhood of $x=x(\theta)$,*

(a) *when $I \neq 0$ for $x=x(\theta)$, there exists no periodic solution;*

(b) *when $I=0$ for $x=x(\theta)$, the existence of a periodic solution is decided by that of a real root of the equation as follows:*

$$v_1(\theta_0+2\pi\,;c,\varepsilon)+v_2(\theta_0+2\pi\,;c,\varepsilon)+\cdots=0, \tag{3.3}$$

where

$$v_{n-1}(\theta_0+2\pi\,;c,\varepsilon)=u_n(\theta_0+2\pi\,;c,\varepsilon)/\varepsilon.$$

§ 4. Deformation of a continuum of the periodic solutions.

In the case where there exists a continuum of the periodic solutions, the equation (3.3) by which the existence of a periodic solution is decided is not of the convenient form since it begins with the term u_2. Therefore, in this paragraph, as in §5 of [P], we deduce the equation of the different form which is convenient to decide the existence of a periodic solution.

In the case where there exists a continuum of the periodic solutions in the neighborhood of $x=x(\theta)$, the solution of (E_2) in the neighborhood of $x=x(\theta)$ is expressed as follows:

$$x=x(\theta)+u(\theta\,;c) \tag{4.1}$$

where

$$u(\theta\,;c)=u(\theta\,;c,0)=ce^h+c^2\dot{u}_2(\theta)+\cdots+c^n u_n(\theta)+\cdots.$$

Making use of the letter a instead of c, we express the equation (4.1) as follows:

$$x=x(\theta\,;a). \tag{4.2}$$

Then, evidently, for sufficiently small $|a|$, $x(\theta\,;a)$ is analytic with regard to a. Consequently, if we write the function $h(\theta)$ for $x=x(\theta\,;a)$ as $h(\theta\,;a)$, $h(\theta\,;a)$ is analytic with regard to a. If we write the function $u(\theta\,;c,\varepsilon)$ for $x=x(\theta\,;a)$ as $u(\theta\,;c,\varepsilon,a)$, then, from the analyticity of the equation (1.4), $u(\theta\,;c,\varepsilon,a)$ is also analytic with regard to a. Let the integral (3.2) for $x=x(\theta\,;a)$ be $I(a)$, then it is readily seen that $I(a)$ is also analytic with regard to a.

If we express the quantity $\Phi(c, \varepsilon)$ for $x=x(\theta; a)$ as $\Phi(c, \varepsilon, a)$, then, from (3.1), it follows that

$$\Phi(0, \varepsilon, a)=\varepsilon\left[I(a)+\varepsilon \pi_1(a)+\cdots+\varepsilon^{m-1} \pi_{m-1}(a)+\cdots\right], \tag{4.3}$$

where

$$\pi_{m-1}(a)=u_m(\theta_0+2\pi ; 0, \varepsilon, a) / \varepsilon^m .$$

Then, as in §5 of [P], we have the conclusion:

I. The case where $I(a) \not\equiv 0$. *When there exists no real root of $I(a)=0$, there exists no periodic solution of the deformed equation in the neighborhood of $x=x(\theta)$. When there exists a real root a_0 of small absolute value of the equation $I(a)=0$, let the multiplicity of the root a_0 be p. Then we have;*

1° *when p is odd, there exists at least one periodic solution of the deformed equation which is stable or unstable according as $\varepsilon I^{(p)}(a_0)<0$ or >0; specially when $p=1$ or $\pi_1(a_0) \neq 0$, the periodic solution is unique;*

2° *when p is even,*

(a) *when $\varepsilon \pi_1(a_0) / I^{(p)}(a_0)<0$, there exist two and only two periodic solutions of the deformed equation having the opposite absolute stability;*

(b) *when $\varepsilon \pi_1(a_0) / I^{(p)}(a_0)>0$, there exists no periodic solution of the deformed equation;*

(c) *when $\pi_1(a_0)=0$, the existence of a periodic solution of the deformed equation is decided by the existence of a real root of the equation* (4.3); *when there exist the periodic solutions, their stability is of the same character as in the case* (ii) (c) *of* §3.

II. The case where $I(a) \equiv 0$. *When $\pi_1(a) \equiv \cdots\cdots \equiv \pi_{m-1}(a) \equiv \cdots\cdots\cdots \equiv 0$, there appears again a continuum of the periodic solutions of the deformed equation. When $\pi_1(a) \equiv \cdots\cdots \equiv \pi_{p-1}(a) \equiv 0$, $\pi_p(a) \not\equiv 0$, we have:*

(i) *when there exists no real root of $\pi_p(a)=0$, there exists no periodic solution of the deformed equation in the neighborhood of $x=x(\theta)$;*

(ii) *when there exists a real root a_0 of multiplicity q of the small absolute value of the equation $\pi_p(a)=0$, we have:*

1° *when q is odd, there exists at least one periodic solution of the deformed equation, which is stable or unstable according as $\varepsilon^{p+1} \pi_p^{(q)}(a_0)<0$ or >0; specially when $q=1$ or $\pi_{p+1}(a_0) \neq 0$, the periodic solution is unique;*

2° *when q is even,*

(a) *when $\varepsilon \pi_{p+1}(a_0) / \pi_p^{(q)}(a_0)<0$, there exist two and only two periodic solutions* *of the deformed equation having the opposite absolute stability;*

(b) *when* $\varepsilon\pi_{p+1}(a_0)/\pi_p^{(q)}(a_0)>0$, *there exists no periodic solution of the deformed equation;*

(c) *when* $\pi_{p+1}(a_0)=0$, *the existence of a periodic solution of the deformed equation is decided by the existence of a real root of the equation* (4.3); *when there exist the periodic solutions, their stability is of the same character as in the case* (ii) (c) *of* §3.

§ 5. Motion of the periodic solution.

In this paragraph, we consider the system of the equations depending on one parameter as follows:

$$\frac{dx}{dt}=P(x,\theta,\alpha),\quad \frac{d\theta}{dt}=Q(x,\theta,\alpha), \tag{5.1}$$

where $P(x,\theta,\alpha)$ and $Q(x,\theta,\alpha)$ satisfy the conditions as follows:

1° *they are continuous with regard to* θ *for* $|\theta|<\infty$;

2° *they are analytic with regard to* (x,α) *for* $|x|$, $|\alpha|<\infty$;

3° *they have a common period independent of* α *with regard to* θ;

4° *for the equations* (5.1), *the conditions of uniqueness of the solutions are fulfilled.*

Corresponding to (5.1), we consider the equation as follows:

$$\frac{dx}{d\theta}=\frac{P(x,\theta,\alpha)}{Q(x,\theta,\alpha)}\equiv X(x,\theta,\alpha), \tag{5.2}$$

then, in any point except for the points where $Q(x,\theta,\alpha)=0$, namely the singularities of (5.2), $X(x,\theta,\alpha)$ is continuous with regard to θ and analytic with regard to (x,α) for $|x|$, $|\theta|$, $|\alpha|<\infty$. Consequently, for (5.2), the Lipschitz condition is locally fulfilled in any point except for the singularities. Without loss of generality, we may assume that the common period of $P(x,\theta,\alpha)$ and $Q(x,\theta,\alpha)$ be 2π. We denote the periodic solution of (5.2) by $x=x(\theta,\alpha)$. If there exists a periodic solution $x=x(\theta,\alpha_0)$ for $\alpha=\alpha_0$, then, for the equation (5.2) for $\alpha=\alpha_0+\delta\alpha$, the discussions of §§3 and 4 are applied by putting $\delta\alpha=\varepsilon$ and $\partial X/\partial\alpha_0=L_0$.

Then, as in §6 of [P], we have the following theorems.

Theorem 1. *When, in the neighborhood of the periodic solution* $x=x(\theta,\alpha_0)$, *there exist the continuums of the periodic solutions for* $\alpha=\alpha_0$ *and* $\alpha=\alpha_0+\delta\alpha$, *each periodic solution* $x=x(\theta,\alpha_0)$ *varies continuously forming a periodic solution from the initial solution to the periodic solution for* $\alpha=\alpha_0+\delta\alpha$.

Theorem 2. *If there exist k periodic solutions* $x=x(\theta,\ \alpha_0+\delta\alpha)$ *in the neighborhood of* $x=x(\theta,\alpha_0)$, *then k periodic solutions* $x=x(\theta,\ \alpha_0+\varepsilon)$ *varies continuously keeping the stability unaltered from* $x=x(\theta,\alpha_0)$ *to* $x=x(\theta,\ \alpha_0+\delta\alpha)$ *as* ε *varies monotonely.*

In Theorem 2, if $I\neq 0$ for any point of $x=x(\theta,\alpha_0)$, then, in the phase (θ,x)-plane, the trajectory $x=x(\theta,\ \alpha_0+\varepsilon)$ varies monotonely upwards or downwards without intersecting with each other.

As in §6 of [P], when there exists always at least one periodic solution $x=x(\theta,\ \alpha+\delta\alpha)$ in the neighborhood of the periodic solution $x=x(\theta,\alpha)$ for $\delta\alpha$ of the fixed sign of the sufficiently small absolute value, we say that *the condition of positive or negative continuation* is fulfilled according as $\delta\alpha>0$ or <0.

Here, for example, let us assume the condition of positive continuation and, increasing α from α_0, suppose that, for any α such that $\alpha_0\leqq\alpha<\alpha'<\infty$, there exists a periodic solution $x=x(\theta,\alpha)$. We assume that, in the phase (θ,x)-plane, the trajectories $\{x=x(\theta,\alpha)\}$ are uniformly bounded and $x=x(\theta,\alpha)$ does not tend to the singularities of (5.2) for $\alpha=\alpha'$ as $\alpha\to\alpha'$.

In the phase (θ,x)-plane, let any point be P, to which $x=x(\theta,\alpha)$ tends as $\alpha\to\alpha'$. Then there exist the numbers α_n's and the points P_n's of $x=x(\theta,\alpha_n)$ such that $\alpha_0\leqq\alpha_n<\alpha'$ and $P_n\to P$ as $\alpha_n\to\alpha'$. From our assumption, P lies at the finite distance and is an ordinary point of the equation (5.2) for $\alpha=\alpha'$. Then there exists a unique trajectory $x=x(\theta,\alpha')$ of (5.2) for $\alpha=\alpha'$ in the neighborhood of P. Extending the trajectory $x=x(\theta,\alpha')$ in both sides of P, let the maximal interval be (θ',θ'') in which $x(\theta,\alpha')$ is defined. We shall show that $\theta'=-\infty$ and $\theta''=+\infty$. For example, let θ'' be finite. Let any point be Q, to which the trajectory $x=x(\theta,\alpha')$ tends as $\theta\to\theta''$. Then there exists a sequence $\{\theta_n\}$ such that $A(\alpha',\ P,\ \theta_n)\to Q$ as $\theta_n\to\theta''$, where $A(\alpha',\ P,\ \theta_n)$ denotes the point at $\theta=\theta_n$ of the trajectory $x=x(\theta,\alpha')$ passing through the point P. Namely, when Q lies at infinity, for any given large positive number G, it is valid that $|x(\theta_n,\alpha')|>G+1$ for sufficiently large n, and, when Q lies at the finite distance, for any given small positive number $\eta\,(\eta<1)$, it is valid that $\overline{A(\alpha',P,\theta_n)Q}<\eta/2$[1] for sufficiently large n. Now, from the continuity of the solutions, for sufficiently large m_n, it is valid that $\overline{A(\alpha_{m_n},P_{m_n},\theta_n)\,A(\alpha',P,\theta_n)}<\eta/2$. Then, when Q lies at infinity, $|x(\theta_n,\alpha_{m_n})|>G+(1-\frac{\eta}{2})>G$. This contradicts the assumption that $\{x(\theta,\alpha)\}$ are uniformly bounded. When Q lies at the finite distance, $\overline{A(\alpha_{m_n},P_{m_n},\theta_n)Q}<\eta$. This means that Q is a point to which the trajectory $x=x(\theta,\alpha_{m_n})$ tends. Therefore, by the assumption, Q is an ordinary point of (5.2) for $\alpha=\alpha'$. Then we can extend

1) The upper bar denotes the distance.

the trajectory $x=x(\theta, \alpha')$ beyond $\theta=\theta''$. This contradicts the assumption that (θ', θ'') is a maximal interval of definition of $x(\theta, \alpha')$. Thus it must be that $\theta''=+\infty$. Likewise it must be that $\theta'=-\infty$.

