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On the Non-Commutative Solutions of the Exponential Equation $e^x e^y = e^{x+y}$ II

By

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In our previous paper [2] [1], we have obtained all the non-commutative solutions of $e^x e^y = e^{x+y}$ for the complex algebras of degree two, and then, as the special cases, for the ternary algebras of degree two, the quaternion algebra and the total matric algebra of order two.

In this paper, we shall first consider the non-commutative solutions of $e^x e^y = e^{x+y}$ for certain iteration pairs, which, as a special case, contains the case for the complex algebras of degree two, secondly for the algebra composed of the triangular matrices of order n, and finally we shall obtain all the non-commutative solutions of $e^x e^y = e^y e^x = e^{x+y}$ for the total matric algebra of order three.

I. NON-COMMUTATIVE SOLUTIONS OF $e^x e^y = e^{x+y}$ FOR CERTAIN ITERATION PAIRS

1. Preliminary. Let $\mathfrak{A}$ be an algebra with the unit 1 over the complex field $\mathfrak{C}$, and let us denote by Greek letters $\alpha, \beta, \ldots, \lambda, \mu, \ldots$ the elements of $\mathfrak{C}$.

We shall consider the set $\mathfrak{S}$ of all the pairs (x, y) of elements $x, y \in \mathfrak{A}$ such that

$$x^3 = \alpha^2 x,\ y^3 = \beta^2 y,\ x^2 y = y x^2 = \alpha^2 y \text{ and } x y^2 = y^2 x = \beta^2 x, \tag{1.1}$$

where $\alpha, \beta \in \mathfrak{C}$. Here it is easily seen that $(x, x+y) \in \mathfrak{S}$ is equivalent to $(y, x+y) \in \mathfrak{S}$, and for this it is necessary and sufficient that $x(x+y)^2 = \gamma^2 x$ and $y(x+y)^2 = \gamma^2 y$. But, in the following we shall never assume that $(x, x+y) \in \mathfrak{S}$. We shall investigate the non-commutative solutions of $e^x e^y = e^{x+y}$ for the set $\mathfrak{S}$.

For $(x, y) \in \mathfrak{S}$, the following are easily seen:

$$e^x = 1 + p(\alpha)x + r(\alpha)x^2 \tag{1.2}$$

where

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1) Numbers in brackets refer to the references at the end of the paper.

$$(1.3)\qquad \begin{cases} p(\alpha)=\dfrac{sh\alpha}{\alpha} & (\alpha\neq 0), \\ \quad = 1 & (\alpha=0), \\ r(\alpha)=\dfrac{ch\alpha-1}{\alpha} & (\alpha\neq 0), \\ \quad =\dfrac{1}{2} & (\alpha=0); \end{cases}$$

and

$$(1.4)\qquad \beta^2x^2=\alpha^2y^2.$$

Lemma 1. *For $(x, y)\in\mathfrak{S}$ such that $xy\neq yx$, it is necessary and sufficient for $e^xe^y=e^ye^x$ that $x^3+l^2\pi^2x=0$ or $y^3+m^2\pi^2y=0$ where l and m are non-zero integers, in other words, that $e^x=1+\frac{1-(-1)^l}{l^2\pi^2}x^2$ or $e^y=1+\frac{1-(-1)^m}{m^2\pi^2}y^2$.*

Proof. Since $(x, y)\in\mathfrak{S}$, by means of (1.1) and (1.2) we have

$$(1.5)\qquad e^xe^y-e^ye^x=p(\alpha)p(\beta)(xy-yx)\,;$$

hence, for x and y such that $xy\neq yx$, $p(\alpha)p(\beta)=0$ if and only if $e^xe^y=e^ye^x$. By (1.3), if $p(\alpha)=0$, then $\alpha=\sqrt{-1}\,l$, where l is a non-zero integer, and then, by (1.3), $r(\alpha)=\frac{1-(-1)^l}{l^2\pi^2}$. Therefore we have

$$(1.6)\qquad e^x=1+\frac{1-(-1)^l}{l^2\pi^2}x^2.$$

If $p(\beta)=0$, then similarly we have

$$(1.7)\qquad e^y=1+\frac{1-(-1)^m}{m^2\pi^2}y^2.$$

Thus this lemma is proved.

Next we shall calculate e^{x+y} for $(x, y)\in\mathfrak{S}$. By the mathematical induction we have

$$(1.8)\qquad \begin{cases} (x+y)^{2r-1}=\sigma_r(x+y)+\tau_r(x-y)+\sigma_{r-1}(x-y)z+u_r, \\ (x+y)^{2r}=\sigma_r(x+y)^2+\tau_r(x^2-y^2)+v_r, \end{cases} \qquad (r\geqq 1)$$

where $z=yx-xy$, and

$$(1.9)\qquad \sigma_{r+1}=\sigma\sigma_r-\tau\tau_r,\quad \tau_{r+1}=\tau\sigma_r,\qquad (r\geqq 1),$$

$$(1.10)\qquad \sigma_0=\tau_1=0,\ \sigma_1=1,\ \sigma_2=\sigma=2(\alpha^2+\beta^2),\ \tau_2=\tau=(\beta^2-\alpha^2),$$

$$(1.11)\qquad v_r=\sigma_{r-1}z^2+(x+y)u_r,\ u_{r+1}=(x+y)v_r,$$

$$(1.12)\qquad u_1=v_1=0.$$

Also, by the mathematical induction we have

$$(1.13)\qquad \begin{cases} [x+y]^{2r-1}=\sigma_r(x+y)+\tau_r(x-y), \\ [x+y]^{2r}=\sigma_r[x+y]^2+\tau_r(x^2-y^2), \end{cases} \qquad (r\geqq 1),$$

where $[x+y]^n$ means $x^n+{}_nC_1x^{n-1}y+{}_nC_2x^{n-2}y+\cdots\cdots+y^n$.
From (1.8) and (1.13) it follows that

$$(1.14)\qquad \begin{cases} (x+y)^{2r-1}=[x+y]^{2r-1}+\sigma_{r-1}(x-y)z+u_r, \\ (x+y)^{2r}=[x+y]^{2r}+\sigma_rz+v_r, \end{cases} \qquad (r\geqq 1).$$

Since it is easily seen that

$$(1.15)\qquad \sum_{n=0}^{\infty}\frac{1}{n!}[x+y]^n=e^xe^y,$$

from (1.14) and (1.15), it follows that

$$(1.16)\qquad e^{x+y}=e^xe^y+\kappa z+\rho(x-y)z+w,$$

where

$$(1.17)\qquad \kappa=\sum_{r=1}^{\infty}\frac{\sigma_r}{(2r)!},\quad \rho=\sum_{r=1}^{\infty}\frac{\sigma_{r-1}}{(2r-1)!}$$

and

$$(1.18)\qquad w=\sum_{r=1}^{\infty}\frac{u_r}{(2r-1)!}+\sum_{r=1}^{\infty}\frac{v_r}{(2r)!}.$$

Here if we put

$$\alpha+\beta=\gamma \text{ and } \beta-\alpha=\delta$$

then we have

$$\sigma=\gamma^2+\delta^2,\ \tau=\gamma\delta \text{ and } 4\alpha\beta=\gamma^2-\delta^2;$$

by means of (1.9) it is easily seen that

$$(1.19)\qquad (\gamma^2-\delta^2)\sigma_r=\gamma^{2r}-\delta^{2r},\ (\gamma^2-\delta^2)\tau_{r+1}=\gamma\delta(\gamma^{2r}-\delta^{2r}), \qquad (r\geqq 1).$$

For $\alpha\beta\neq 0$, by (1.19) we have

$$(1.20)\qquad \kappa=\frac{1}{2}p(\alpha)p(\beta);$$

also for $\alpha\beta=0$, it is easily seen that (1.20) holds. Similarly, after some computations we have

(1.21)
$$\rho=\frac{1}{2(\beta^2-\alpha^2)}(p(\alpha)q(\beta)-p(\beta)q(\alpha)), \qquad (\alpha^2\neq\beta^2),$$
$$=\frac{1}{(2\alpha)^2}(p(2\alpha)-1), \qquad (\alpha^2=\beta^2\neq 0),$$
$$=\frac{1}{6}, \qquad (\alpha=\beta=0),$$

where

(1.22)
$$q(\alpha)=ch\alpha.$$

Moreover, since $xz=-zx$ and $yz=-zy$ for $(x,y)\in\mathfrak{S}$, from (1.11) and (1.12) it follows that u_r and v_r are commutative with $x+y$, consequently by (1.18) we have

(1.23)
$$w(x+y)=(x+y)w.$$

Now we shall consider the condition for

(1.24)
$$(x+y)e^xe^y=e^xe^y(x+y).$$

From (1.16) and (1.23) it follows that (1.24) is epuivalent to

(1.25)
$$\kappa(x+y)z-(\beta^2-\alpha^2)\rho z=0.$$

By (1.20) and (1.21), (1.25) is equivalent to

(1.26)
$$\theta z-\omega(x+y)z=0,$$

where

(1.27)
$$\theta=p(\alpha)q(\beta)-p(\beta)q(\alpha), \quad \omega=p(\alpha)p(\beta).$$

By multiplying x to (1.26) from left and right respectively, and by adding these results together, we have

(1.28)
$$\omega z^2=0;$$

from (1.26) and (1.28) it follows that

(1.29)
$$\theta z^2=0.$$

Therfore if $\theta=p(\alpha)q(\beta)-p(\beta)q(\alpha)\neq0$ or $\omega=p(\alpha)p(\beta)\neq0$, then $z^2=0$. And then, by multiplying $(x-y)$ to (1.26) from left, and by using of (1.28), we have

(1.30)
$$(\beta^2-\alpha^2)\omega z+\theta(x-y)z=0.$$

From (1.26) and (1.30) it follows that

$$((\beta^2-\alpha^2)\omega^2-\theta^2)z+2\omega\theta xz=0\,; \tag{1.31}$$

by multiplying x to this from left, we have

$$2\alpha^2\omega\theta z+((\beta^2-\alpha^2)\omega^2-\theta^2)xz=0. \tag{1.32}$$

From (1.31) and (1.32) we have

$$\chi z=0 \tag{1.33}$$

where $\chi=((\beta^2-\alpha^2)\omega^2-\theta^2)^2-4\alpha^2\omega^2\theta^2$. After some computations we have $\chi=(\theta+\omega(\alpha+\beta))\,(\theta-\omega(\alpha+\beta))\,(\theta+\omega(a-\beta))\,(\theta-\omega(\alpha-\beta))$, and also

$$\theta-\omega(\alpha+\beta)=e^{\alpha-\beta}(e(-2\alpha)-e(2\beta)), \tag{1.34}$$

where $e(\xi)=\dfrac{e^{\xi}-1}{\xi,}$, $(\xi\neq 0)$ and $e(0)=1$, so we have

$$\chi=(e(2\alpha)-e(2\beta))\,(e(2\alpha)-e(-2\beta))\,(e(-2\alpha)-e(2\beta))\,(e(-2\alpha)-e(-2\beta)). \tag{1.35}$$

From (1.33) it follows that for $(x,y)\in\mathfrak{S}$ such that $xy\neq yx$, χ must be zero.