From the continuity of the solution, it follows that $A(\alpha_n, P_n, \theta_0) \to A(\alpha', P, \theta_0)$, $A(\alpha_n, P_n, \theta_0+2\pi) \to A(\alpha', P, \theta_0+2\pi)$ as $\alpha_n \to \alpha'$. Now, from the periodicity, $x(\theta_0, \alpha_n)=x(\theta_0+2\pi, \alpha_n)$. Consequently it must be that $x(\theta_0, \alpha')=x(\theta_0+2\pi, \alpha')$, that is to say that there exists a periodic solution $x=x(\theta, \alpha')$ of (5.2) for $\alpha=\alpha'$. Then, by the condition of positive continuation, for sufficiently small positive $\delta\alpha'$, there exists at least one periodic solution of (5.2) for $\alpha=\alpha'+\delta\alpha'$.

When the condition of negative continuation is assumed, the similar results are also obtained. Thus, corresponding to Theorem 3 of [P], we have

Theorem 3. *When the condition of positive continuation is fulfilled, if $\alpha'<\infty$ is a least upper bound of $\overline{\alpha}$ such that, for any α such that $\alpha_0 \leqq \alpha \leqq \overline{\alpha}$, the periodic solution $x=x(\theta, \alpha)$ of (5.2) exists, then, in the phase (θ, x)-plane, as $\alpha \to \alpha'$, the trajectory $x=x(\theta, \alpha)$ tends to either singularities of (5.2) or points at infinity. When the condition of negative continuation is fulfilled, the conclusion is also valid if $\alpha'>-\infty$ is a greatest lower bound of $\underline{\alpha}$ such that, for any α such that $\underline{\alpha} \leqq \alpha \leqq \alpha_0$, the periodic solution $x=x(\theta, \alpha)$ of (5.2) exists.*

Chapter II. The periodic solution of the equation of a pendulum.

§ 6. Preliminaries.

In this chapter, we apply the general theory of the preceding chapter to the equation of a pendulum as follows:

$$\frac{d^2\theta}{dt^2}+\alpha f(\theta)\frac{d\theta}{dt}+g(\theta)=0, \tag{6.1}$$

where $f(\theta)$ and $g(\theta)$ are the periodic integral functions and $f(\theta)>0$. Without loss of generality, we may assume that $\alpha \geqq 0$ and the period of $f(\theta)$ and $g(\theta)$ is 2π. The equation (6.1) is as usual transformed to the simultaneous equations of the first order as follows:

$$\begin{cases} \dfrac{d\theta}{dt}=z, \\[2ex] \dfrac{dz}{dt}=-\alpha f(\theta)z-g(\theta). \end{cases} \tag{6.2}$$

Corresponding to (6.2), we consider the equation as follows:

$$\frac{dz}{d\theta}=\frac{-\alpha f(\theta)z-g(\theta)}{z}. \tag{6.3}$$

The singularities of (6.3) are the points where $z=0$.

Put $z=1/x$, then the equation (6.3) is transformed to the equation as follows:

$$\frac{dx}{d\theta}=\alpha f(\theta)x^2+g(\theta)x^3=X(x,\theta,\alpha). \tag{6.4}$$

For the equation (6.4), there does not exist any singularity at the finite distance. Besides, it is readily seen that, in the phase (θ, x)-plane, the trajectories of (6.4) do not intersect with the line $x=0$ except for the trajectory $x=0$, consequently the trajectories lie in a half plane separated by the line $x=0$. If we suppose α as a parameter, then, for $\alpha=\alpha_0+\varepsilon$, from (1.2), it follows that

$$X(x,\theta)=\alpha_0 f(\theta)x^2+g(\theta)x^3,\quad L(x,\theta,\varepsilon)=f(\theta)x^2, \tag{6.5}$$

consequently, from (1.10) and (3.2), putting $\theta_0=0$, we have:

$$h(\theta)=\int_0^\theta [2\alpha_0 f(\theta)x+3g(\theta)x^2]\,d\theta, \tag{6.6}$$

and

$$I=\int_0^{2\pi} e^{-h(\theta)}f(\theta)\,x^2 d\theta. \tag{6.7}$$

From (6.7), it is readily seen that I of (3.2) is positive for any θ_0 except for the solution $x=0$.

§ 7. The solutions of (6.4) for $\alpha=0$.

In this case the equation (6.4) is easily integrated, and the solution such that $x(0)=k$ is given as follows:

$$x=\frac{k}{\sqrt{1-2k^2G(\theta)}}, \tag{7.1}$$

where

$$G(\theta)=\int_0^\theta g(\theta)\,d\theta. \tag{7.2}$$

When $G(2\pi)=0$, $G(\theta)$ is periodic and $x(2\pi)=x(0)=k$. Consequently, except for the solutions which tend to infinity, all the solutions are periodic, that is to say that the solutions for certain range of k containing zero constitute a continuum of the periodic solutions. When $G(2\pi) \neq 0$, the solution $x=0$ is a unique periodic solution and it is stable or unstable according as $G(2\pi)<0$ or >0. The feature of the trajectories in the phase (θ, x)-plane is shown in Fig. 1.

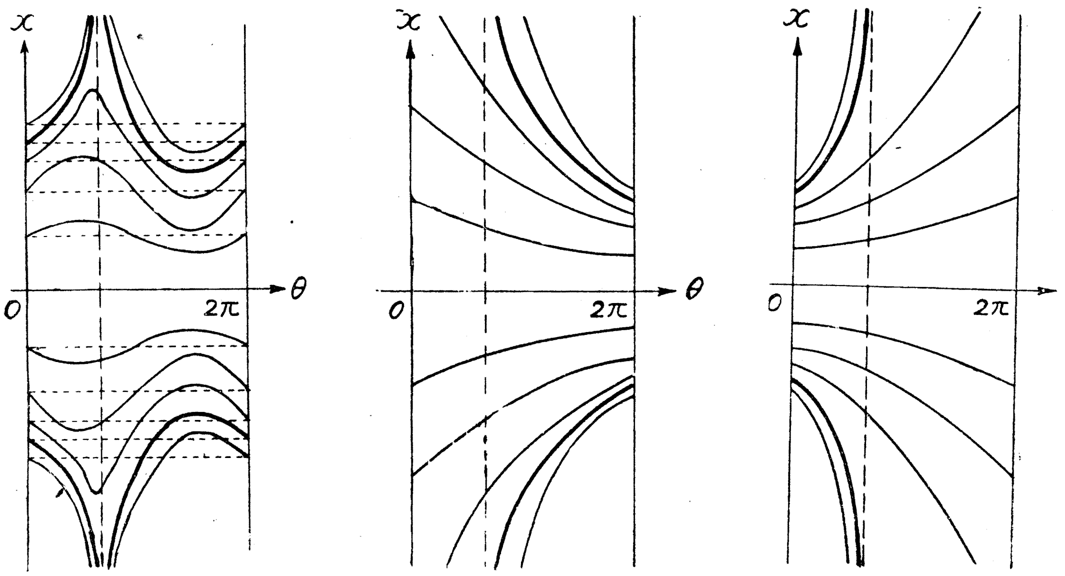


Fig. 1.

§ 8. The periodic solutions of (6.4) for $0<\alpha \leqslant 1$.

In this paragaph we seek for the periodic solutions of (6.4) for $0<\alpha \leqslant 1$ by means ofthe theory of the preceding chapter.

First we consider the case where $G(2\pi) \neq 0$. According to the notations of the preceding chapter, we write ε instead of α. In the neighborhood of the solution $x=0$, we consider the solution $x=x(\theta;\ c,\ \varepsilon)$ of (6.4) such that $x(0;\ c,\ \varepsilon)=c$. For (6.4), the periodic solution $x=x(\theta)$ of § 3 is $x=0$, consequently the function $u(\theta;\ c,\ \varepsilon)$ defined by (1.3) is equal to $x(\theta;\ c,\ \varepsilon)$. Now $u(\theta;0,\varepsilon)=x(\theta;0,\varepsilon)=0$, consequently $u(\theta;\ c,\ \varepsilon)$ can be written as follows:

(8.1) $$u(\theta;c,\varepsilon)=c\left[w_0(\theta)+w_1(\theta;c,\varepsilon)+w_2(\theta;c,\varepsilon)+\cdots\right].$$

From the condition that $x(0;c,\varepsilon)=c$, it must be that

(8.2) $$w_0(0)=1, \quad w_1(0;c,\varepsilon)=w_2(0;c,\varepsilon)=\cdots=0.$$

Substitute (8.1) into (6.4), then, comparing the terms of the same degrees with regard to (c, ε), we have:

$$\begin{cases} \dfrac{dw_0}{d\theta} = 0, \quad \dfrac{dw_1}{d\theta} = 0, \\ \dfrac{dw_2}{d\theta} = c\,[\varepsilon f w_0^2 + c g w_0^3], \\ \cdots\cdots\cdots\cdots, \\ \dfrac{dw_n}{d\theta} = c\,[\cdots\cdots], \\ \cdots\cdots\cdots\cdots. \end{cases}$$

Consequently, from (8.2), it follows that

$$w_0 = 1, \quad w_1 = 0, \quad w_2 = c\,[\varepsilon \int_0^\theta f(\theta)\,d\theta + cG(\theta)], \cdots,$$

$$w_n = c\,[\cdots\cdots], \quad \cdots\cdots,$$

that is to say that

$$u(\theta\,; c, \varepsilon) = c + c^2\Big[\Big\{cG(\theta) + \varepsilon\int_0^\theta f(\theta)\,d\theta\Big\} + \Big\{\cdots\cdots\Big\} + \cdots\cdots\Big], \tag{8.3}$$

where the unwritten terms in the square brackets are those of the second and higher orders with regard to (c, ε).

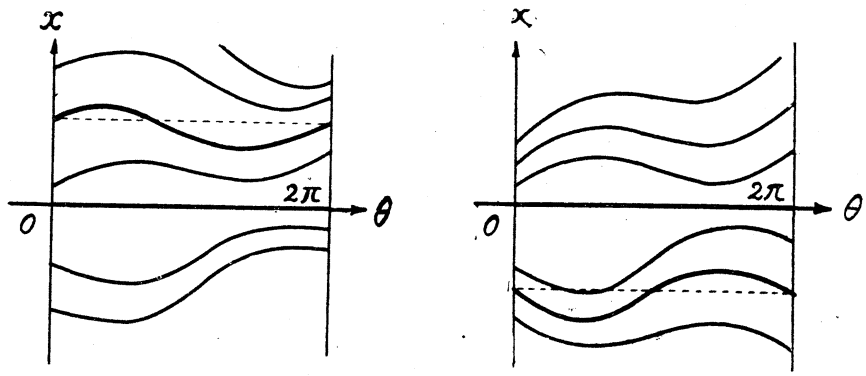
From (8.3), the function $\Phi(c, \varepsilon) = u(2\pi\,; c, \varepsilon) - u(0\,; c, \varepsilon)$ becomes

$$\Phi(c, \varepsilon) = c^2\Big[\Big\{cG(2\pi) + \varepsilon\int_0^{2\pi} f(\theta)\,d\theta\Big\} + \Big\{\cdots\cdots\Big\} + \cdots\cdots\Big]. \tag{8.4}$$

Consequently the equation $\Phi(c, \varepsilon) = 0$ has a unique non-vanishing real root c_0 which is written as follows:

$$c_0 = -\frac{\varepsilon}{G(2\pi)}\int_0^{2\pi} f(\theta)\,d\theta + \cdots. \tag{8.5}$$

Since $f(\theta) > 0$ and $\varepsilon > 0$, we see that, *except for* $x=0$, *there exists a unique periodic solution which is stable and positive or unstable and negative according as* $G(2\pi) < 0$ *or* > 0. From (8.4), it is seen that *the solution* $x = 0$ *is stable for the negative side and is unstable for the positive side, namely that* $x=0$ *is semi-stable.* The feature of the solutions is shown in Fig. 2.



$G(2\pi) < 0$ $\qquad$ $G(2\pi) > 0$

Fig. 2.

Now we shall show that, for any $\alpha > 0$, there can not exist the periodic solutions more than one except for $x=0$. If there exist two periodic solutions $x = x_i(\theta)$ $(i = 1, 2)$ except for $x = 0$, then, from (6.4), for these solutions, it is valid that

$$\frac{1}{x_i^3}\frac{dx_i}{d\theta} = \alpha f(\theta) \cdot \frac{1}{x_i} + g(\theta).$$

Integrating both sides of these equations from $\theta = 0$ to $\theta = 2\pi$, we have

$$\alpha \int_0^{2\pi} \frac{f(\theta)}{x_i(\theta)}\, d\theta + G(2\pi) = 0,$$

since $x_i(2\pi) = x_i(0)$. Then it follows that

$$\alpha \int_0^{2\pi} f(\theta) \left(\frac{1}{x_1(\theta)} - \frac{1}{x_2(\theta)}\right) d\theta = 0.$$

Since α, $f(\theta) > 0$ and $(1/x_1(\theta)) - (1/x_2(\theta))$ is of the fixed sign, it must be that $x_1(\theta) = x_2(\theta)$, namely the periodic solution must be unique except for $x = 0$. Thus we see that, *for* $0 < \alpha \ll 1$, *there does not exist any periodic solution besides the periodic solutions obtained above in the neighborhood of the line* $x = 0$.

Next we consider the case where $G(2\pi) = 0$. In this case, it can be shown that, for any $\alpha > 0$, there exists no periodic solution except for $x = 0$. If there exists a periodic solution $x = x(\theta) \neq 0$, then, from (6.4), for this solution, it is valid that

$$\frac{1}{x^3}\frac{dx}{d\theta} = \alpha f(\theta) \cdot \frac{1}{x} + g(\theta).$$

Integrating both sides of this equation from $\theta = 0$ to $\theta = 2\pi$, we have

$$\alpha\int_0^{2\pi}\frac{f(\theta)}{x(\theta)}\,d\theta = 0.$$

This is a contradiction since α, $f(\theta) > 0$ and $x(\theta)$ is of the fixed sign. Thus we see that *there exists no periodic solution except for* $x=0$. For the solution $x=0$, from (8.4), we see that the periodic solution $x=0$ has the same stability as in the case where $G(2\pi)\neq 0$.

§ 9. Motion of the periodic solution.

First we investigate the motion of the solution $x=0$ for $\alpha=\alpha_0$. By §8, for $0<\alpha_0\leqslant 1$, the solution $x=0$ is semi-stable and, for this solution, from (6.7), $I=0$. Consequently, in order to investigate the motion of the periodic solution $x=0$, we must consider the terms of the second and higher orders in the expression (3.1). For this purpose, we seek for the expansion formula of the solution $x=x(\theta\,;c,\varepsilon)$ of (6.4) for $\alpha=\alpha_0+\varepsilon$ such that $x(0:c,\varepsilon)=c$. Then, in like manner as (8.3) is deduced, the following formula is deduced:

$$x(\theta\,;c,\varepsilon) = c + c^2\left[\alpha_0\int_0^{\theta} f(\theta)\,d\theta + \left\{\cdots\cdots\right\} + \cdots\cdots\right],$$

where the unwritten terms in the square brackets are those of the first and higher orders with regard to (c,ε). Consequently the function $\Phi(c,\varepsilon)$ defined by (3.1) becomes

$$\Phi(c,\varepsilon) = c^2\left[\alpha_0\int_0^{2\pi} f(\theta)\,d\theta + \left\{\cdots\cdots\right\} + \cdots\cdots\right].$$

Since $f(\theta)>0$, $\Phi(c,\varepsilon)>0$ for $c\neq 0$. Thus we see that, when α increases from $\alpha_0>0$, the periodic solution $x=0$ stays fixed and keeps the stability unaltered.

Next we investigate the motion of the periodic solution $x=x(\theta)$ distinct from $x=0$. By §8, for $0<\alpha_0\leqslant 1$, there exists one and only one periodic solution, which has the absolute stability. Now, from (6.7), for any point of such solution, $I>0$. Therefore, by §3, the condition of both continuations is valid, consequently the trajectory of such periodic solution moves monotonely upwards or downwards keeping the stability unaltered as α increases. Then, from §§2 and 8, when $h_0\neq 0$, h_0 and $G(2\pi)$ are at the same time negative or positive according as the periodic solution is stable or unstable. Since the unique real solution of (3.1) is $c=\dfrac{\varepsilon e^{h_0}}{1-e^{h_0}}I+\cdots$, the trajectory of periodic solution moves upwards or downwards according as the solution is stable or unstable. When $h_0 = 0$, the unique real solution of (3.1) is $c = \left(-\frac{\varepsilon}{a_m} I \right)^{\frac{1}{m}} + \cdots$, where m is odd since the periodic solution has the absolute stability. Now $a_m < 0$ or > 0 according as the periodic solution is stable or unstable. Consequently the periodic solution moves in like manner as when $h_0 \neq 0$. Thus we have the conclusion:

The unique periodic solution $x = x(\theta)$ distinct from $x = 0$ which appears for $0 < \alpha \ll 1$, moves monotonely upwards or downwards from $x = 0$ according as $G(2\pi) < 0$ or > 0 and this periodic solution is always stable or unstable according as $G(2\pi) < 0$ or > 0.

If we return to the equation (6.3) from (6.4) by means of the substitution $z = 1/x$, then the solution $x = 0$ of (6.4) becomes a line at infinity, consequently it does not give a periodic solution of (6.3). Thus the results obtained on (6.4) are stated on (6.3) as follows:

When $G(2\pi) = 0$, there exists no periodic solution except for the case where $\alpha = 0$. In the case where $\alpha = 0$, there appears a continuum of the periodic solutions.

When $G(2\pi) \neq 0$, for a suitable range of α such that $0 < \alpha < \alpha'$, there exists a unique periodic solution, the trajectory oj which lies in the half plane of the phase (θ, z)-plane separated by $z = 0$. This periodic solution is stable and positive or unstable and negative according as $G(2\pi) < 0$ or > 0. When α increases from 0, the trajectory of the periodic solution moves monotonely downwards or upwards from infinity according as $G(2\pi) < 0$ or > 0. For $\alpha = 0$, there exists no periodic solution.

Chapter III. The least upper bound of the value of the parameter for which a periodic solution exists.

§ 10. Preliminaries.

In this chapter, we seek for the least upper bound of α for which a periodic solution of (6.3) exists. For this purpose, first, we seek for the least upper bound α' of $\bar{\alpha}$ such that, for $0 \leqq \alpha \leqq \bar{\alpha}$, there exists a periodic solution of (6.3). When $G(2\pi) = 0$, from § 9, it is evident that $\alpha' = 0$. Now, if we put $\theta = -\theta'$ in (6.1), we have:

$$\frac{d\theta'}{dt^2} + \alpha f(-\theta') \frac{d\theta'}{dt} - g(-\theta') = 0.$$

Put $f(-\theta') = f_1(\theta')$ and $-g(-\theta') = g_1(\theta')$, then the above equation becomes

$$\frac{d^2\theta'}{dt^2} + \alpha f_1(\theta')\frac{d\theta'}{dt} + g_1(\theta') = 0.$$

Of course $f_1(\theta')$ and $g_1(\theta')$ have the period 2π and $f_1(\theta')>0$. Now $G_1(2\pi)$ corresponding to the function $g_1(\theta')$ becomes $G_1(2\pi) = -\int_0^{2\pi} g(-\theta')\,d\theta' = \int_0^{-2\pi} g(\theta)\,d\theta$ $= -G(2\pi)$. Consequently, when $G(2\pi) \neq 0$, without loss of generality, we may assume that $G(2\pi) < 0$. In the following we assume this. Then, in the phase (θ, z)-plane, the trajectory of the periodic solution of (6.3) lies in the upper half plane separated by the line $z=0$.

By Theorem 3 of §5 and the results of §9, if α' is finite, the periodic solution of (6.3) must tend to the point on the line $z=0$ as $\alpha \to \alpha' - 0$. Now, for any finite value of α, the solution of (6.3) crosses the line $z=0$ at the right angle in any point except for the critical points of (6.2). Consequently, when $\alpha' < \infty$, the periodic solutions of (6.3) can not tend to the ordinary point on the line $z=0$, namely *they must tend to some of the critical points of* (6.2). Now the critical points of (6.2) are the points where $z=0$ and $g(\theta)=0$. Therefore we see that, *if* $g(\theta) \neq 0$ *for any* θ, *then it must be that* $\alpha' = +\infty$, for, in this case, there exists no critical point.

§ 11. The character of the critical points of (6.2).

From (6.3),

$$\frac{dz}{d\theta} = -\frac{\alpha f}{z}\left(z + \frac{g}{\alpha f}\right). \tag{11.1}$$

Let the curve $z + \dfrac{g}{\alpha f} = 0$ in the phase (θ, z)-plane be Δ. Then, from $\alpha f > 0$, it is seen that $dz/d\theta > 0$ in the region Ω bounded by the line $z=0$ and the curve Δ, and that $dz/d\theta < 0$ in the complementary region Ω' of Ω. Then, making use of the method by which S. Lefschetz has dealt with the critical points[1], we can see the behaviour of the trajectories in the neighborhood of the critical points. For our purpose it needs only to know the behaviour of the trajectories in the upper half plane separated by θ-axis, therefore the results are stated only for the upper half plane:

The critical point $(\theta_0, 0)$ such that $g(\theta) < 0$ for $\theta < \theta_0$ and $g(\theta) > 0$ for $\theta > \theta_0$,

1) S. Lefschetz, *Notes on Differential Equations*, Contributions to the Theory of Nonlinear Oscillations, Vol. II edited by S. Lefschetz, (1952), 61-67.

is of the character of a focus, a center, a node or of a multiple point of these as shown in Fig. 3. We shall call such a critical point a *focus-like* point.

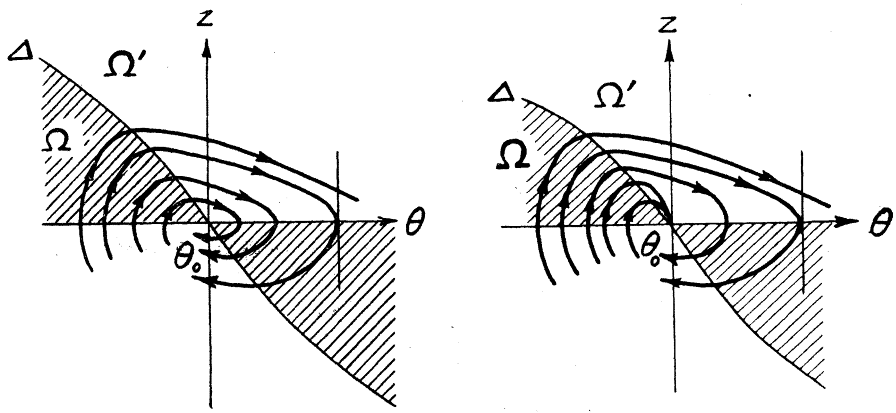


Fig. 3.