We shall consider the non-commutative solutions of $(x+y)e^xe^y=e^xe^y(x+y)$ according to the following classification of the cases.

(I). The case where $e^xe^y=e^ye^x$. In this case, from Lemma 1, we have $\omega=p(\alpha)p(\beta)$. And from $\chi=0$ it follows that $\theta=p(\alpha)q(\beta)-p(\beta)q(\alpha)=0$; hence $p(\alpha)=p(\beta)=0$, i. e.,

$$\alpha=\sqrt{-1}\,l\pi,\ \ \beta=\sqrt{-1}\,m\pi, \tag{1.36}$$

where l and m are non-zero integers. Conversely, for (1.36), (1.26) is satisfied, that is, $(x+y)e^xe^y=e^xe^y(x+y)$ is satisfied. And then we have

$$e^x=1+\frac{1-(-1)^l}{l^2\pi^2}x^2,\ \ e^y=1+\frac{1-(-1)^m}{m^2\pi^2}y^2. \tag{1.37}$$

(II). The case where $e^xe^y\neq e^ye^x$. In this case, from Lemma 1, we have $\omega=p(\alpha)p(\beta)\neq 0$.

(II$_1$). The case where $\omega\neq 0$ and $\theta=0$. In this case, $\chi=0$ is equivalent to $\alpha^2=\beta^2$. And also (1.26) is equivalent to

$$(x+y)z=0. \tag{1.38}$$

(II$_2$). The case where $\omega\neq 0$ and $\theta\neq 0$. In this case, we have $\alpha^2\neq\beta^2$ because $\alpha^2=\beta^2$ implies $\theta=p(\alpha)q(\beta)-p(\beta)q(\alpha)=0$. Under the condition $\chi=0$, taking, for example, $\theta=\omega(\alpha+\beta)$, (1.26) is equivalent to

$$(x+y)z=(\alpha+\beta)z, \tag{1.39}$$

and also (1.30) is equivalent to

$$(x-y)z=(\alpha-\beta)z\,; \tag{1.40}$$

hence we have

$$xz=\alpha z,\ \ yz=\beta z, \tag{1.41}$$

where $e(-2\alpha)=e(2\beta)$ and $\alpha^2\neq\beta^2$. It is easily seen that from the other cases for $\chi=0$ the same result (1.41) is obtained.

Summarizing these results we have the following lemmas.

LEMMA 2. *For* $(x,\ y)\in\mathfrak{S}$ *such that* $xy\neq yx$, *it is necessary and sufficient for* $(x+y)e^xe^y=e^xe^y(x+y)$ and $e^xe^y=e^ye^x$ that $x^3+l^2\pi^2x=0$, $x^2y=yx^2=-l^2\pi^2y$, $xy^2=y^2x=-m^2\pi^2x$ *and* $y^3+m^2\pi^2y=0$ *where* l *and* m *are non-zero integers, in other words, that* $e^x=1+\frac{1-(-1)^l}{l^2\pi^2}x^2$ *and* $e^y=1+\frac{1-(-1)^m}{m^2\pi^2}y^2$.

LEMMA 3. *For* $(x,y)\in\mathfrak{S}$ *such that* $xy\neq yx$, *it is necessary and sufficient for* $(x+y)e^xe^y=e^xe^y(x+y)$ *and* $e^xe^y\neq e^ye^x$ *that*

(II$_1$) $x^3=\alpha^2x$, $x^2y=yx^2=\alpha^2y$, $xy^2=y^2x=\beta^2x$, $y^3=\beta^2y$ *and* $(x+y)\ (xy-yx)=0$;

or (II$_2$) $x^3=\alpha^2x$, $x^2y=yx^2=\alpha^2y$, $xy^2=y^2x=\beta^2x$, $y^3=\beta^2y$ $x(xy-yx)=\alpha(xy-yx)$ *and* $y(xy-yx)=\beta(xy-yx)$, *where* $e(-2\alpha)=e(2\beta)$ *and* $\alpha^2\neq\beta^2$. *And then* $(xy-yx)^2=0$ *is satisfied.*

2. Non-commutative solutions of $e^xe^y=e^{x+y}$.

Since $e^xe^y=e^{x+y}$ implies $(x+y)e^xe^y=e^xe^y(x+y)$, according to the above classification of cases (I), (II$_1$) and (II$_2$) we shall consider the non-commutative soltuions of $e^xe^y=e^ye^x$.

(I). The case where $e^xe^y=e^ye^x$. From Lemma 2 we have

THEOREM 1. *For* $(x,\ y)\in\mathfrak{S}$ *such that* $xy\neq yx$, *it is necessary and sufficient for* $e^xe^y=e^{x+y}=e^ye^x$ *that* $x^3+l^2\pi^2x=0$, $x^2y=yx^2=-l^2\pi^2y$, $xy^2=y^2x=-m^2\pi^2x$, $y^3+m^2\pi^2y=0$ *and* $e^{x+y}=1+\frac{1-(-1)^{l+m}}{l^2\pi^2}x^2$, *where* l *and* m *are non-zero integers. And then* $e^{2x}=e^{2y}=e^{2(x+y)}=1$ *is satisfied.*

(II). The case where $e^xe^y\neq e^ye^x$. From Lemma 3, we have $z^2=0$. Therefore, by means of (1.11) and (1.12) we have $u_r=v_r=0$ $(r\geqq 1)$, and hence $w=0$. So, from (1.16) it follows that $e^xe^y=e^{x+y}$ is equivalent to

$$\kappa z+\rho(x-y)z=0. \tag{2.1}$$

(II$_1$). The case where $\omega\neq 0$ and $\theta=0$. In this case $\alpha^2=\beta^2$, and then we have

$$\kappa=\frac{1}{2}\omega=\frac{1}{2}p(\alpha)^2\neq 0,$$

and from (1.21) we have

$$\begin{aligned}\rho&=\frac{1}{(2\alpha)^2}(p(2\alpha)-1), && (\alpha\neq 0),\\ &=\frac{1}{6}, && (\alpha=0).\end{aligned}$$

Therefore (2.1) is written as

$$(2.2)\qquad \begin{cases} p(\alpha)^2 z+\dfrac{1}{2\alpha^2}(p(2\alpha)-1)\,(x-y)z=0, & (\alpha\neq 0),\\ z+\dfrac{1}{3}(x-y)z=0, & (\alpha=0).\end{cases}$$

By Lemma 3, $(x+y)z=0$, so (2.2) is equivalent to

$$(2.3)\qquad \begin{cases} p(\alpha)^2 z+\dfrac{1}{\alpha^2}(p(2\alpha)-1)xz=0, & (\alpha\neq 0),\\ z+\dfrac{2}{3}xz=0, & (\alpha=0).\end{cases}$$

For the case where $\alpha=0$, we have $x^2z=0$; so by multiplying x to (2.3) from left we have $xz=0$, and hence from (2.3) $z=0$; that is, for this case there is no non-commutative solution of $e^x e^y=e^{x+y}\neq e^y e^x$. For the case where $\alpha\neq 0$, by multiplying x to (2.3) from left we have

$$(2.4)\qquad p(\alpha)^2 xz+(p(2\alpha)-1)z^2=0.$$

From (2.3) and (2.4) we have

$$(2.5)\qquad \{\alpha p(\alpha)^2-(p(2\alpha)-1)\}\ \{\alpha p(\alpha)^2+(p(2\alpha)-1)\}\ z=0\,;$$

and for $z\neq 0$ we have

$$(2.6)\qquad \alpha p(\alpha)^2+(p(2\alpha)-1)=0 \ \text{ or } \ \alpha p(\alpha)^2-(p(2\alpha)-1)=0.$$

For example we shall take $\alpha p(\alpha)^2+(p(2\alpha)-1)=0$, i. e.,

$$(2.7)\qquad e(2\alpha)=1,$$

then by substituting $p(2\alpha)-1=-\alpha p(\alpha)^2$ into (2.3) we have $xz=\alpha z$, since $\omega=p(\alpha)^2\neq 0$. And by means of $(x+y)\,z=0$ we have $yz=-\alpha z$, and hence we have

$$(2.8)\qquad xz=\alpha z,\ \ yz=-\alpha z,$$

where $e(2\alpha)=1$, $\alpha\neq 0$. Starting from $\alpha p\,(\alpha)^2-(p\,(2\alpha)-1)=0$, we also arrive at the same result (2.8).

(II_2). The case where $\omega \neq 0$ and $\theta \neq 0$. In this case, by Lemma 3 we have $\alpha^2 \neq \beta^2$, and then we have

$$\kappa = \frac{1}{2}\omega = \frac{1}{2} p(\alpha)\, p(\beta) \neq 0,$$

and from (1.21) we have

$$\rho = \frac{1}{2(\beta^2-\alpha^2)}(p(\alpha)\, q(\beta) - p(\beta)\, q(\alpha)) \neq 0.$$

Therefore (2.1) is written as

$$(\beta^2 - \alpha^2)\,\omega z + \theta\,(x-y)\, z = 0. \tag{2.9}$$

This is the same one as (1.30); that is, (1.26) implies (2.9), in other words, $(x+y)\, e^x e^y = e^x e^y (x+y)$ implies $e^x e^y = e^{x+y}$..