The critical point $(\theta_0, 0)$ such that $g(\theta) > 0$ for $\theta < \theta_0$ and $g(\theta) < 0$ for $\theta > \theta_0$, is of the character of a saddle as shown in Fig. 4. We shall call such a critical point *a saddle-like point* and the trajectories passing through the critical point the *separatrices*. Of the separatrices, those which make the upper and lower boundaries will be called the greatest and least separatrices respectively.

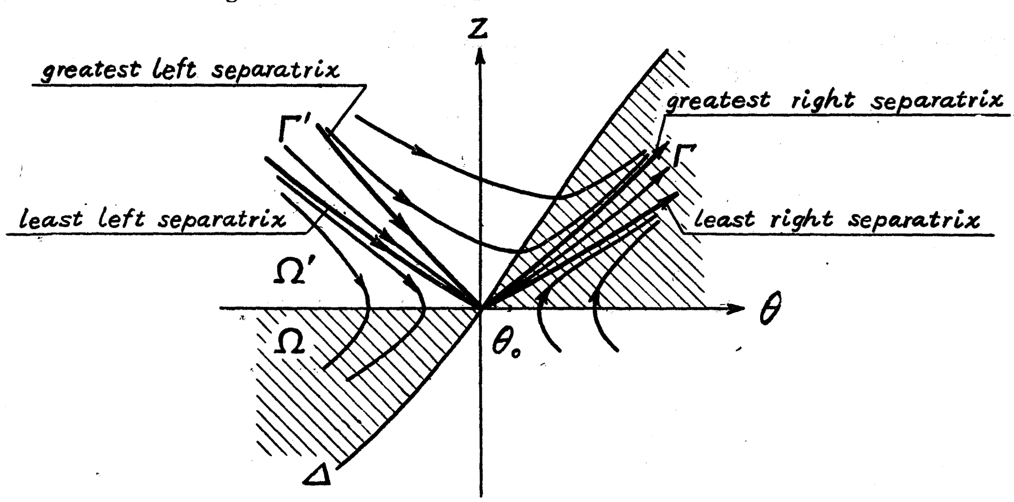


Fig. 4.

The critical point $g(\theta) > 0$ for $\theta \gtrless \theta_0$ is a mixed point consisting of left half of a saddle-like point and of right half of a focus-like point as shown in Fig. 5, consequently it has the left separatrices, but none of the right separatrices.

The critical point $(\theta_0, 0)$ such that $g(\theta) < 0$ for $\theta \gtrless \theta_0$ is a mixed point consisting of left half of a focus-like point and of right half of a saddle-like point, consequently it has certainly the right separatrices. However, the existence of the left separatrices is uncertain. In the sequel, we shall show that it has certainly also the left separatrices. Take any negative number c such that $0 > c > -\alpha f(\theta_0)$, and consider the curve Δ': $z = -g(\theta) / [c + \alpha f(\theta)]$. Then, for $|\theta - \theta_0| \ll 1$, Δ' lies above Δ. Now

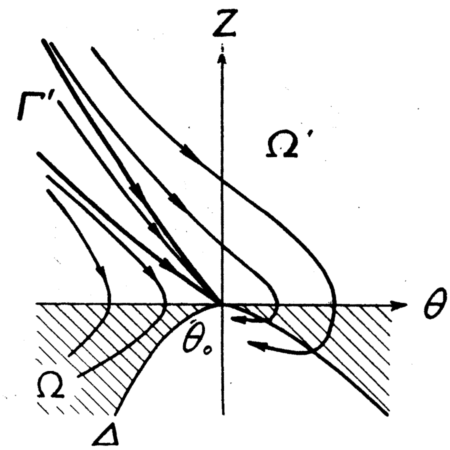


Fig. 5.

$$c - \frac{dz}{d\theta} = c - \left\{ -\alpha f(\theta) - \frac{g(\theta)}{z} \right\} = \frac{c + \alpha f(\theta)}{z} \cdot \left\{ z + \frac{g(\theta)}{c + \alpha f(\theta)} \right\} > 0$$

for any point above Δ', consequently, in this domain, in the neighborhood of the critical point $(\theta_0, 0)$, the left half of a trajectory passing through a point (θ_0, k) for any $k > 0$ lies above the line $z - k = c(\theta - \theta_0)$. We consider the intersections of the line $z = c(\theta - \theta_0)$ with Δ'. They are determined by the equation as follows:

(11.2) $$-\frac{g(\theta)}{c + \alpha f(\theta)} = c(\theta - \theta_0).$$

From our assumption, for $|\theta - \theta_0| \ll 1$, $g(\theta)$ can be expanded as follows:

$$g(\theta) = \frac{g^{(n)}(\theta_0)}{n!} \cdot (\theta - \theta_0)^n + \cdots,$$

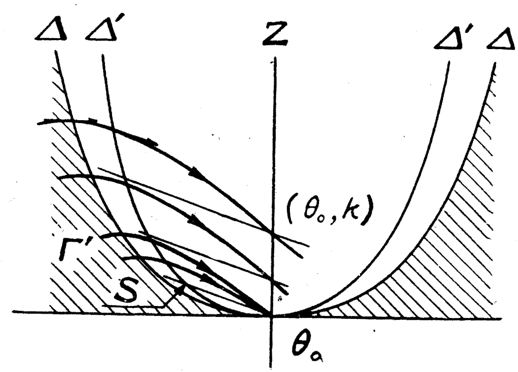


Fig. 6.

where n is even and $g^{(n)}(\theta_0) < 0$. Consequently the equation (11.2) becomes

$$\frac{g^{(n)}(\theta_0)}{n!} \cdot (\theta - \theta_0)^n + \cdots + (\theta - \theta_0)\, c \left\{ c + \alpha f(\theta_0) + \alpha f'(\theta_0)(\theta - \theta_0) + \cdots \right\} = 0.$$

Therefore the intersections except for the critical point $(\theta_0, 0)$, are determined by

$$\frac{g^{(n)}(\theta_0)}{n!} \cdot (\theta - \theta_0)^{n-1} + \cdots + c \left\{ c + \alpha f(\theta_0) + \alpha f'(\theta_0)(\theta - \theta_0) + \cdots \right\} = 0.$$

If we take sufficiently small $|c|$, then, by the Weierstrass preparation theorem, the above equation is reduced to the equation of the form as follows:

$$(11.3) \qquad \frac{g^{(n)}(\theta_0)}{n!} \cdot (\theta-\theta_0)^{n-1} + P_1(c)\,(\theta-\theta_0)^{n-2} + \cdots + P_{n-1}(c) = 0,$$

where $P_i(c)$'s $(i=1, 2, \cdots, n-1)$ are the analytic functions vanishing with c and in particular

$$P_{n-1}(c) = \alpha f(\theta_0)\, c + \cdots.$$

Since n is even and $g^{(n)}(\theta_0) < 0$, by the Newton's polygon method, it is seen that (11.3) has a unique real root

$$\theta - \theta_0 = \left(- \frac{n!\,\alpha f(\theta_0)}{g^{(n)}(\theta_0)} \right)^{1/(n-1)} c^{1/(n-1)} + \cdots < 0.$$

Namely the line $z=c(\theta-\theta_0)$ intersects the left half of $\varDelta'$ in the neighborhood of the critical point. Let this intersection be S. Then the left half of a trajectory passing through a point (θ_0, k) for any $k>0$ crosses $\varDelta'$ in the left side of S, consequently there exists a trajectory crossing $\varDelta'$ in the left side of S and passing through the critical point, namely there exists a left separatrix. It is evident that the trajectories crossing the arc of $\varDelta'$ limited by the critical point and the intersection of the greatest left separatrix with $\varDelta'$ are all the left separatrices. The feature of such a critical point is shown in Fig. 7. We shall call such a point also the *saddle-like point.*

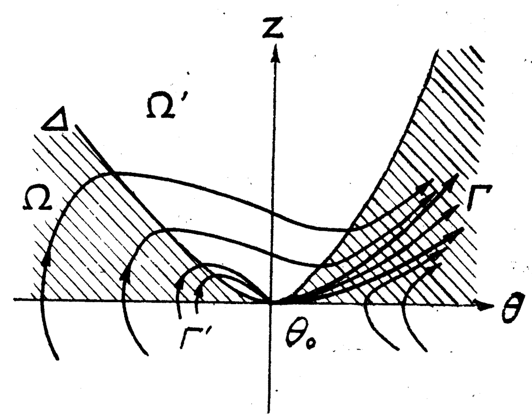


Fig. 7.

Let the right separatrix be Γ. From (6.3), it is seen that $dz/d\theta$ increases on Γ and $\varDelta$ when α decreases, consequently, by means of the method of Lefschetz[1], we see that Γ *moves upwards when* α *decreases.*[2] In like manner, it is readily seen

1) S. Lefschetz, ibid.

2) Here we mean that there exists at least one right separatrix lying above Γ. Consequently, when the right separatrix is unique, Γ moves upwards in the literal sense as α decreases.

that *the left separatrix Γ' moves downwards when α decreases.*

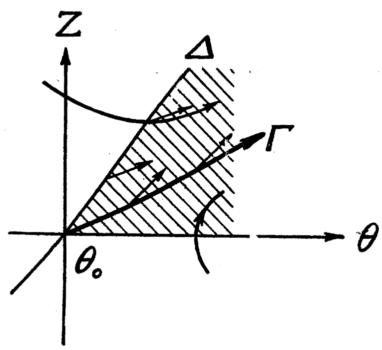


Fig. 8.

The arrows at the point of Γ and Δ denote $dz/d\theta$ for decreased α.

§ 12. Condition for finite least upper bound α'.

From the preceding paragraph, it is seen that, any trajectory passing through any point Q which lies on the perpendicular h of the θ–axis at the critical point P and moreover lies near P, crosses the θ–axis at the right-angle near P so long as P is not a saddle-like point. Therefore, when α' is finite, any trajectory for α near α' passing through the point Q crosses the θ–axis near P so long as P is not a saddle-like point. Thus we see that, *when α' is finite, the trajectory of the periodic solution can not tend to the critical point except for the saddle-like point.* From this, it is seen that, *when there exists no saddle-like point, it must be that $\alpha' = +\infty$.*

We assume that α' is finite. Then, from the above result, as $\alpha \to \alpha' - 0$, the trajectory of the periodic solution moves monotonely downwards and tends to some of the saddle-like points. Let any one of the saddle-like points be P, to which the trajectory of the periodic solution tends. The trajectory of the periodic solution lies above the θ–axis. Therefore there exists a limiting set C to which the trajectory of the periodic solution tends.

Take a point A near P on the θ–axis, and draw perpendicular k to the θ–axis through A. Let the point of C on k be R. Then $R \neq A$, for, otherwise, the trajectory of the periodic solution will tend to the ordinary point on the θ–axis, which is contrary to the result of §10. Let the trajectory for α' passing through R be C'. Then, by virtue of the continuity, the trajectory of the periodic solution must tend to C' in the neighborhood of R, that is to say that $C = C'$ in the neighborhood of R. Continuing this process, we see that C becomes the solution for α' except

1) S. Lefschetz, ibid.

for the saddle-like points to which the trajectory of the periodic solution tends, namely that C coincides with the separatrices passing through the saddle-like points to which the periodic solution tends.