Thus summarizing the above results we have

THEOREM 2. *For* $(x, y) \in \mathfrak{S}$ *such that* $xy \neq yx$, *it is necessary and sufficient for* $e^x e^y = e^{x+y} \neq e^y e^x$ *that*

(II_1) $x^3 = \alpha^2 x$, $x^2 y = yx^2 = \alpha^2 y$, $xy^2 = y^2 x = \alpha^2 x$, $y^3 = \alpha^2 y$, $x(xy-yx) = \alpha(xy-yx)$ and $y(xy-yx) = -\alpha(xy-yx)$ *where* $e(2\alpha) = 1$ *and* $\alpha \neq 0$, *or*

(II_2) $x^3 = \alpha^2 x$, $x^2 y = yx^2 = \alpha^2 y$, $xy^2 = y^2 x = \beta^2 x$, $y^3 = \beta^2 y$ $x(xy-yx) = \alpha(xy-yx)$ *and* $y(xy-yx) = \beta(xy-yx)$ *where* $e(-2\alpha) = e(2\beta)$ *and* $\alpha^2 \neq \beta^2$.

REMARK. For $(x, y) \in \mathfrak{S}$ such that $e^x e^y = e^{x+y}$, $(x, x+y) \in \mathfrak{S}$ does not necessarily occur. For example we shall consider the case (II). $(x, x+y) \in \mathfrak{S}$ is equivalent to $(y, x+y) \in \mathfrak{S}$, for this it is necessary and sufficient that

$$x(x+y)^2 = \gamma^2 x \text{ and } y(x+y)^2 = \gamma^2 y, \tag{2.10}$$

by means of (2.8), from this we have

$$\begin{cases} (\alpha^2 + \beta^2 - \gamma^2)\, x + 2\alpha^2 y - \alpha(xy-yx) = 0, \\ 2\beta^2 x + (\alpha^2 + \beta^2 - \gamma^2)\, y + \beta(xy-yx) = 0, \end{cases} \tag{2.11}$$

so we have

$$((\alpha+\beta)^2 - \gamma^2)\,(\beta x + \alpha y) = 0,$$

since $xy \neq yx$, and α and β are not both zero, we have $\beta x + \alpha y \neq 0$, and hence $(\alpha+\beta)^2 = \gamma^2$. Therefore (2.11) is equivalent to

$$xy - yx = -2\beta x + 2\alpha y \tag{2.12}$$

and by means of (2.8), from this it follows that

$$\alpha\beta(-\beta x+\alpha y)=0\,;$$

since $-\beta x+\alpha y\neq 0$, we have $\alpha\beta=0$. That is, in this case, $(x, x+y)\in\mathfrak{S}$, if and only if $\alpha\beta=0$.

3. Matrices representing the non-commutative solutions of $e^xe^y=e^{x+y}$. In this section we shall determine the matrices representing the non-commutative solutions of $e^xe^y=e^{x+y}$ for the set $\mathfrak{S}$. We shall denote 1_r and 0_r the unit matrix and the zero matrix of order r respectively.

Now let x and y be the complex matrices of order n such that $(x, y)\in\mathfrak{S}$, then we have

$$x^3=\alpha^2x,\ x^2y=yx^2=\alpha^2y,\ xy^2=y^2x=\beta^2x \text{ and } y^3=\beta^2y. \tag{3.1}$$

If $\alpha\neq 0$, then from $x^3=\alpha^2x$ we have

$$x=p^{-1}x_0p,\quad x_0=\begin{pmatrix}x_1 & 0\\ 0 & 0\end{pmatrix} \text{ and } x_1=\begin{pmatrix}\alpha 1_{n_1} & 0\\ 0 & -\alpha 1_{n_2}\end{pmatrix}, \tag{3.2}$$

where p is a regular matrix of order n. From $x^2y=yx^2=\alpha^2y$ it follows that

$$y=p^{-1}y_0p,\quad y_0=\begin{pmatrix}y_1 & 0\\ 0 & 0\end{pmatrix}, \tag{3.3}$$

where y_1 is a matrix of order n_1+n_2. And also from $xy^2=y^2x=\beta^2x$, i. e., $x_1y_1^2=y_1^2x_1=\beta^2x_1$ we have

$$y_1^2=\beta^2 1_{n_1+n_2}, \tag{3.4}$$

and then $y^3=\beta^2y$ is satisfied automatically.

(I). The case where $e^xe^y=e^ye^x$. By Theorem 1, the above matrices x_1 and y_1 must satisfy

$$x_1^2=-l^2\pi^2 1_{n_1+n_2},\quad y_1^2=-m^2\pi^2 1_{n_1+n_2} \text{ and } e^{x_1+y_1}=(-1)^{l+m}1_{n_1+n_2}, \tag{3.5}$$

where l and m are non-zero integers. And from $e^{x_1+y_1}=(-1)^{l+m}1_{n_1+n_2}$ we have

$$\begin{gathered} x_1+y_1=s^{-1}fs,\\ f=\sqrt{-1}\,\pi\Big[\sum_{a=1}^{p}\oplus f_a 1_{r_a}\oplus\sum_{\alpha=p+1}^{p+q}\oplus f\begin{pmatrix}1_{s_\alpha} & 0\\ 0 & -1_{t_\alpha}\end{pmatrix}\oplus 0\Big]\end{gathered} \tag{3.6}$$

where $\oplus$ means the direct sum of matrices, $f_1,\cdots,\ f_{p+q}$ are all distinct non-zero even numbers for $l+m\equiv 0 \pmod{2}$, and all distinct odd numbers for $l+m\equiv 1$ (mod. 2); the last term can occur only for the case where $l+m\equiv 0$ (mod. 2).

If we put

$$\bar{x} = s x_1 s^{-1} \text{ and } \bar{y} = s y_1 s^{-1}, \tag{3.7}$$

then from (3.5) and (3.6) we have

$$\bar{x}^2 = -l^2 \pi^2 1_{n_1+n_2},\ \bar{y}^2 = -m^2 \pi^2 1_{n_1+n_2} \text{ and } \bar{x} + \bar{y} = f. \tag{3.8}$$

From this it follows that $\bar{y}^2 = (f - \bar{x})^2 = f^2 + \bar{x}^2 - (f\bar{x} + \bar{x}f)$, that is, by means of (3.5),

$$f\bar{x} + \bar{x}f = f^2 - l^2 \pi^2 \cdot 1_{n_1+n_2} + m^2 \pi^2 \cdot 1_{n_1+n_2}. \tag{3.9}$$

If we put $\bar{x} = \|x_{ij}\|$, then from (3.9) we have

$$(\bar{f}_i + \bar{f}_j)\, x_{ij} = \sqrt{-1}\, \pi\, (\bar{f}_i^{\,2} + l^2 - m^2)\, \delta_{ij} \tag{3.10}$$

where

$$f = \sqrt{-1}\, \pi \begin{pmatrix} \bar{f}_1 & & 0 \\ & \bar{f}_2 \ddots & \\ 0 & & \bar{f}_{n_1+n_2} \end{pmatrix}, \quad i, j = 1, 2, \cdots, n_1 + n_2.$$

From this we obtain

$$\begin{aligned} x_{ij} &= \frac{\sqrt{-1}\,\pi}{2\bar{f}_i}(\bar{f}_i^{\,2} + l^2 - m^2) && \text{for } \bar{f}_i \neq 0, \\ x_{ij} &= 0 && \text{for } \bar{f}_i + \bar{f}_j \neq 0, \end{aligned} \tag{3.11}$$

and x_{ii} is undetermined for $\bar{f}_i$, this can occur only for the case where $l^2 = m^2$, and also $x_{ij}\,(i \neq j)$ *is* undetermined for $\bar{f}_i + \bar{f}_j = 0$. By means of (3.6) and (3.11) we have

$$\bar{x} = \sqrt{-1}\, \pi \left[\sum_{a=1}^{p} \oplus\, \lambda_a 1_{ra} \oplus \sum_{\alpha=p+1}^{p+q} \oplus \begin{pmatrix} 1_{s_\alpha} & a \\ b & -1_{t_\alpha} \end{pmatrix} \oplus c \right], \tag{3.12}$$

where $a_\alpha, b_\alpha\, (\alpha = p+1, \cdots, p+q)$ and c are arbitrary matrices, this matrix c can occur only for the case where $l^2 = m^2$, and $\lambda_k = \frac{1}{2f_k}(f_k^{\,2} + l^2 - m^2)$ $(k = 1, 2, \cdots, p+q)$. And then, from $\bar{x}^2 = -l^2 \pi^2 1_{n_1+n_2}$, it follows that

$$\lambda_a = \pm l,\ \delta_a \equiv \frac{-1}{4f_a^{\,2}}(f_a + l + m)\,(f_a + l - m)\,(f_a - l + m)\,(f_a - l - m) = 0, \tag{3.13}$$
$$a = 1, 2, \cdots, p;$$

$$\left\{ \begin{aligned} & \begin{pmatrix} 0 & a_\alpha \\ b_\alpha & 0 \end{pmatrix}^2 = \begin{pmatrix} a_\alpha b_\alpha & 0 \\ 0 & b_\alpha a_\alpha \end{pmatrix} = \delta_\alpha \cdot 1_{s_\alpha + t_\alpha} \\ & \delta_\alpha \equiv l^2 - \lambda_\alpha^2 \equiv \frac{-1}{4f_\alpha^2}(f_\alpha + l + m)\,(f_\alpha + l - m)\,(f_\alpha - l + m)\,(f_\alpha - l - m) \end{aligned} \right. \tag{3.14}$$
$$\alpha = p+1, \cdots, p+q;$$

and

$$c^2 = l^2 \cdot 1_w, \tag{3.15}$$

where w is the order of the matrix c. Here if we put

$$i_\alpha = \begin{pmatrix} 1_{s_\alpha} & 0 \\ 0 & -1_{t_\alpha} \end{pmatrix} \text{ and } h_\alpha = \begin{pmatrix} 0 & a_\alpha \\ b_\alpha & 0 \end{pmatrix}$$

then we have

$$\begin{pmatrix} \lambda_\alpha 1_{s_\alpha} & a_\alpha \\ b_\alpha & -\lambda_\alpha 1_{t_\alpha} \end{pmatrix} = \lambda_\alpha i_\alpha + h_\alpha, \tag{3.16}$$

where $i_\alpha^2 = 1_{s_\alpha + t_\alpha}$, $h_\alpha^2 = \delta_\alpha \cdot 1_{s_\alpha + t_\alpha}$ and $i_\alpha h_\alpha + h_\alpha i_\alpha = 0$.
From (3.14) we have

$$\left(\det\begin{pmatrix} 0 & a_\alpha \\ b_\alpha & 0 \end{pmatrix}\right)^2 = \delta_\alpha^{\ s_\alpha + t_\alpha}.$$

If $\delta_\alpha \neq 0$, then we have $\det\begin{pmatrix} 0 & a_\alpha \\ b_\alpha & 0 \end{pmatrix} \neq 0$; and hence $s_\alpha = t_\alpha$. If we put

$$i_\alpha = \begin{pmatrix} 1_{s_\alpha} & 0 \\ 0 & -1_{s_\alpha} \end{pmatrix} \text{ and } j_\alpha = \frac{1}{\sqrt{\delta_\alpha}} h_\alpha = \frac{1}{\sqrt{\delta_\alpha}} \begin{pmatrix} 0 & a_\alpha \\ b_\alpha & 0 \end{pmatrix},$$

then we have

$$\begin{pmatrix} \lambda_\alpha 1_{s_\alpha} & a_\alpha \\ b_\alpha & -1_{s_\alpha} \end{pmatrix} = \lambda_\alpha i_\alpha + \sqrt{\delta_\alpha}\, j_\alpha$$

where $i_\alpha^2 = j_\alpha^2 = 1_{2s_\alpha}$ and $i_\alpha j_\alpha + j_\alpha i_\alpha = 0$. And moreover we have

$$a_\alpha b_\alpha = b_\alpha a_\alpha = \delta_\alpha \cdot 1_{s_\alpha}, \tag{3.17}$$

that is, there exist a_α^{-1} and b_α^{-1}. Since

$$\begin{pmatrix} p_\alpha & 0 \\ 0 & q_\alpha \end{pmatrix}^{-1} \begin{pmatrix} 1_{s_\alpha} & a \\ b & -1_{t_\alpha} \end{pmatrix} \begin{pmatrix} p_\alpha & 0 \\ 0 & q_\alpha \end{pmatrix} = \begin{pmatrix} \lambda_\alpha 1_{s_\alpha} & p_\alpha^{-1} a_\alpha q_\alpha \\ q_\alpha^{-1} b_\alpha p_\alpha & -\lambda_\alpha 1_{t_\alpha}, \end{pmatrix}, \tag{3.18}$$

where p_α and q_α are regular matrices of order s_α and t_α respectively, if we take $p_\alpha = \frac{1}{\sqrt{\delta_\alpha}} a_\alpha$ and $q_\alpha = 1_{s_\alpha}$ (this is possible when $s_\alpha = t_\alpha$), then we have

$$p_\alpha^{-1} a_\alpha q_\alpha = \sqrt{\delta_\alpha}\, 1_{s_\alpha}, \quad q_\alpha^{-1} b_\alpha p_\alpha = \frac{1}{\sqrt{\delta_\alpha}} b_\alpha a_\alpha = \sqrt{\delta_\alpha}\, 1_{s_\alpha}. \tag{3.19}$$

Therefore, for the case where $\delta_\alpha \equiv l^2 - \lambda_\alpha \neq 0$, we have

$$q^{-1} \begin{pmatrix} \lambda_\alpha 1_{s_\alpha} & a_\alpha \\ b_\alpha & -\lambda_\alpha 1_{s_\alpha} \end{pmatrix} q = \begin{pmatrix} \lambda_\alpha 1_{s_\alpha} & \sqrt{\delta_\alpha}\, 1_{s_\alpha} \\ \sqrt{\delta_\alpha}\, 1_{s_\alpha} & -\lambda_\alpha 1_{s_\alpha} \end{pmatrix} = \begin{pmatrix} \lambda_\alpha & \sqrt{\delta_\alpha} \\ \sqrt{\delta_\alpha} & -\lambda_\alpha \end{pmatrix} \times 1_{s_\alpha} \tag{3.20}$$

$$= (\lambda_\alpha i + \sqrt{\delta_\alpha}\, j) \times 1_{s_\alpha},$$

where $q=\begin{pmatrix} \frac{1}{\sqrt{\delta_\alpha}} a_\alpha & 0 \\ 0 & 1_{s_\alpha} \end{pmatrix}$, $i=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $j=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

If $\delta_\alpha=0$, then from (3.14) we have

$$\lambda_\alpha = \pm l, \quad a_\alpha b_\alpha = 0 \quad \text{and} \quad b_\alpha a_\alpha = 0. \tag{3.21}$$

Here also, by taking a suitable matrices p_α and q_α, we have

$$a'_\alpha = p_\alpha^{-1} a_\alpha q_\alpha = \begin{pmatrix} 1_{u_\alpha} & 0 \\ 0 & 0 \end{pmatrix} \tag{3.22}$$

where u_α is the rank of a_α, and hence $u_\alpha \leqq \min\ (s_\alpha, t_\alpha)$; and then by means of $a_\alpha b_\alpha = 0$ and $b_\alpha a_\alpha = 0$ we have

$$b'_\alpha = q_\alpha^{-1} b_\alpha p_\alpha = \begin{pmatrix} 0 & 0 \\ 0 & b_\alpha^* \end{pmatrix}, \tag{3.23}$$

where b_α^* is an arbitrary matrix of $(t_\alpha - u_\alpha)$-rows and $(s_\alpha - u_\alpha)$-columns. Furhtermore, by taking the matrices

$$p'_\alpha = \begin{pmatrix} 1_{u_\alpha} & 0 \\ 0 & p_\alpha^* \end{pmatrix} \quad \text{and} \quad q'_\alpha = \begin{pmatrix} 1_{u_\alpha} & 0 \\ 0 & q_\alpha^* \end{pmatrix}, \tag{3.24}$$

where p_α^* and q_α^* are the regular matrices of order $s_\alpha - u_\alpha$ and $t_\alpha - u_\alpha$ respectively, such that $q_\alpha^{*-1} b_\alpha^* p_\alpha^* = \begin{pmatrix} 0 & 0 \\ 0 & 1_{v_\alpha} \end{pmatrix}$, v_α being the rank of the matrix b_α^* i. e., b_α and hence $v_\alpha \leqq \min (t_\alpha - u_\alpha, s_\alpha - u_\alpha)$; we have

$$p'^{-1}_\alpha a'_\alpha q'_\alpha = \begin{pmatrix} 1_{u_\alpha} & 0 \\ 0 & 0 \end{pmatrix}, \quad q'^{-1}_\alpha b'_\alpha p'_\alpha = \begin{pmatrix} 0 & 0 \\ 0 & 1_{v_\alpha} \end{pmatrix}, \tag{3.25}$$

where u_α and v_α are the ranks of matrices a_α and b_α respectively.

As for the matrix c in (3.12), since $c^2 = l^2 \cdot 1_w$, we have

$$t^{-1} c t = \begin{pmatrix} l \cdot 1_{w_1} & 0 \\ 0 & -l \cdot 1_{w_2} \end{pmatrix}. \tag{3.26}$$

Thus we have

$$x_1 = r^{-1} \mathring{x} r, \quad \mathring{x} = \sqrt{-1}\,\pi \left[\sum_{a=1}^{p} \oplus \lambda_a 1_{r_a} \oplus \sum_{\alpha=p+1}^{p+q} \oplus \begin{pmatrix} \lambda_\alpha 1_{s_\alpha} & \mathring{a}_\alpha \\ \mathring{b}_\alpha & -\lambda_\alpha 1_{t_\alpha} \end{pmatrix} \oplus \begin{pmatrix} l \cdot 1_{w_1} & 0 \\ 0 & -l \cdot 1_{w_2} \end{pmatrix} \right] \tag{3.27}$$

$$= \sqrt{-1}\,\pi \left[\sum_{a=1}^{p} \oplus \lambda_a 1_{r_a} \oplus \sum_{\alpha=p+1}^{p+q} \oplus (\lambda_\alpha i_\alpha + h_\alpha) \oplus l \cdot 1_{w_1} \oplus (-l) \cdot 1_{w_2} \right],$$

where

$$(3.28)\quad \begin{cases} \lambda_a = \pm l, \quad \delta_a = 0, \quad a = 1, 2, \cdots, p; \\ \lambda_\alpha = \dfrac{1}{2f_\alpha}(f_\alpha{}^2 + l^2 - m^2),\ \mathring{a}_\alpha = \sqrt{\delta_\alpha}\, 1_{s_\alpha},\ \mathring{b}_\alpha = \sqrt{\delta_\alpha} \cdot 1_{s_\alpha},\ s_\alpha = t_\alpha \text{ for } \delta_\alpha \neq 0; \\ \lambda_\alpha = \pm l,\ \mathring{a}_\alpha = \begin{pmatrix} 1_{u_\alpha} & 0 \\ 0 & 0 \end{pmatrix},\ \mathring{b}_\alpha = \begin{pmatrix} 0 & 0 \\ 0 & 1_{v_\alpha} \end{pmatrix},\ u_v + v_\alpha \leqq \min(s_\alpha, t_\alpha) \text{ for } \delta_\alpha = 0, \\ \qquad \alpha = p+1, \cdots, p+q; \\ \delta_k \equiv l^2 - \lambda_k^2 = -\dfrac{1}{4f_k{}^2}(f_k + l + m)\,(f_k + l - m)\,(f_k - l + m)\,(f_k - l - m), \\ \qquad k = 1, \cdots, p+q; \end{cases}$$

and the last term in (3.27) can occur only for the case where $l^2 = m^2$.

And then, from the above results, (3.6) and (3.8) we have

$$(3.29)\quad y_1 = r^{-1}\mathring{y}r,\ \mathring{y} = \sqrt{-1}\,\pi\left[\sum_{a=1}^{p} \oplus \mu_a 1_{r_a} \oplus \sum_{\alpha=p+1}^{p+q} \oplus \begin{pmatrix} \mu_\alpha 1_{s_\alpha} & -\mathring{a}_\alpha \\ -\mathring{b}_\alpha & -\mu_\alpha 1_{t_\alpha} \end{pmatrix} \oplus \begin{pmatrix} -l \cdot 1_{w_1} & 0 \\ 0 & l \cdot 1_{w_2} \end{pmatrix}\right],$$

$$= \sqrt{-1}\,\pi\left[\sum_{a=1}^{p} \oplus \mu_a 1_{r_a} \oplus \sum_{\alpha=p+1}^{p+q} \oplus (\mu_\alpha i_\alpha - h_\alpha) \oplus l \cdot 1_{w_1} \oplus (-l) \cdot 1_{w_2}\right],$$

where

$$(3.30)\quad \begin{cases} \mu_a = \pm m, \quad \delta_a = 0, \quad a = 1, 2, \cdots, p; \\ \mu_\alpha = \dfrac{1}{2f_\alpha}(f_\alpha{}^2 - l^2 + m^2), & \text{for } \delta_\alpha \neq 0, \\ \mu_\alpha = \pm m, & \text{for } \delta_\alpha = 0, \end{cases}$$

and the last term in (3.29) can occur only for the case where $l^2 = m^2$.