Now, on the right side of h, $0 < \alpha' - \alpha \ll 1$, the trajectory of the periodic solution for α lies above the right greatest separatrix $\Gamma(\alpha)$ for α and, by the results of §11, $\Gamma(\alpha)$ lies above the right greatest separatrix $\Gamma(\alpha')$ for α'. Therefore the periodic soluton for α lies above $\Gamma(\alpha')$ on the right side of h, consequently C must lie on or above $\Gamma(\alpha')$ on the right side of h. Thus, on the right side of h, C must coincide with the right greatest separatrix $\Gamma(\alpha')$, namely C is composed of the left separatrix and the right greatest separatrix passing through the saddle-like points to which the periodic solution tends. In other words, *when α' is finite, there must exist a curve C extending over $-\infty < \theta < \infty$ composed of the left separarices and the right greatest separatrices.* We shall call such a curve the *seperatrix-curve.* Then, from this result, it is seen that, *if there does not exist any separatrix-curve, it must be that α' is infinite.*

Making use of these results, in the subsequent paragraphs, we shall seek for α'.

§ 13. The equations of the separatrices.

In the following we assume that *any saddle-like critical point of (6.2) is an elementary critical point, namely a saddle.*

Let any saddle of (6.2) be P, which corresponds to θ_0. Then, in the neighborhood of P, the equations (6.2) are written as follows:

$$\text{(13.1)} \qquad \begin{cases} \dfrac{dz}{dt} = -\alpha f(\theta_0)\, z - \xi g'(\theta_0) + \cdots, \\[2ex] \dfrac{d\xi}{dt} = z, \end{cases}$$

where $\xi = \theta - \theta_0$, and the unwritten terms are those of the second and higher orders with regard to ξ and z. The characteristic equation of (13.1) is

$$\text{(13.2)} \qquad \lambda^2 + \alpha f(\theta_0)\, \lambda + g'(\theta_0) = 0.$$

Since P is a saddle, it must be that

$$\text{(13.3)} \qquad g'(\theta_0) < 0.$$

Since the function $g(\theta)$ is periodic with the period 2π, it can be expanded in a Fourier series as follows:

$$(13.4)\qquad g(\theta) = -\beta + \varphi(\theta),$$

where

$$(13.5)\qquad \beta = -\frac{1}{2\pi}\int_0^{2\pi} g(\theta)\,d\theta = -\frac{G(2\pi)}{2\pi}, \qquad (\geqq 0 \text{ by } \S 10)$$

$$\varphi(\theta) = \sum_{n=1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta).$$

Then, since $g(\theta_0) = 0$ and $g'(\theta_0) < 0$, it holds that

$$(13.6)\qquad \varphi(\theta_0) = \beta, \quad \varphi'(\theta_0) = g'(\theta_0) < 0.$$

Considering β as a parameter, we denote the root of the equation $g(\theta) = 0$, namely $\varphi(\theta) = \beta$ by $\theta_0(\beta)$. Then, if $\theta_0(\beta_0)$ corresponds to a saddle for $\beta = \beta_0$, from (13.6), it is seen that, for β sufficiently near to β_0, the root $\theta_0(\beta)$ becomes an analytic function of $(\beta - \beta_0)$, which is expanded as follows:

$$(13.7)\qquad \theta_0(\beta) = \theta_0(\beta_0) + \frac{1}{\varphi'\{\theta_0(\beta_0)\}}(\beta - \beta_0) + \cdots.$$

In the following, we assume that $\theta_0(\beta_0) = \theta_0$ corresponds to a saddle. Then, from (13,7), it is evident that, for β sufficiently near β_0, the critical point corresponding to $\theta_0(\beta)$ is also a saddle. If we put $\theta_0 = \theta_0(\beta)$ in (13.2), from (13.7), we see that the characteristic roots λ_1, λ_2 of the equation (13.1) for $\theta_0 = \theta_0(\beta)$ are analytic with regard to $(\beta - \beta_0)$. Consequently, if we consider the characteristic roots λ_1, λ_2 as the functions of α and β, then, from (13.2), it follows that $\lambda_1(\alpha, \beta)$ and $\lambda_2(\alpha, \beta)$ are analytic with regard to α and β in the region Δ: $|\beta - \beta_0| \ll 1$, $0 \leqq \alpha \leqq M$, where M is an arbitrary positive number.

Making use of this result, we shall investigate the analyticity of the equations of the separatrices passing through the saddle corresponding to $\theta_0(\beta)$. By the linear transformation of the variables as follows:

$$(13.8)\qquad \begin{cases} \bar{z} = -\dfrac{1}{\lambda_2(\alpha, \beta)} z + \xi, \\[2ex] \bar{\xi} = -\dfrac{1}{\lambda_1(\alpha, \beta)} z + \xi; \end{cases}$$

$$\begin{cases} z = \dfrac{g'\{\theta_0(\beta)\}}{\lambda_2(\alpha, \beta) - \lambda_1(\alpha, \beta)}(\bar{z} - \bar{\xi}), \\[2ex] \xi = \dfrac{1}{\lambda_2(\alpha, \beta) - \lambda_1(\alpha, \beta)}\left\{\lambda_2(\alpha, \beta)\,\bar{z} - \lambda_1(\alpha, \beta)\,\bar{\xi}\right\}, \end{cases}$$

the equations (13.1) are transformed to the equations of the forms as follows:

$$\begin{cases} \dfrac{d\bar{z}}{dt} = \lambda_1(\alpha, \beta)\,\bar{z} + \cdots\cdots = Z, \\[2ex] \dfrac{d\bar{\xi}}{dt} = \lambda_2(\alpha, \beta)\,\bar{\xi} + \cdots\cdots = \Xi, \end{cases}$$

where the unwritten terms are those of the second and higher orders with regard to $(\bar{z}, \bar{\xi})$ with the coefficients which are analytic with regard to (α, β) in Λ. Then, since $\lambda_1(\alpha, \beta)\,\lambda_2(\alpha, \beta) < 0$, as is well known, the equations of the separatrices are given by $\bar{z} = \bar{z}(\bar{\xi})$ and $\bar{\xi} = \bar{\xi}(\bar{z})$, where $\bar{z}(\bar{\xi})$ and $\bar{\xi}(\bar{z})$ are the regular solutions of the equations

$$\text{(13.9)} \qquad \Xi\frac{d\bar{z}}{d\bar{\xi}} = Z \quad\text{and}\quad Z\frac{d\bar{\xi}}{d\bar{z}} = \Xi$$

respectively, and vanish with their derivatives of the first order when the arguments vanish. Now, if we substitute the Taylor series of $\bar{z}(\bar{\xi})$ and $\bar{\xi}(\bar{z})$ with regard to the arguments into (13.9), it is readily seen that the coefficients of these series are real and analytic with regard to (α, β) in Λ. Then, since, for sufficiently small positive number ε, $|n\lambda_2(\alpha, \beta) - \lambda_1(\alpha, \beta)|$, $|n\lambda_1(\alpha, \beta) - \lambda_2(\alpha, \beta)| > \varepsilon(n-1)$ for any positive integer $n \geqq 2$, it follows that the solutions $\bar{z}(\bar{\xi})$ and $\bar{\xi}(\bar{z})$ of (13.9) considered as the functions of (α, β) are analytic in Λ.[1] Consequently, if we return to the variables (z, ξ) by means of the transformation (13.8), the equations of the separatrices are written as follows:

$$\text{(13.10)} \qquad \begin{cases} -\dfrac{1}{\lambda_2(\alpha, \beta)}z + \xi = \bar{z}\left\{-\dfrac{1}{\lambda_1(\alpha, \beta)}z + \xi;\ \alpha, \beta\right\}, \\[2ex] -\dfrac{1}{\lambda_1(\alpha, \beta)}z + \xi = \bar{\xi}\left\{-\dfrac{1}{\lambda_2(\alpha, \beta)}z + \xi;\ \alpha, \beta\right\}, \end{cases}$$

where $\bar{z}(0;\ \alpha, \beta) = \partial\bar{z}/\partial\bar{\xi}\,|_{\bar{\xi}=0} = 0$ and $\bar{\xi}(0;\ \alpha, \beta) = \partial\bar{\xi}/\partial\bar{z}\,|_{\bar{z}=0} = 0$. Solving these equations with regard to z, we have the equation of the form $z = z(\xi;\ \alpha, \beta)$, where the function $z(\xi;\ \alpha, \beta)$ is analytic with regard to ξ and (α, β) in the region: $|\xi| \ll 1$, $(\alpha, \beta) \in \Lambda$. Now, since

$$\xi = \theta - \theta_0(\beta) = \theta - \theta_0(\beta_0) - \left[\frac{1}{\varphi'\{\theta_0(\beta_0)\}}(\beta - \beta_0) + \cdots\right]$$

from (13.7), the equations of the separatrices are ultimately written as follows:

$$\text{(13.11)} \qquad z = z(\theta;\ \alpha, \beta),$$

1) M. Urabe. *On solutions of the linear homogeneous partial differential equations in the vicinity of the singularity*, II. J. Sci. Hiroshima Univ., Ser. A, 14 (1950), 195-207.

where the function of the right-hand side is analytic with regard to the arguments in the region: $|\theta-\theta_0(\beta_0)|\ll 1$, $(\alpha,\beta)\in\Lambda$. Then, by continuation of the solution, we see that, for $0\leqq\alpha<\infty$ and $|\beta-\beta_0|\ll 1$, the equation of the separatrix of the form (13.11) is analytic with regard to (α,β) in the interval I of the θ-axis containing the saddle corresponding to $\theta_0(\beta_0)$, in which the separatrix does not intersect with the θ-axis in the point except for the saddle corresponding to $\theta_0(\beta)$.

§ 14. Infinitesimal deformation of the separatrix.

We consider the neighborhood of (α_0,β_0) and put

$$\alpha-\alpha_0=a,\quad \beta-\beta_0=b,\quad \theta_0(\beta_0)=\theta_0. \tag{14.1}$$

Then the equations (6.2) are written as follows:

$$\begin{cases} \dfrac{dz}{dt}=\Big\{-\alpha_0 f(\theta)z+\beta_0-\varphi(\theta)\Big\}-af(\theta)z+b, \\ \dfrac{d\theta}{dt}=z. \end{cases} \tag{14.2}$$

Let the equation of the separatrix for $\alpha=\alpha_0$ passing through the saddle corresponding to θ_0 be $z=z_0(\theta)$. For $|a|$, $|b|\ll 1$, we consider the separatrix passing through the saddle corresponding to $\theta_0(\beta)=\theta_0(\beta_0+b)$, then, by §13, the equation of that separatrix is written as follows: $z=z(\theta;a,b)$, where $z(\theta;a,b)$ is analytic with regard to (a,b) for $\theta\in I$ explained at the end of the preceding paragraph. Therefore, since $z(\theta;0,0)=z_0(\theta)$ by the assumption, the function $z(\theta;a,b)$ can be expanded in a power series of (a,b) as follows:

$$z(\theta;a,b)=z_0(\theta)+z_1(\theta;a,b)+z_2(\theta;a,b)+\cdots+z_n(\theta;a,b)+\cdots, \tag{14.3}$$

where $z_n(\theta;a,b)$ is a homogeneous polynomial of the n-th degree with regard to a and b. Since the separatrix $z=z(\theta;a,b)$ passes through the saddle corresponding to $\theta_0(\beta_0+b)$, it must be that $z\{\theta_0(\beta_0+b);a,b\}=0$. From (13.7) and (14.3), this condition is written as follows:

$$\left\{\frac{z_0'(\theta_0)}{\varphi'(\theta_0)}b+z_1(\theta_0;a,b)\right\}+\Big\{\cdots\cdots\Big\}+\cdots=0, \tag{14.4}$$

where the unwritten terms are those of the second and higher orders with regard to a and b. Now, from (13.10), it follows that

$$z_0'(\theta_0)=\lim_{\theta\to\theta_0}\frac{z_0(\theta)}{\theta-\theta_0}=\lambda(\alpha_0,\beta_0)\qquad(=\lambda_0), \tag{14.5}$$

consequently, from (14.4), it must be that

$$z_1(\theta_0; a, b) = -\frac{\lambda_0}{\varphi'(\theta_0)} b, \quad z_2(\theta_0; a, b) = \cdots, \ \cdots. \tag{14.6}$$