Moreover, since

$$(3.31)\quad \mathring{x}\mathring{y} - \mathring{y}\mathring{x} = 2\pi^2 \sum_{\alpha=p+1}^{p+q} \oplus f_\alpha \cdot \begin{pmatrix} 0 & \mathring{a}_\alpha \\ -\mathring{b}_\alpha & 0 \end{pmatrix},$$

it is necessary and sufficient for $xy \neq yx$ that either $\mathring{a}_\alpha$ or $\mathring{b}_\alpha$ is not the zero matrix for some α, $\alpha = p+1, \cdots, p+q$, that is, if $\delta_\alpha \neq 0$ for some α, then $xy \neq yx$, and if $\delta_\alpha = 0$ for all α, then $xy \neq yx$ if and only if either u_α or v_α is not zero for some α.

Thus we have

THEOREM 3. *All the matrices of order n representing the non-commutative solutions of $e^x e^y = e^{x+y} = e^y e^x$ for the set $\mathfrak{S}$ are given by $x = u^{-1}\overset{*}{x}u$, $y = u^{-1}\overset{*}{y}u$ (u is a regular matrix),*

$$\overset{*}{x} = \sqrt{-1}\,\pi\left[l 1_{p_1} \oplus (-l) 1_{p_2} \oplus \sum_{\alpha=1}^{q_1} \oplus \left(\begin{pmatrix} \lambda_\alpha & \sqrt{\delta_\alpha} \\ \sqrt{\delta_\alpha} & -\lambda_\alpha \end{pmatrix} \times 1_{s_\alpha}\right) \oplus \sum_{\beta=q_1+1}^{q_1+q_2} \oplus \begin{pmatrix} \lambda_\beta 1_{s_\beta} & \mathring{a}_\beta \\ \mathring{b}_\beta & -\lambda_\beta 1_{t_\beta} \end{pmatrix}\right],$$

$$\overset{*}{y} = \sqrt{-1}\,\pi \Big[m 1_{p_{11}} \oplus (-m) 1_{p_{12}} \oplus m 1_{p_{12}} \oplus (-m) 1_{p_{22}} \oplus$$

$$\sum_{\alpha=1}^{q_1} \oplus \left(\begin{pmatrix} \mu_\alpha & -\sqrt{\delta_\alpha} \\ -\sqrt{\delta_\alpha} & -\mu_\alpha \end{pmatrix} \times 1_{s_\alpha} \right) \oplus \sum_{\beta=q_1+1}^{q_1+q_2} \oplus \begin{pmatrix} \mu_\beta 1_{s_\beta} & -\mathring{a}_\beta \\ -\mathring{b}_\beta & -\mu_\beta 1_{t_\beta} \end{pmatrix} \Big],$$

where

$$\lambda_\alpha = \frac{1}{2f_\alpha}(f_\alpha^2 + l^2 - m^2),\ \mu_\alpha = \frac{1}{2f_\alpha}(f_\alpha^2 - l^2 + m^2), \delta_\alpha = l^2 - \lambda_\alpha^2 = m^2 - \mu_\alpha^2 \neq 0,$$

$$\alpha = 1, 2, \cdots, q_1,$$

$$\lambda_\beta = \pm l,\ \mu_\beta = \pm m,\ \mathring{a}_\beta = \begin{pmatrix} 1_{u_\beta} & 0 \\ 0 & 0 \end{pmatrix},\ \ \mathring{b}_\beta = \begin{pmatrix} 0 & 0 \\ 0 & 1_{v_\beta} \end{pmatrix},\ \beta = q_1 + 1, \cdots, q_1 + q_2,$$

l, m and f_α are non-zero integers such that $l + m + f_\alpha \equiv 0$ (mod 2), p_{11}, p_{12}, p_{21}, p_{22}, p_1, p_2, *s's and t's are arbitrary natural numbers such that*

$$p_{11} + p_{12} = p_1,\ p_{21} + p_{22} = p_2,\ p_1 + p_2 + 2\sum_{\alpha=1}^{q_1} s_\alpha + \sum_{\beta=q_1+1}^{q_1+q_2} (s_\beta + t_\beta) = n \text{ and } q_1 + q_2 \neq 0\,;$$

and if $q_1 = 0$, then either u_β or v_β must not be zero, at least, for some β, $\beta = q_1 + 1, \cdots, q_1 + q_2$.

(II). The case where $e^x e^y \neq e^y e^x$. We shall first consider the case (II$_1$) and the case for $\alpha\beta \neq 0$ of (II$_2$). We have to determine the matrices x and y of order n such that

$$\begin{aligned} &x^3 = \alpha^2 x,\quad x^2 y = y x^2 = \alpha^2 y,\quad x y^2 = y^2 x = \beta^2 x,\quad y^3 = \beta^2 y, \\ &x(xy - yx) = \alpha(xy - yx) \text{ and } y(xy - yx) = \beta(xy - yx), \end{aligned} \tag{3.32}$$

from which the case (II$_1$) is obtained by taking $\beta = -\alpha$. From (3.2), (3.3) and (3.4) we have

$$\begin{aligned} &x = p^{-1} x_0 p,\quad x_0 = \begin{pmatrix} x_1 & 0 \\ 0 & 0 \end{pmatrix},\quad x_1 = \begin{pmatrix} \alpha 1_{n_1} & 0 \\ 0 & -\alpha 1_{n_2} \end{pmatrix}, \\ &y = p^{-1} y_0 p,\quad y_0 = \begin{pmatrix} y_1 & 0 \\ 0 & 0 \end{pmatrix},\quad y_1^2 = \beta^2 \cdot 1_{n_1+n_2}. \end{aligned} \tag{3.33}$$

And then we have

$$x_1(x_1 y_1 - y_1 x_1) = \alpha(x_1 y_1 - y_1 x_1),\ y_1(x_1 y_1 - x_1 y_1) = \beta(x_1 y_1 - y_1 x_1)\,; \tag{3.34}$$

if we put $y_1 = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}$ where y_{11} and y_{22} are the matrices of order n_1 and n_2 respectively, then from the first of (3.34) we have

$$y_{21} = 0, \tag{3.35}$$

and then from the second of (3.34) we have

(3.36) $$y_{11}y_{12} = \beta y_{12}.$$

And also from $y_1^2=\beta^2\cdot 1_{n_1+n_2}$ we have

(3.37) $$y_{11}^2=\beta^2 1_{n_1},\ y_{22}^2=\beta^2 1_{n_2} \text{ and } y_{11}y_{12}+y_{12}y_{22}=0\,;$$

from (3.36) and the last of (3.37) it follows that

(3.38) $$y_{12}y_{22} = -\beta y_{12}.$$

By taking the suitable regular matrices q_1 and q_2 we have from the first two of (3.37)

(3.39) $$y'_{11}=q_1^{-1}y_{11}q_1=\begin{pmatrix}-\beta 1_{n_{11}} & 0\\ 0 & \beta 1_{n_{12}}\end{pmatrix},\ y'_{22}=q_2^{-1}y_{22}q_2=\begin{pmatrix}-\beta 1_{n_{21}} & 0\\ 0 & \beta 1_{n_{22}}\end{pmatrix},$$

where $n_{11}+n_{12}=n_1$ and $n_{21}+u_{22}=n_2$. Then by the matrix $q=\begin{pmatrix}q_1 & 0\\ 0 & q_2\end{pmatrix}$, we have

(3.40) $$x'_1=q^{-1}x_1q=x_1,\ y'_1=q^{-1}y_1q=\begin{pmatrix}y'_{11} & y'_{12}\\ 0 & y'_{22}\end{pmatrix},\ y'_{12}=q_1^{-1}y_{12}q_2\,;$$

and also (3.36) and (3.38) are written as

(3.41) $$y'_{11}y'_{12}=\beta y'_{12},\ y'_{12}y'_{22}=-\beta y'_{12}.$$

Here if we put

(3.42) $$y'_{12}=\begin{pmatrix}u_{11} & u_{12}\\ u_{21} & u_{22}\end{pmatrix},$$

where u_{11} is the matrix of n_{11}–rows and n_{21}–columus, then from the first of (3.41) we have

(3.43) $$u_{11}=0 \text{ and } u_{12}=0,$$

and then from the second of (3.41) we have

(3.44) $$u_{22}=0.$$

Hence we have

(3.45) $$y'_{12}=\begin{pmatrix}0 & 0\\ u_{21} & 0\end{pmatrix};$$

furthermore we can take the regular matrices r_{12} and r_{21} such that

$$r_{12}^{-1}u_{21}r_{21}=\begin{pmatrix}0 & 0\\ 1_u & 0\end{pmatrix},$$

where u is the rank of the matrix u_{21}, the orders of r_{12} and r_{21} are n_{12} and n_{21} respectively. So, by taking the regular matrix

$$r = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}, \text{ where } r_1 = \begin{pmatrix} 1_{n_{11}} & 0 \\ 0 & r_{12} \end{pmatrix} \text{ and } r_2 = \begin{pmatrix} r_{21} & 0 \\ 0 & 1_{n_{21}} \end{pmatrix},$$

we have

$$\text{(3.46)} \quad \left\{ \begin{aligned} & x_1'' = r^{-1} x_1' r = x_1' = x_1, \ y_1'' = r^{-1} y_1' r = \begin{pmatrix} y_{11}' & y_{12}'' \\ 0 & y_{22}' \end{pmatrix}, \\ & y_{12}'' = r_1^{-1} y_{12}' r_2 = \begin{pmatrix} 0 & 0 \\ r_{12}^{-1} u_{21} r_{21} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1_u & 0 \end{pmatrix}, \end{aligned} \right.$$

where $u \leqq \min(n_{12}, n_{21})$.

Thus we have

$$\text{(3.47)} \quad \left\{ \begin{aligned} & x = t^{-1} \mathring{x} t, \quad y = t^{-1} \mathring{y} t, \qquad (t \text{ is a regular matrix}), \\ & \mathring{x} = \alpha 1_{n_1} \oplus (-\alpha) 1_{n_2} \oplus 0_{n_3}, \\ & \mathring{y} = [(-\beta) 1_{n_{11}} \oplus \beta 1_{n_{12}} \oplus (-\beta) 1_{n_{21}} \oplus \beta 1_{n_{22}} \oplus 0_{n_3}] + j_u, \\ & j_u = \left(\begin{array}{c|c} 0_{n_1} & \begin{matrix} 0 & 0 \\ 1_u & 0 \end{matrix} \\ \hline 0 & 0 \end{array} \right), \end{aligned} \right.$$

where u, n_1, n_2, n_3, n_{11}, n_{12}, n_{21} and n_{22} are the non-negative integers such that $n_{11} + n_{12} = n_1$, $n_{21} + n_{22} = n_2$, $n_1 + n_2 + n_3 = n$ and $u \leqq \min(n_{12}, n_{21})$. It is easily seen that $xy \neq yx$ if and only if $1 \leqq u$.

Next we have to consider the case where $\alpha\beta = 0$ in (II$_2$). Since in the case (II$_2$) $\alpha^2 \neq \beta^2$, either α or β is not zero. So we shall first consider the case where $\alpha \neq 0$ and $\beta = 0$. Then in (3.33) we have

$$\text{(3.48)} \qquad y_1{}^2 = 0_{n_1+n_2};$$

(3.34) is replaced by

$$\text{(3.49)} \qquad x_1 (x_1 y_1 - y_1 x_1) = \alpha (x_1 y_1 - y_1 x_1), \quad y_1 (x_1 y_1 - y_1 x_1) = 0_{n_1+n_2}.$$

If we put $y_1 = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}$ where y_{11} and y_{22} are the matrices of order n_1 and n_2 respectively, then similarly as in the previous considerations, we have from (3.48) and (3.49),

$$\text{(3.50)} \qquad y_{21} = 0, \quad y_{11} y_{12} = 0, \quad y_{12} y_{22} = 0, \quad y_{11}^2 = 0 \text{ and } y_{22}^2 = 0.$$

By choosing the suitable regular matrices q_1 and q_2, we have from $y_{11}^2 = 0$ and $y_{22}^2 = 0$,

$$y'_{11}=q_1^{-1}y_{11}q_1=\begin{pmatrix}0 & 0_{s_{11}} & 0\\ 1_{s_{11}} & 0 & 0\\ 0 & 0 & 0_{s_{12}}\end{pmatrix},\quad y'_{22}=q_2^{-1}y_{22}q_2=\begin{pmatrix}0_{s_{22}} & 0 & 0\\ 0 & 0_{s_{21}} & 0\\ 0 & 1_{s_{21}} & 0_{s_{21}}\end{pmatrix}, \tag{3.51}$$

where $2s_{11}+s_{12}=n_1$ and $2s_{21}+s_{22}=n_2$. And then, by the matrix $q=\begin{pmatrix}q_1 & 0\\ 0 & q_2\end{pmatrix}$ we have

$$x'_1=q^{-1}x_1q=x_1,\quad y'_1=q^{-1}y_1q=\begin{pmatrix}y'_{11} & y'_{12}\\ 0 & y'_{22}\end{pmatrix},\quad y'_{12}=q_1^{-1}y_{12}q_2. \tag{3.52}$$

And moreover we have

$$y'_{11}y'_{12}=0,\quad y'_{12}y'_{22}=0. \tag{3.53}$$

Here if we put

$$y'_{12}=\begin{array}{ccc|c} \overset{s_{22}}{\leftrightarrow} & \overset{s_{21}}{\leftrightarrow} & \overset{s_{21}}{\leftrightarrow} & \\ v_{11} & v_{12} & v_{13} & \updownarrow s_{11}\\ v_{21} & v_{22} & v_{23} & \updownarrow s_{11}\\ v_{31} & v_{32} & v_{33} & \updownarrow s_{12}\end{array}, \tag{3.54}$$

then from (3.53) we have

$$y'_{12}=\begin{pmatrix}0 & 0 & 0\\ v_{21} & v_{22} & 0\\ v_{31} & v_{32} & 0\end{pmatrix},$$

that is,

$$y'_{12}=\begin{pmatrix}0 & 0\\ v & 0\end{pmatrix}, \tag{3.55}$$

where v is an arbitrary matrix of $(s_{11}+s_{12})$–rows and $(s_{21}+s_{22})$–columns.

Thus we have

$$\left\{\begin{aligned} & x=t^{-1}\mathring{x}t,\quad y=t^{-1}\mathring{y}t,\qquad (t \text{ is a regular matrix}),\\ & \mathring{x}=\alpha 1_{n_1}\oplus(-\alpha)1_{n_2}\oplus 0_{n_3},\\ & \mathring{y}=\left[\left(\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\times 1_{s_{11}}\right)\oplus 0_{s_{12}+s_{22}}\oplus\left(\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\times 1_{s_{21}}\right)\oplus 0_{n_3}\right]+J_v,\\ & J_v=\left(\begin{array}{c|cc} 0_{n_1} & \begin{matrix}0\\ v\end{matrix} & \begin{matrix}0\\ 0\end{matrix}\\ \hline 0 & \multicolumn{2}{c}{0}\end{array}\right), \end{aligned}\right. \tag{3.56}$$

where v is an arbitrary matrix of $(s_{11}+s_{12})$–rows and $(s_{21}+s_{22})$–columns, $2s_{11}+s_{12}=n_1$, $2s_{21}+s_{22}=n_2$ and $n_1+n_2+n_3=n$. And $xy\neq yx$, if and only if $v\neq 0$.

As for the case where $\alpha=0$ and $\beta\neq 0$, by exchanging the role of x and y, we have from the above result

$$\left\{\begin{aligned} & x = s^{-1}\overset{\circ}{x}s, \quad y = s^{-1}\overset{\circ}{y}s, \qquad (s \text{ is a regular matrix}), \\ & \overset{\circ}{x} = \left[\left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \times 1_{s_{11}}\right) \oplus 0_{s_{12}+s_{22}} \oplus \left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \times 1_{s_{21}}\right) \oplus 0_{n_3}\right] + J_v, \\ & \overset{\circ}{y} = \beta 1_{n_1} \oplus (-\beta) 1_{n_2} \oplus 0_{n_3}, \end{aligned}\right.$$

where v is an arbitrary matrix of $(s_{11}+s_{12})$-rows and $(s_{21}+s_{22})$-columns, $2s_{11}+s_{12}=n_1$, $2s_{21}+s_{22}=n_2$ and $n_1+n_2+n_3=n$. And $xy \neq yx$, if and only of $v \neq 0$.

Summarizing the above results we have

Theorem 4. *All the matrices of order n representing the non-commutative solutions of $e^xe^y = e^{x+y} \neq e^xe^y$ for the set $\mathfrak{S}$ of iteration pairs are given by*

$$x = u^{-1}\overset{*}{x}u, \quad y = u^{-1}\overset{*}{y}u, \qquad (u \textit{ is a regular matrix}),$$

$$\text{(i)} \quad \overset{*}{x} = \alpha 1_{n_1} \oplus (-\alpha) 1_{n_2} \oplus 0_{n_3}, \quad \overset{*}{y} = \left[(-\beta) 1_{n_{11}} \oplus \beta 1_{n_{12}} \oplus (-\beta) 1_{n_{21}} \oplus \beta 1_{n_{22}} \oplus 0_{n_3}\right] + j_u,$$

$$j_u = \left(\begin{array}{c|c} 0_{n_1} & \begin{matrix} 0 & 0 \\ 1_u & 0 \end{matrix} \\ \hline 0 & 0 \end{array}\right),$$

where $e(-2\alpha)=e(2\beta)$, $\alpha^2 \neq \beta^2$ *and* $\alpha\beta \neq 0$, or $e(2\alpha)=1$ *and* $\beta=-\alpha \neq 0$; $n_{11}+n_{12}=n_1$, $n_{21}+n_{22}=n_2$, $n_1+n_2+n_3=n$ *and* $1 \leqq u \leqq \min(n_{12}, n_{21})$;
and

$$\text{(ii)} \quad \overset{*}{x} = \alpha 1_{n_1} \oplus (-\alpha) 1_{n_2} \oplus 0_{n_3},$$

$$\overset{*}{y} = \left[\left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \times 1_{s_{11}}\right) \oplus 0_{s_{12}+s_{22}} \oplus \left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \times 1_{s_{21}}\right) \oplus 0_{n_3}\right] + J_v;$$

$$\text{(iii)} \quad \overset{*}{x} = \left[\left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \times 1_{s_{11}}\right) \oplus 0_{s_{12}+s_{22}} \oplus \left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \times 1_{s_{21}}\right) \oplus 0_{n_3}\right] + J_v,$$

$$\overset{*}{y} = \beta 1_{n_1} \oplus (-\beta) 1_{n_2} \oplus 0_{n_3};$$

$$J_v = \left(\begin{array}{c|c} 0_{n_1} & \begin{matrix} 0 & 0 \\ v & 0 \end{matrix} \\ \hline 0 & 0 \end{array}\right);$$

where $e(-2\alpha)=1$ *and* $\alpha \neq 0$; $e(2\beta)=1$ *and* $\beta \neq 0$; $2s_{11}+s_{12}=n_1$, $2s_{21}+s_{22}=n_2$, $n_1+n_2+n_3=n$; *v is an arbitrary non-zero matrix of $(s_{11}+s_{12})$-rows and $(s_{21}+s_{22})$-columns.*

II. NON-COMMUTATIVE SOLUTIONS OF $e^xe^y=e^{x+y}$ FOR AN ALGEBRA REPRESENTED BY THE TRIANGULAR MATRICES

4. Actual construction of the non-commutative solutions of $e^xe^y=e^{x+y}$.

In this section we shall consider the non-commutative solutions of $e^xe^y=e^{x+y}$ for an algebra which is represented by the triangular matrices. We have only to confine ourselves to a subalgebra of the algebra $\mathfrak{T}_n$ which is composed of all the complex triangular matrices of order n. An example of such an algebra is given by a matric algebra which is a solvable Lie algebra.