Here, since λ_0 is a root of the equation (13.2) for $\alpha = \alpha_0$, it holds that

$$1 + \frac{\alpha_0 f(\theta_0)}{\lambda_0} = -\frac{g'(\theta_0)}{\lambda_0^2} > 0. \tag{14.7}$$

Now, by the substitution (14.1), the equation (6.3) corresponding to (6.2) is written corresponding to (14.2) as follows:

$$z\frac{dz}{d\theta} = (-\alpha_0 fz + \beta_0 - \varphi) - afz + b. \tag{14.8}$$

Since the separatrix $z = z(\theta; a, b)$ is a solution of (14.8), substituting (14.3) into (14.8), we have:

$$\begin{aligned}(z_0 + z_1 + z_2 + \cdots + z_n + \cdots)\left(\frac{dz_0}{d\theta} + \frac{dz_1}{d\theta} + \frac{dz_2}{d\theta} + \cdots + \frac{dz_n}{d\theta} + \cdots\right) \\ = \left\{-\alpha_0 f(z_0 + z_1 + z_2 + \cdots + z_n + \cdots) + \beta_0 - \varphi\right\} \\ - af(z_0 + z_1 + z_2 + \cdots + z_n + \cdots) + b.\end{aligned}$$

Comparing the terms of the same degrees with regard to a and b, we have:

$$\left\{\begin{array}{ll}\text{(i)} & z_0\dfrac{dz_0}{d\theta} = -\alpha_0 fz_0 + \beta_0 - \varphi, \\ \text{(ii)} & z_0\dfrac{dz_1}{d\theta} + \left(\dfrac{dz_0}{d\theta} + \alpha_0 f\right)z_1 = -afz_0 + b, \\ \text{(iii)} & z_0\dfrac{dz_n}{d\theta} + \left(\dfrac{dz_0}{d\theta} + \alpha_0 f\right)z_n = -\left(z_1\dfrac{dz_{n-1}}{d\theta} + \cdots + z_{n-1}\dfrac{dz_1}{d\theta} + afz_{n-1}\right). \\ & \hfill (n \geqq 2)\end{array}\right. \tag{14.9}$$

Since $z = z_0(\theta)$ is a solution of (14.8) for $a = b = 0$, (i) of (14.9) is automatically satisfied. The equations (ii) and (iii) of (14.9) are written as follows:

$$\left\{\begin{array}{l}\dfrac{dz_1}{d\theta} + \left(\dfrac{1}{z_0}\dfrac{dz_0}{d\theta} + \dfrac{\alpha_0 f}{z_0}\right)z_1 = -af + \dfrac{b}{z_0}, \\ \dfrac{dz_n}{d\theta} + \left(\dfrac{1}{z_0}\dfrac{dz_0}{d\theta} + \dfrac{\alpha_0 f}{z_0}\right)z_n = -\left(\dfrac{z_1}{z_0}\dfrac{dz_{n-1}}{d\theta} + \cdots + \dfrac{z_{n-1}}{z_0}\dfrac{dz_1}{d\theta} + af\dfrac{z_{n-1}}{z_0}\right). \\ \hfill (n \geqq 2)\end{array}\right. \tag{14.10}$$

These are the linear equations, consequently, in the interval I, except for the point corresponding to θ_0, these equations are easily integrated as follows:

$$z_n(\theta; a, b) = \frac{1}{z_0} e^{-\alpha_0 \int_c^\theta \frac{f}{z_0} d\theta} \left[\int_{c'}^\theta e^{\alpha_0 \int_c^\theta \frac{f}{z_0} d\theta} R_n \, d\theta + k_n \right], \quad (n \geqq 1) \tag{14.11}$$

where

$$R_1 = -afz_0 + b,$$

$$R_n = -z_1 \frac{dz_{n-1}}{d\theta} - z_2 \frac{dz_{n-2}}{d\theta} - \cdots - z_{n-1} \frac{dz_1}{d\theta} - afz_{n-1}, \qquad (n \geqq 2)$$

and k_n's are the constants of integration. For $|\theta - \theta_0| \ll 1$,

$$\int_c^\theta \frac{f(\theta)}{z_0(\theta)} d\theta = \int_c^\theta \frac{f(\theta_0) + (\theta - \theta_0) f'(\theta_0) + \cdots}{(\theta - \theta_0) z_0'(\theta_0) + \cdots} d\theta$$

$$= \frac{f(\theta_0)}{z_0'(\theta_0)} \log |\theta - \theta_0| + \cdots,$$

consequently,

$$e^{\alpha_0 \int_c^\theta \frac{f}{z_0} d\theta} = |\theta - \theta_0|^{\alpha_0 \frac{f(\theta_0)}{z_0'(\theta_0)}} E(\theta), \tag{14.12}$$

where $E(\theta)$ is analytic with regard to θ and $E(\theta_0) \neq 0$. Then, from (14.5) and (14.7), for any function $R(\theta)$ analytic at $\theta = \theta_0$, the integral $\int_{c'}^\theta e^{\alpha_0 \int_c^\theta \frac{f}{z_0} d\theta} R(\theta) \, d\theta$ converges as $\theta \to \theta_0$, and $z_0(\theta) e^{\alpha_0 \int_c^\theta \frac{f}{z_0} d\theta} \to 0$ as $\theta \to \theta_0$. Then, from (14.11), it follows that

$$\int_{c'}^{\theta_0} e^{\alpha_0 \int_c^\theta \frac{f}{z_0} d\theta} R_n \, d\theta + k_n = 0.$$

Thus (14.11) is written as follows:

$$z_n(\theta; a, b) = \frac{1}{z_0} e^{-\alpha_0 \int_c^\theta \frac{f}{z_0} d\theta} \int_{\theta_0}^\theta e^{\alpha_0 \int_c^\theta \frac{f}{z_0} d\theta} R_n d\theta. \tag{14.13}$$

Here it is easily seen that the functions $z_n(\theta; a, b)$'s are independent of the lower limit c of the integral $\int_c^\theta \frac{f}{z_0} d\theta$.

From (14.12), it follows that

$$z_n = \frac{\dfrac{\eta\,|\theta-\theta_0|^{1+(\alpha_0 f(\theta_0)/z_0'(\theta_0))}}{1+(\alpha_0 f(\theta_0)/z_0'(\theta_0))} E(\theta_0)\, R_n(\theta_0) + \cdots}{z_0'(\theta_0)\,(\theta-\theta_0)\,|\theta-\theta_0|^{\alpha_0 f(\theta_0)/z_0'(\theta_0)}\, E(\theta_0) + \cdots},$$

where $\eta=+1$ or -1 according as $\theta>\theta_0$ or $<\theta_0$, consequently it follows that

$$z_n(\theta_0;\, a, b) = \frac{R_n(\theta_0)}{z_0'(\theta_0) + \alpha_0 f(\theta_0)}. \tag{14.14}$$

For example, for $n=1$, from (14.5), it follows that

$$z_1(\theta_0;\, a, b) = \frac{b}{\lambda_0 + \alpha_0 f(\theta_0)} = \frac{-\lambda_0}{\varphi'(\theta_0)}\, b,$$

since λ_0 is a root of (13.2). Namely the condition (14.6) for $z_1(\theta;\, a, b)$ is automatically satisfied. Since $z_n(\theta;\, a, b)$'s given by (14.13) do not contain the arbitrary constants, it is expected that they satisfy automatically the condition (14.6), even if (14.6) is not verified.

The series (14.3) for which $z_n(\theta;\, a, b)$'s are given by (14.13) denotes the infinitesimal deformation of the separatrix for the infinitesimal variation of α and β.

§ 15. The condition of the continuation for the separatrix-curve.

First, we consider the trajectory passing through two saddles and we shall call the arc of such trajectory bounded by the two saddles the *separatrix-arc*. When the separatrix-arc exists for $\beta=\beta_0$ and $\alpha=\alpha_0$, does the separatrix-arc exist for β and α such that $|\beta-\beta_0|$, $|\alpha-\alpha_0|\ll 1$? In order to study this problem, suppose that, for $\beta=\beta_0$ and $\alpha=\alpha_0$, there exists a separatrix arc $z=z_0(\theta)$ passing through the saddles corresponding to $\theta_1(\beta_0)$ and $\theta_2(\beta_0)$, where $\theta_1(\beta_0)<\theta_2(\beta_0)$ and these are the roots of $\varphi(\theta)=\beta_0$. Then, for $\beta=\beta_0+b$ and $\alpha=\alpha_0+a$ such that $|a|$, $|b|\ll 1$, the separatrices passing through each one of the saddles corresponding to $\theta_1(\beta)$ and $\theta_2(\beta)$ are represented by the equations of the form (14.3). Consequently, the condition that there may exist the separatrix-arc for β and α, becomes that, for φ such that $\theta_1(\beta)<\varphi<\theta_2(\beta)$,

$$z^{(1)}(\varphi;\, a, b) = z^{(2)}(\varphi;\, a, b), \tag{15.1}$$

where $z^{(1)}(\theta;\, a, b)$ and $z^{(2)}(\theta;\, a, b)$ are the functions such that $z=z^{(1)}(\theta;\, a, b)$ and $z=z^{(2)}(\theta;\, a, b)$ represent the separatrices passing through the saddles corresponding to $\theta_1(\beta)$ and $\theta_2(\beta)$ respectively. By (14.3), $z^{(i)}(\theta;\, a, b)$ $(i=1, 2)$ are of the forms as follows:

$$z^{(i)}(\theta;\, a, b) = z_0(\theta) + z_1^{(i)}(\theta;\, a, b) + z_2^{(i)}(\theta;\, a, b) + \cdots,$$

consequently the condition (15.1) is written as follows;

$$\left\{z_1^{(2)}(\varphi;\,a,\,b)\,-\,z_1^{(1)}(\varphi;\,a,\,b)\right\}+\left\{z_2^{(2)}(\varphi;\,a,\,b)\,-\,z_2^{(1)}(\varphi;\,a,\,b)\right\}+\cdots=0. \tag{15.2}$$

From (14.13), the linear part of the left-hand side of (15.2) is written as follows:

$$\begin{aligned}\frac{1}{z_0(\varphi)}e^{-\alpha_0\int_c^\varphi\frac{f}{z_0}d\theta}&\left[\int_{\theta_2}^{\varphi}e^{\alpha_0\int_c^\theta\frac{f}{z_0}d\theta}(-afz_0+b)\,d\theta-\int_{\theta_1}^{\varphi}e^{\alpha_0\int_c^\theta\frac{f}{z_0}d\theta}(-afz_0+b)d\theta\right] \\ =\frac{1}{z_0(\varphi)}e^{-\alpha_0\int_c^\varphi\frac{f}{z_0}d\theta}&\left[a\int_{\theta_1}^{\theta_2}e^{\alpha_0\int_c^\theta\frac{f}{z_0}d\theta}fz_0\,d\theta-b\int_{\theta_1}^{\theta_2}e^{\alpha_0\int_c^\theta\frac{f}{z_0}d\theta}d\theta\right],\end{aligned} \tag{15.3}$$

where $\theta_i=\theta_i(\beta_0)$ $(i=1,2)$. Since $f(\theta)>0$ and $z_0(\theta)>0$ for $\theta_1<\theta<\theta_2$, the coefficient of a in (15.3) is positive, consequently (15.2) is solved uniquely with regard to a as follows:

$$a=b\,\frac{\displaystyle\int_{\theta_1}^{\theta_2}e^{\alpha_0\int_c^\theta\frac{f}{z_0}d\theta}d\theta}{\displaystyle\int_{\theta_1}^{\theta_2}e^{\alpha_0\int_c^\theta\frac{f}{z_0}d\theta}fz_0\,d\theta}+\cdots\cdots, \tag{15.4}$$

where the unwritten terms are those of the second and higher orders with regard to b. Thus we see that, if there exists a separatrix-arc for $\beta=\beta_0$ and $\alpha=\alpha_0$, then, for $|\beta-\beta_0|\ll 1$, there exists always one and only one separatrix-arc lying in the neighborhood of the initial one. This fact means that *the condition of both continuations defined in* §5 *for the periodic solution is also valid for the separatrix-arc.* From (15.4), it is also seen that a increases with b, namely that α increases with β.