We shall state the method to construct actually all the non-commutative solutions of $e^xe^y=e^{x+y}$ for $\mathfrak{T}_n$. Now let x and y be the elements of $\mathfrak{T}_r$, then x and y are written as

$$x=\begin{pmatrix}\lambda & 0\\ u & x_0\end{pmatrix},\quad y=\begin{pmatrix}\mu & 0\\ v & y_0\end{pmatrix}, \tag{4.1}$$

where x_0 and y_0 are the elements of $\mathfrak{T}_{r-1}$; λ and μ are complex numbers. It is easily seen that

$$\begin{cases} e^x=e^{\lambda}\cdot e^{\begin{pmatrix}0 & 0\\ u & x_1\end{pmatrix}},\quad e^y=e^{\mu}\cdot e^{\begin{pmatrix}0 & 0\\ v & y_1\end{pmatrix}},\\ x_1=x_0-\lambda,\quad y_1=y_0-\mu\,; \end{cases} \tag{4.2}$$

and then

$$e^{\begin{pmatrix}0 & 0\\ u & x_1\end{pmatrix}}=\begin{pmatrix}1 & 0\\ u_1 & e^{x_1}\end{pmatrix},\quad e^{\begin{pmatrix}0 & 0\\ v & y_1\end{pmatrix}}=\begin{pmatrix}1 & 0\\ v_1 & e^{y_1}\end{pmatrix}, \tag{4.3}$$

where

$$u_1=(1+\frac{x_1}{2!}+\frac{x_1^2}{3!}+\cdots)\,u=\frac{e^{x_1}-1}{x_1}u,$$

$$v_1=(1+\frac{y_1}{2!}+\frac{y_1^2}{3!}+\cdots)\,v=\frac{e^{y_1}-1}{y_1}v.$$

And then we have

$$e^{x+y}=e^{\lambda+\mu}.\begin{pmatrix}1 & 0\\ w_1 & e^{x_1+y_1}\end{pmatrix}, \tag{4.4}$$

where

$$w_1=\left(1+\frac{1}{2!}(x_1+y_1)+\frac{1}{3!}(x_1+y_1)^2+\cdots\right)(u+v)=\frac{e^{x_1+y_1}-1}{x_1+y_1}(u+v).$$

On the other hand, from (4.2) and (4.3) we have

$$e^xe^y=e^{\lambda+\mu}.\begin{pmatrix}1 & 0\\ u_1+e^{x_1}v_1 & e^{x_1}e^{y_1}\end{pmatrix}. \tag{4.5}$$

Hence, from (4.4) and (4.5) it follows that $e^xe^y=e^{x+y}$ is equivalent to

$$e^{x_1}e^{y_1}=e^{x_1+y_1}\ \text{i. e.,}\quad e^{x_0}e^{y_0}=e^{x_0+y_0}\ \text{and}\ u_1+e^{x_1}v_1=w_1. \tag{4.6}$$

By means of the above expressions for u_1, v_1 and w_1, the latter of (4.6) is written as

$$\frac{e^{x_1}-1}{x_1}\cdot u+e^{x_1}\cdot\frac{e^{y_1}-1}{y_1}v-\frac{e^{x_1+y_1}-1}{x_1+y_1}(u+v)=0, \tag{4.7}$$

$$\text{i. e., }\left(\frac{e^{x_1}-1}{x_1}-\frac{e^{x_1+y_1}-1}{x_1+y_1}\right)u+\left(e^{x_1}\frac{e^{y_1}-1}{y_1}-\frac{e^{x_1+y_1}-1}{x_1+y_1}\right)v=0.$$

But then under the condition $e^{x_1}e^{y_1}=e^{x_1+y_1}$ we have

$$\begin{aligned} e^{x_1}\cdot\frac{e^{y_1}-1}{y_1}-\frac{e^{x_1+y_1}-1}{x_1+y_1} &= (x_1+y_1)^{-1}\left((x_1+y_1)e^{x_1}(e^{y_1}-1)-(e^{x_1+y_1}-1)y_1\right)y_1^{-1}\\ &= (x_1+y_1)^{-1}\left((x_1+y_1)(e^{x_1+y_1}-e^{x_1})-(e^{x_1+y_1}-1)y_1\right)y_1^{-1}\\ &= (x_1+y_1)^{-1}\left((e^{x_1+y_1}-1)x_1-(x_1+y_1)(e^{x_1}-1)\right)y_1^{-1}\\ &= \left(\frac{e^{x_1+y_1}-1}{x_1+y_1}-\frac{e^{x_1}-1}{x_1}\right)x_1y_1^{-1}, \end{aligned}$$

(if necessary, by replacing y_0 by $y_0+\beta$, y_1^{-1} can be considered),

that is,

$$e^{x_1}\cdot\frac{e^{y_1}-1}{y_1}-\frac{e^{x_1+y_1}-1}{x_1+y_1}=\left(\frac{e^{x_1+y_1}-1}{x_1+y_1}-\frac{e^{x_1}-1}{x_1}\right)x_1y_1^{-1}, \tag{4.8}$$

hence we have

$$\left(\frac{e^{x_1+y_1}-1}{x_1+y_1}-\frac{e^{x_1}-1}{x_1}\right)(x_1y_1^{-1}v-u)=0. \tag{4.9}$$

Here if we assume that $x_1y_1=y_1x_1$, then (4.9) is written as

$$\frac{1}{y_1}\left(\frac{e^{x_1+y_1}-1}{x_1+y_1}-\frac{e^{x_1}-1}{x_1}\right)(x_1v-y_1u)=0, \tag{4.10}$$

or in the other form

$$\frac{1}{(x_1+y_1)}\left(\frac{e^{y_1}-1}{y_1}-\frac{e^{-x_1}-1}{-x_1}\right)(x_1v-y_1u)=0. \tag{4.11}$$

And moreover we have from (4.1)

$$xy-yx=\begin{pmatrix} 0 & 0\\ x_1v-y_1u & x_1y_1-y_1x_1\end{pmatrix}. \tag{4.12}$$

Thus we have

THEOREM 5. *All the non-commutative solutions of* $e^xe^y=e^{x+y}$ *for* $\mathfrak{T}_r$ *are given by*

$$x=\begin{pmatrix}\lambda & 0\\ u & x_0\end{pmatrix},\quad y=\begin{pmatrix}\mu & 0\\ v & y_0\end{pmatrix}$$

such that

$$e^{x_0} e^{y_0} = e^{x_0 + y_0},$$

$$\left(\frac{e^{(x_0+y_0)-(\lambda+\mu)}-1}{(x_0+y_0)-(\lambda+\mu)} - \frac{e^{x_0-\lambda}-1}{x_0-\lambda}\right)\left((x_0-\lambda)\,(y_0-\mu)^{-1}\,v-u\right)=0$$

and

$$x_0 y_0 \neq y_0 x_0 \quad \textit{or} \quad (x_0-\lambda)\,v-(y_0-\mu)\,u \neq 0,$$

where x_0, $y_0 \in \mathfrak{T}_{r-1}$ *and* λ, μ *are complex numbers.*

By means of Theorem 5, we can find inductively all the non-commutative solutions of $e^x e^y = e^{x+y}$ for $\mathfrak{T}_n$. For $\mathfrak{T}_1$, the solutions are given by all the pairs of complex numbers. Let x_0 and y_0 be any elements of $\mathfrak{T}_{r-1}$ such that $e^{x_0}e^{y_0}=e^{x_0+y_0}$, then it follows from (4.9) that all the non-commutative solutions $x=\begin{pmatrix}\lambda & 0\\ u & x_0\end{pmatrix}$ and $y=\begin{pmatrix}\mu & 0\\ v & y_0\end{pmatrix}$ of $e^x e^y=e^{x+y}$ for $\mathfrak{T}_r$ is obtained by taking

$$u=(x_0-\lambda)\,(y_0-\mu)^{-1}v+w \tag{4.13}$$

where

$$\left(\frac{e^{(x_0+y_0)-(\lambda+\mu)}-1}{(x_0+y_0)-(\lambda+\mu)} - \frac{e^{x-\lambda}-1}{x_0-\lambda}\right) w = 0, \tag{4.14}$$

for any complex numbers λ and μ, and any matrix v.

But if $x_0 y_0 = y_0 x_0$, then the following condition must be satisfied :

$$\det\left[\frac{1}{(y_0-\mu)}\left(\frac{e^{(x_0+y_0)-(\lambda+\mu)}-1}{(x_0+y_0)-(\lambda+\mu)} - \frac{e^{x_0-\lambda}-1}{x_0-\lambda}\right)\right]=0\,. \tag{4.15}$$

If we put

$$x_0=\begin{pmatrix}\lambda_1 & & 0\\ & \lambda_2 & \\ & & \ddots & \\ \# & & & \lambda_{r-1}\end{pmatrix}, \quad y_0=\begin{pmatrix}\mu_1 & & 0\\ & \mu_2 & \\ & & \ddots & \\ \# & & & \mu_{r-1}\end{pmatrix},$$

then (4.15) is written as

$$\prod_{i=1}^{r-1}\left(\frac{1}{(\mu_i-\mu)}\left(\frac{e^{(\lambda_i+\mu_i)-(\lambda+\mu)}-1}{(\lambda_i+\mu_i)-(\lambda+\mu)} - \frac{e^{\lambda_i-\lambda}-1}{\lambda_i-\lambda}\right)\right)=0, \tag{4.16}$$

or in the other form

$$\prod_{i=1}^{r-1}\frac{1}{(\lambda_i+\mu_i)-(\lambda+\mu)}\left(\frac{e^{\mu_i-\mu}-1}{\mu_i-\mu} - \frac{e^{-(\lambda_i-\lambda)}-1}{-(\lambda_i-\lambda)}\right)=0. \tag{4.17}$$

From this it follows that

$$\frac{1}{(\lambda_i+\mu_i)-(\lambda+\mu)}\left(\frac{e^{\mu_i-\mu}-1}{\mu_i-\mu}-\frac{e^{-(\lambda_i-\lambda)}-1}{-(\lambda_i-\lambda)}\right)=0 \tag{4.18}$$

for some i, $i=1, 2, \cdots, r-1$; and hence

$$\frac{e^{\lambda-\lambda_i}-1}{\lambda-\lambda_i}=\frac{e^{\mu_i-\mu}-1}{\mu_i-\mu} \text{ and } \lambda+\mu\neq\lambda_i+\mu_i. \tag{4.19}$$

for some i, $i=1, 2, \cdots, r-1$. That is, we must take λ and μ satisfying (4.19).