Next, we consider the continuation of the *separatrix-curve.* Suppose that there exists a separatrix-curve C for $\beta=\beta_0$ and $\alpha=\alpha_0$. Let the saddles passed by C lying in $[0,\,2\pi)$ be $P_1, P_2, \cdots, P_n$, which correspond to $0\leqq\theta_1(\beta_0)<\theta_2(\beta_0)<\cdots<\theta_n(\beta_0)<2\pi$, and let the saddes corresponding to $\theta_i(\beta_0)+2k\pi$ be P_{i+kn}. Let the separatrix-arc passing through P_i and P_{i+1} be Γ_i. Then, by the preceding results, for $\beta=\beta_1$ such that $0<\beta_1-\beta_0\ll 1$, corresponding to Γ_i, there exists a separatrix-arc $\Gamma_i^{(1)}$ for $\alpha=\alpha_i>\alpha_0$ which passes through the saddles P_i' and P_{i+1}' corresponding to $\theta_i(\beta_1)$ and $\theta_{i+1}(\beta_1)$. In this case, in general, α_i's are not necessarily equal to each other. Suppose that, for certain i, $\alpha_i>\alpha_{i+1}$. Let the right and left separatrices for α and $\beta=\beta_1$ passing through P_i' be $\Gamma_i^{(1)}(\alpha)$ and $\Gamma_i'^{(1)}(\alpha)$ respectively. Then, from the remarks at the end of §11, in the neighborhood of the saddles, when α is decreased from α_i, β being fixed, the right separatrix $\Gamma_i^{(1)}(\alpha)$ moves upwards from $\Gamma_i^{(1)}$ and the left separatrix $\Gamma_{i+1}^{\prime(1)}(\alpha)$ moves downwards from $\Gamma_i^{(1)}$. Now, from (6.3), in any point, $dz/d\theta$ increases when α decreases, consequently $\Gamma_i^{(1)}(\alpha)$ for $0 \leqq \alpha < \alpha_i$ can not cross $\Gamma_i^{(1)}$ except in P_i' and P_{i+1}'. Thus we see that $\Gamma_i^{(1)}(\alpha)$ for $0 \leqq \alpha < \alpha_i$ runs over P_{i+1}'. In the same manner, $\Gamma_{i+2}^{\prime(1)}(\alpha)$ for $\alpha > \alpha_{i+1}$ runs over P_{i+1}'. Let the equations of $\Gamma_i^{(1)}(\alpha)$ and $\Gamma_{i+2}^{\prime(1)}(\alpha)$ be $z = z(\theta, \alpha)$ and $z = z'(\theta, \alpha)$ respectively. Then, for the quantity $\Delta(\alpha) = z[\theta_{i+1}(\beta_1), \alpha] - z'[\theta_{i+1}(\beta_1), \alpha]$, it is valid that

$$\Delta(\alpha_i) = -z'[\theta_{i+1}(\beta_1), \alpha_i] < 0,$$

$$\Delta(\alpha_{i+1}) = z[\theta_{i+1}(\beta_1), \alpha_{i+1}] > 0.$$

Now, from §14, it is readily seen that $\Delta(\alpha)$ is analytic and monotone decreasing for $\alpha_{i+1} < \alpha < \alpha_i$ and moreover is continuous at $\alpha = \alpha_{i+1}$ and $\alpha = \alpha_i$. Consequently, there exists a unique α such that $\alpha_{i+1} < \alpha < \alpha_i$ and $\Delta(\alpha) = 0$, that is to say that, for $\beta = \beta_1$, there exists a unique separatrix-arc $\Gamma_i^{(2)}$ passing through P_i' and P_{i+2}' which corresponds to α such that $\alpha_{i+1} < \alpha < \alpha_i$. Since the separatrix-arcs appear periodically, if we continue the above process, then, after finite times of the procedures, we get a separatrix-curve C' for $\beta = \beta_1$ and α such that min. $\alpha_i \leqq \alpha \leqq$ max. α_i. This fact means that *the condition of positive continuation is valid for the separatrix-curve.* Since $\alpha_i > \alpha_0$ and $\alpha_i \to \alpha_0$ as $\beta_1 \to \beta_0$, *the value of* α *ensuring the separatrix-curve increases continuously and monotonely as* β *increases.* When $0 < \beta_1 - \beta_0 \ll 1$, $0 < \alpha - \alpha_0 \ll 1$, consequently C' lies near C, namely *the separatrix-curve moves continuously as* β *increases.* In the same manner, it is proved that the condition of negative continuation is also valid.

§ 16. Relation between the least upper bound α' and the separatrix-curve.

Suppose that there exists a separatrix-curve C for $\beta = \beta_0$ and $\alpha = \alpha_0 > 0$. Let the saddles passed by C lying in $(0, 2\pi)$ be $P_1, P_2, \cdots, P_n$, which correspond to $0 \leqq \theta_1 < \theta_2 < \cdots < \theta_n < 2\pi$, and let the saddles corresponding to $\theta_i + 2k\pi$ be P_{i+kn}. Let the right and left separatrices for α and $\beta = \beta_0$ pasing through P_i be $\Gamma_i(\alpha)$ and $\Gamma_i'(\alpha)$ respectively. Then, as is remarked in §15, for $\alpha \geqq 0$ such that $\alpha < \alpha_0$, $\Gamma_i(\alpha)$ runs over P_{i+1}. Consequently the right separatrix $\Gamma_0(\alpha)$ passing through P_0 can be extended to $\theta = 2\pi$. Let the solution corresponding to this extended $\Gamma_0(\alpha)$ be $z = z(\theta, \alpha)$. Then it is seen that $z(0, \alpha) < z(2\pi, \alpha)$. Now, for $\alpha > 0$, let the maximum of $-g(\theta)/\alpha f(\theta)$ be M. Then the solution $z = z(\theta)$ passing

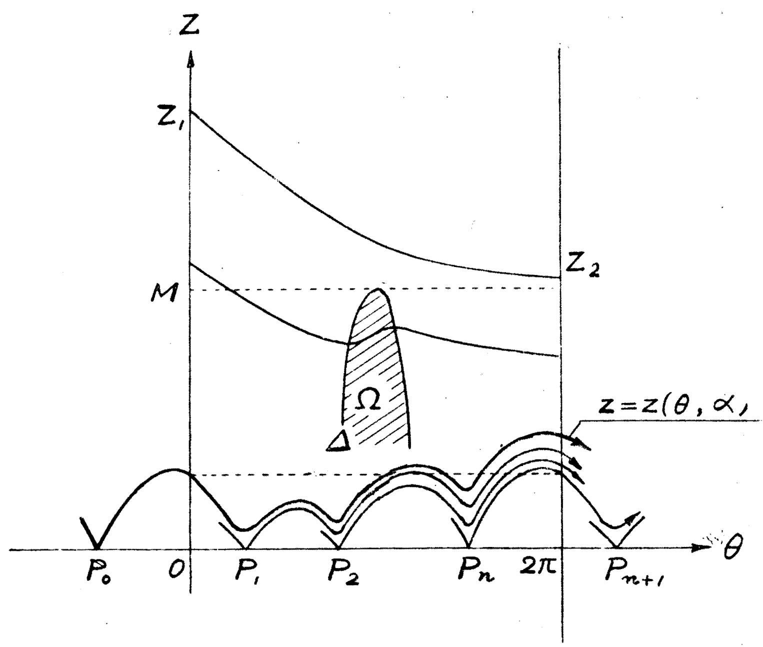


Fig. 9.

through the point $(0, z_1)$ such that $z_1 > M$ attains the value z_2 at $\theta = 2\pi$, which is smaller than z_1 by § 11. Then, by the continuity of the solution, there exists at least one periodic solution for α. Thus it must be that $\alpha_0 \leqq \alpha'$.

Suppose that, for $\alpha = \alpha_1 > \alpha_0$, there exists a periodic solution $z = z_0(\theta)$. In the same manner as before, it is readily seen that, for α_1, the left separatrix $\Gamma'_{n+1}(\alpha_1)$: $z = z(\theta, \alpha_1)$ passing through P_{n+1} can be extended to $\theta = 0$ and that $z(0, \alpha_1) > z(2\pi, \alpha_1)$. Then, from the uniqueness of the periodic solution, $z = z_0(\theta)$ must be semi-stable, for the trajectory $z = z_0(\theta)$ lies below the solution $z = z(\theta)$ passing through the point $(0, z_1)$ explained above. By (6.7), for $z = z_0(\theta)$, $I \neq 0$, consequently, by continuation of the semi-stable periodic solution, we see that, for α sufficiently near to α_1, there exists two distinct periodic solutions. This contradicts the uniqueness of the periodic solution. Thus, for any $\alpha > \alpha_0$, there exists no periodic solution, consequently $\alpha' \leqq \alpha_0$.

Summarizing the results, we have

Theorem 4. *If there exists a separatrrix-curve C for $\beta = \beta_0$ and $\alpha = \alpha_0 > 0$, then the least upper bound α' is equal to α_0 and, moreover α' becomes the least upper bound of α for which a periodic solution exists.*

By this theorem, the problem to seek for the least upper bound α' is reduced to the problem to seek for α for which a separatrix-curve exists.

For $\beta=\alpha=0$, the equation (6.3) is easily integrated as follows:

(16.1) $$z = \pm \sqrt{2\{c - G(\theta)\}},$$

where c is a constant of integration. In this case, the saddle corresponds to the value of θ for which $G(\theta)$ becomes maximum. The trajectories of the solutions (16.1) are shown in Fig. 10. From this figure, it is easily seen that there exists a

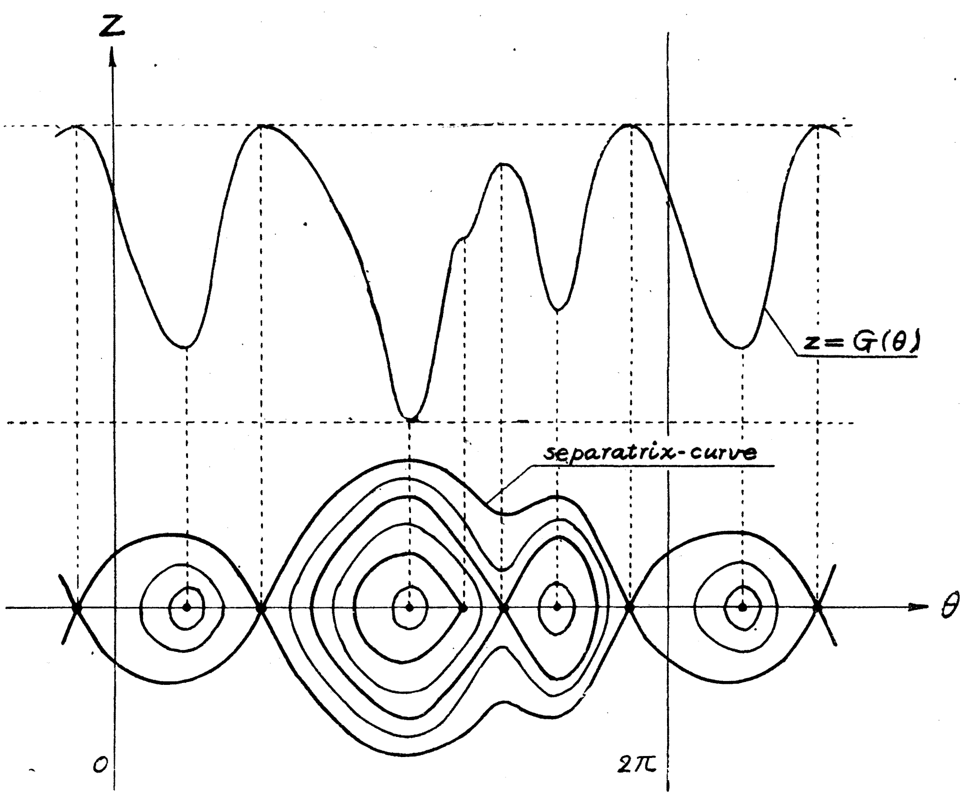


Fig. 10.

separatrix-curve. Now, from §9, it follows that $\alpha'=0$ when $\beta=0$. If there exists a separatrix-curve for $\beta=0$ and $\alpha>0$, then, by Theorem 4, it must be that $\alpha'=\alpha>0$ for $\beta=0$. This is a contradiction. Thus, when $\beta=0$, there exists one and only one separatrix-curve C_0, which corresponds to $\alpha=0$.

Next we increase β continuously from zero. Then, by §15, we see that there appears a separatrix-curve C such that C moves continuously from the initial position C_0 and the corresponding α increases continuously from zero. Let the value α' of

α corresponding in this way to β be $\alpha(\beta)$. Then, by Theorem 4, $\alpha(\beta)$ is the least upper bound of α for which a periodic solution exists for β given. Since the least upper bound of α ensuring a periodic solution is uniquely determined, *the function* $\alpha(\beta)$ *is one-valued.* Besides, from §15, it is seen that *the function* $\alpha(\beta)$ *is continuous with regard to* β *and is monotone increasing.*

Remark. The function $\alpha(\beta)$ was studied first by Tricomi[1] for the equation where

$$f(\theta) = 1 \quad \text{and} \quad g(\theta) = -\beta + \sin\theta, \tag{16.2}$$

and the estimation of $\alpha(\beta)$ was given first by him. Since then, for the equation of the above form, the estimation of $\alpha(\beta)$ has been improved gradually by Amerio[2], Seifert[3] and Hayes[4]. The results of Seifert are certainly better than those of Tricomi and Amerio. For $0<\beta<1$, his results are

$$\left[\frac{(3\pi-4\theta_0)\beta+2(1-\cos\theta_0)}{\pi(\pi-2\theta_0)}\right]^{1/2} \geqq \alpha(\beta) > \frac{\beta}{\pi-\theta_0}\left[\frac{\beta}{\pi-\theta_0}+\cos\theta_0\right]^{-1/2}, \tag{16.3}$$

where $\beta=\sin\theta_0$ and $0<\theta_0<\pi/2$. The results of Hayes which are better than those of Seifert are

$$2\sin\frac{\theta_0}{2} \geqq \alpha(\beta) \geqq \left[\sqrt{3\cos^2\theta_0+1} - 2\cos\theta_0\right]^{1/2}. \tag{16.4}$$

For $0<\beta\leqslant 1$, these results become

Seifert's: $\quad 0.9772\sqrt{\beta} = \sqrt{\frac{3}{\pi}}\sqrt{\beta} \geqq \alpha(\beta) > \frac{\beta}{\pi} = 0.3183\beta;$

Hayes': $\quad \beta \geqq \alpha(\beta) \geqq \frac{1}{2}\beta = 0.5\beta.$

Now, for the equation of the form (16.2), from (15.4), it follows that

$$\alpha(\beta) = \frac{\pi}{4}\beta = 0.7854\beta,$$

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1) F. Tricomi, *Integrazione di un'equazione differenziale presentatasi in elettrotecnica*, Ann. R. Sci. Norm. Sup. di Pisa, (1933), 1—20. This paper we could not refer to our regret, since we could not receive it.

2) L. Amerio, *Determinazione delle condizione di stabilità per gli di un'equazione interessante l'elettrotecnica*, Ann. Math. pura appl. [4] 30, (1949), 75-90. This paper we could not refer directly to our regret, since we could not receive it.

3) G. Seifert, *On the existence of certain solutions of a nonlinear differential equation*, Z. angew. Math. Phys., 3 (1952), 468-471.

do., *On certain solutions of a pendulum-type equation*, Quart. Appl. Math., 11 (1953), 127-131.

4) W. D. Hayes, *On the equation for a damped pendulum under constant torque*, Z. angew. Math. Phys., 4 (1953), 398-401.

since $\alpha_0 = 0$ and $z_0(\theta) = 2\cos(\theta/2)$ for $-\pi \leqq \theta \leqq \pi$. Thus, for $\beta \ll 1$, it is seen that the result of Hayes is considerably accurate than that of Seifert, but the former is still pretty rough compared with the true value calculated by our method.

For the equation of the general form, the estimation is not yet given by anyone, except for the lower bound given by Seifert, which turns to the right-hand side of (16.3) when the equation becomes that of the special form (16.2).

Our results do not give directly the estimation over the whole range of β, but they give the method of calculating the true value of $\alpha(\beta)$ successively in the general case, and they make clear the functional properties of the function $\alpha(\beta)$ in the general case.

The actual calculation of the true value of $\alpha(\beta)$ by means of our method will be illustrated in the next paper for the equation of the form (16.2).

§ 17. The bounary behaviour of the function $\alpha(\beta)$.

In this paragraph, we shall consider the least upper bound β' of β for which a separatrix-curve exists, when we increase β from zero.

Suppose that β' is finite and that the critical points to which the separatrix-curve for β tends, are also the saddles. When there exists a separatrix-curve passing through these saddles, the condition of the positive continuation is fulfilled for the separatrix-curve for β', consequently there exists a separatrix-curve for $\beta > \beta'$. This contradicts the assumption for β'. When there exists no separatrix-curve passing through the saddles explained above, it is seen that there does not exist any separatrix-curve for β'. For, suppose that there exists a separatrix-curve C' for β' which passes through the saddles distinct from the saddles explained above. Then, by the condition of negative continuation, there exists a separatrix-curve C lying arbitrarily near C' for β sufficiently near and less than β', consequently the saddles which C' passes must be the saddles to which C tends. Now, since $\alpha(\beta)$ is one-valued, the separatrix-curve for β is unique. Therefore, the saddles which C' passes must be the saddles to which the separatrix-curve for $\beta < \beta'$ tends. This contradicts the definition of C'. Thus we see that there does not exist any separatrix-curve for β'. Then, by §12, the least upper bound of α for which a periodic solution for β' exists is infinity. Next, suppose that β' is finite and that at least one of the critical points to which the separatrix-curve for $\beta < \beta'$ tends is not a saddle. Then, as before, it is again seen that there does not exist any separatrix-curve for β', consequently, as before, the least upper bound of

α for which a periodic solution for β' exists is infinity. Thus it is concluded that, *when β' is finite, for any finite α, there exists always a periodic solution for β'.*

Now, if we substitute (13.4) into (6.4) and think β as the parameter instead of α, then, corresponding to (6.5), we have:

$$X(x, \theta) = \alpha f(\theta)\, x^2 + \{\varphi(\theta) - \beta_0\}\, x^3, \quad L(x, \theta, \varepsilon) = -x^3.$$

Consequently, from (3.2), we have:

$$I = -\int_0^{2\pi} e^{-h} x^3\, d\theta, \tag{17.1}$$

where, from (1.10),

$$h(\theta) = \int_0^{\theta} [2\alpha f(\theta)\, x + 3\,\{\varphi(\theta) - \beta_0\}\, x^2]\, d\theta.$$

From (17.1), $I < 0$ except for $x = 0$. Then, as in §9, we see that *the condition of both continuations is fulfilled with regard to β.*

Making use of this result, we shall show that, when β' is finite, $\alpha(\beta) \to \infty$ as $\beta \to \beta'$. Since $\alpha(\beta)$ is motone increasing, if $\alpha(\beta)$ does not tend to infinity, there exists a finite number α' such that $\alpha(\beta) \to \alpha'$ as $\beta \to \beta'$. Since β' is finite, there exists a periodic solution for (α', β'). Then, by the result just obtained, for $0 < \beta' - \beta \ll 1$, there exists a periodic solution for (α', β). Then, by the condition of the continuation with regard to α, there exists a periodic solution for (α, β) such that $\alpha(\beta) < \alpha < \alpha'$ and $\alpha' - \alpha \ll 1$. This contradicts the fact that there exists no periodic solution for $\alpha > \alpha(\beta)$. Thus we see that $\alpha(\beta) \to \infty$ as $\beta \to \beta'$, namely that, even when $\beta = \beta'$, $\alpha(\beta)$ is equal to the least upper bound of α for which a periodic solution exists.

If there exists a value of $\beta > \beta'$ for which there exists a separatrix-curve, let the greatest lower bound of such β be $\beta'' \geqq \beta'$. Then, in the same manner as in β', letting $\beta \to \beta'' + 0$, we see that the least upper bound of α for which a periodic solution exists for β'' is infinity. Let $\lim_{\beta \to \beta'' + 0} \alpha(\beta) = \alpha''$. Then, by the continuation with regard to β, for $\alpha > \alpha''$, there exists a periodic solution for β such that $0 < \beta - \beta'' \ll 1$ and $\alpha(\beta) < \alpha$. This is a contradiction. Thus, for any $\beta > \beta'$, there exists no separatrix-curve, namely, for $\beta > \beta'$, by §12, the least upper bound of α for which a periodic solution exists is infinity. Consequently, *if we extend $\alpha(\beta)$ so that $\alpha(\beta) = \infty$ for $\beta \geqq \beta'$, then, for any $\beta \geqq 0$, the function $\alpha(\beta)$ gives the least upper bound of α for which a periodic solution exists.*

Now, $\varphi(\theta)$ is a periodic function, consequently it is bounded for $-\infty<\theta<\infty$. Therefore, for sufficiently large β, there exists no root of the equation $\varphi(\theta)=\beta$, namely there exists no critical point. Consequently, of course, there exists no separatrix-curve. Thus we see that the least upper bound β' of β for which a separatrix-curve exists is finite. This says that there exists a finite value β' of β such that, for $\beta\geqq\beta'$, there always exists a periodic solution for any finite value of α and for $\beta<\beta'$, there exists a periodic solution for α less than the certain positive number determined corresponding to β. In the first part of this paragraph, it is shown that *the value β' of β of the above properties is the value of β such that either the critical points to which the separatrix-curve for $\beta<\beta'$ tends are the saddles and there exists no separatrix-curve passing through these saddles, or at least one of the critical points to which the separatrix-curve for $\beta<\beta'$ tends is not a saddle.*

Remark. In §13, we have assumed that any saddle-like point is an elementary critical point, namely a saddle. Consequently the conclusion of this paper is not necessarily valid when there exists a saddle-like point which is not a saddle. For example, we suppose that, as $\beta\to\beta'$, the separatrix-curve for $\beta<\beta'$ tends to a saddle-like point P which is not a saddle. Then there may exist a separatrix-curve passing through P, consequently, for β', it may be that the least upper bound of α for which a periodic solution exists is finite. In fact, as will be shown in the next paper, this occurs for the equation of the form (16. 2), for which $\beta'=1$ and $\alpha(\beta')\doteqdot 1.193$.

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