Thus we see that all the non-commutative solutions of $e^x e^y=e^{x+y}$ for $\mathfrak{T}_n$ are obtained inductively.

By the method of construction we have

COROLLARY. *Any non-commutative solution* $x=\begin{pmatrix}\lambda_1 & & 0\\ & \lambda_2 & \\ & & \ddots & \\ \# & & & \lambda_n\end{pmatrix}$ *and* $y=\begin{pmatrix}\mu_1 & & 0\\ & \mu_2 & \\ & & \ddots & \\ \# & & & \mu_n\end{pmatrix}$ *must satisfy* $\dfrac{e^{\lambda_i-\lambda_j}-1}{\lambda_i-\lambda_j}=\dfrac{e^{\mu_i-\mu_j}-1}{\mu_i-\mu_j}$ *and* $\lambda_i-\lambda_j\neq\mu_i-\mu_j$ *for some pair* i, j; $i, j=1, 2,\cdots,n$.

5. Some properties of the solutions of $e^x e^y=e^{x+y}$ for $\mathfrak{T}_n$. In this section we shall consider some properties of the solutions of $e^x e^y=e^{x+y}$ for $\mathfrak{T}_n$.

Let $x\in\mathfrak{T}_n$ be written as

$$x=\begin{pmatrix}* & & & 0\\ * & \bar{x}_\alpha & & \\ * & \bar{x}_{\alpha\beta} & \bar{x}_\beta & \\ * & * & * & *\end{pmatrix}, \tag{5.1}$$

(where $\bar{x}_\alpha$ and $\bar{x}_\beta$ themselves need not be triangular, but may be any square matrices in the following considerations), then we shall write

$$\left\{\begin{array}{l} x_\alpha=\begin{pmatrix}0 & & & 0\\ 0 & \bar{x}_\alpha & & \\ 0 & 0 & 0 & \\ 0 & 0 & 0 & 0\end{pmatrix},\quad x_\beta=\begin{pmatrix}0 & & & 0\\ 0 & 0 & & \\ 0 & 0 & \bar{x}_\beta & \\ 0 & 0 & 0 & 0\end{pmatrix},\quad x_{\alpha\beta}=\begin{pmatrix}0 & & & 0\\ 0 & 0 & & \\ 0 & \bar{x}_{\alpha\beta} & 0 & \\ 0 & 0 & 0 & 0\end{pmatrix} \\ \\ \text{and } x_\gamma = x_\alpha + x_\beta + x_{\alpha\beta}; \end{array}\right. \tag{5.2}$$

and furthermore by P_α and $P_{\alpha\beta}$ we shall denote the mappings:

$$P_\alpha:\ x\rightarrow x_\alpha \quad \text{and} \quad P_{\alpha\beta}:\ x\rightarrow x_{\alpha\beta} \tag{5.3}$$

respectively, and call P_α and $P_{\alpha\beta}$ the projection operations.

The following are easily seen:

$$x_\alpha y_\beta = x_\beta y_\alpha = x_\alpha y_{\alpha\beta} = x_{\alpha\beta} y_\beta = x_{\alpha\beta} y_{\alpha\beta} = 0, \tag{5.4}$$

i. e., $P_\alpha(x)\, P_\beta(y) = P_\beta(x)\, P_\alpha(y) = P_\alpha(x)\, P_{\alpha\beta}(y) = P_{\alpha\beta}(x)\, P_\beta(y) = P_{\alpha\beta}(x)\, P_{\alpha\beta}(y) = 0,$

$$\begin{cases} P_\gamma(x+y) = P_\gamma(x) + P_\gamma(y), \\ P_\gamma(xy) = P_\gamma(x)\cdot P_\gamma(y), \end{cases} \tag{5.5}$$

$$P_\gamma(x) = P_\alpha(x) + P_\beta(x) + P_{\alpha\beta}(x), \quad P_\alpha P_\beta(x) = P_\beta P_\alpha(x) = 0, \tag{5.6}$$

and

$$P_\gamma e^x = \dot{e}^{x_\gamma} \tag{5.7}$$

where $\dot{e}^{x_\gamma} = P_\gamma\cdot 1 + \sum_{r=1}^{\infty} \frac{x_\gamma{}^r}{r!}$.

If $e^xe^y=e^{x+y}$, then from (5.5) we have

$$e^{x_\gamma}e^{y_\gamma} = e^{(x+y)_\gamma} = e^{x_\gamma+y_\gamma}; \tag{5.8}$$

and if $xy=yx$, then $x_\gamma y_\gamma=y_\gamma x_\gamma$.
And furthermore by means of (5.4), it follows from $x_\gamma=x_\alpha+x_\beta+x_{\alpha\beta}$ that

$$(x_\gamma)^m = x_\alpha{}^m + x_\beta{}^m + \sum_{m_1+m_2=m-1} x_\beta{}^{m_1} x_{\alpha\beta} x_\alpha{}^{m_2},$$

and hence

$$\begin{aligned} e^{x_\gamma} &= \dot{e}^{x_\alpha} + \dot{e}^{x_\beta} + \sum_{m_1+m_2=m-1} \frac{1}{m!} x_\beta{}^{m_1} x_{\alpha\beta} x_\alpha{}^{m_2}, \\ &= P_\alpha e^x + P_\beta e^x + \sum_{m=1}^{\infty} \sum_{m_1+m_2=m-1} \frac{1}{m!} x_\beta{}^{m_1} x_{\alpha\beta} x_\alpha{}^{m_2}. \end{aligned} \tag{5.9}$$

But, on the other hand, from (5.6) and (5.7) we have

$$e^{x_\gamma} = P_\gamma e^x = P_\alpha e^x + P_\beta e^x + P_{\alpha\beta} e^x, \tag{5.10}$$

therefore we have

$$P_{\alpha\beta} e^x = \sum_{m=1}^{\infty} \sum_{m_1+m_2=m-1} \frac{1}{m!} x_\beta{}^{m_1} x_{\alpha\beta} x_\alpha{}^{m_2}. \tag{5.11}$$

By means of (5.4), we have from (5.10)

$$P_\gamma(e^xe^y) = e^{x_\gamma}e^{y_\gamma} = P_\alpha e^x\cdot P_\alpha e^y + P_\beta e^x P_\beta e^y + P_\beta e^x\cdot P_{\alpha\beta} e^y + P_{\alpha\beta} e^x\cdot P_\alpha e^y, \tag{5.12}$$

and by (5.5)

$$= P_\alpha(e^xe^y) + P_\beta(e^xe^y) + P_\beta e^x P_{\alpha\beta} e^y + P_{\alpha\beta} e^x P_\alpha e^y;$$

and, on the other hand, since

$$P_\gamma(e^xe^y) = P_\alpha(e^xe^y) + P_\beta(e^xe^y) + P_{\alpha\beta}(e^xe^y), \tag{5.13}$$

we have

(5.14) $$P_{\alpha\beta}(e^x e^y) = P_\beta e^x P_{\alpha\beta} e^y + P_{\alpha\beta} e^x P_\alpha e^y.$$

Moreover, from (5.11) we have

(5.15) $$x_\beta (P_{\alpha\beta} e^x) - (P_{\alpha\beta} e^x)\, x_\alpha = (e^{x_\beta} - 1)\, x_{\alpha\beta} - x_{\alpha\beta}\,(e^{x_\alpha} - 1) = e^{x_\beta} x_{\alpha\beta} - x_{\alpha\beta} e^{x_\alpha}.$$

Here if $e^x e^y = e^{x+y}$, then $P_{\alpha\beta}(e^x e^y) = P_{\alpha\beta} e^{x+y}$,and hence by (5.14) we have

(5.16) $$P_{\alpha\beta} e^{x+y} = e^{x_\beta} P_{\alpha\beta} e^y + P_{\alpha\beta} e^x \cdot e^{y_\alpha}.$$

Thus we have

LEMMA 4. *If* $e^x e^y = e^{x+y}$, *then* $e^{x_\alpha} e^{y_\alpha} = e^{x_\alpha + y_\alpha}$, $e^{x_\beta} e^{y_\beta} = e^{x_\beta + y_\beta}$ *and* $P_{\alpha\beta} e^{x+y} = e^{x_\beta} \cdot P_{\alpha\beta} e^y + P_{\alpha\beta} e^x \cdot e^{y_\alpha}$, *where* $P_{\alpha\beta}(e^x) = \sum_{m=1}^{\infty} \sum_{m_1+m_2=m-1} \frac{1}{m!} x_\beta^{\ m_1} x_{\alpha\beta} x_\alpha^{\ m_2}$.

III. NON-COMMUTATIVE SOLUTIONS OF $e^x e^y = e^{x+y} = e^y e^x$ FOR THE TOTAL MATRIC ALGEBRA

6. Equation $e^x e^y = e^{x+y} = e^y e^x$ for the complex total matric algebra. Let $\mathfrak{A}$ be the complex total matric algebra of order n, $\mathfrak{A}_{(0)}$ the set of all the matrices of order n all of whose distinct characteristic values μ_i have the imaginary part $I(\mu_i)$ such that $-\pi \leqq I(\mu_i) < \pi$, $\mathfrak{A}_s$ the set of all the matrices of orther n some two of whose characteristic values have the difference $2l\pi\sqrt{-1}$ (l is a non-zero integer), and $\mathfrak{A}_0 = \mathfrak{A} - \mathfrak{A}_s$. We shall call the elements of $\mathfrak{A}_0$ and $\mathfrak{A}_s$ the ordinary elements and the singular elements of $\mathfrak{A}$ with respect to the exponential function e^x respectively. From this definition, it is easily seen that $\mathfrak{A}_0$ is open in $\mathfrak{A}$ and $\mathfrak{A}_s$ is closed in $\mathfrak{A}$, and moreover that $\mathfrak{A}_{(0)} \subset \mathfrak{A}_0$.

From the results obtained in our previous paper [1], we have the following lemmas:

LEMMA 5. *Any element* x *of* $\mathfrak{A}$ *is uniquely expressed as* $x = x_{(0)} + x_{(p)}$ *where* $x_{(0)}$ *is the element of* $\mathfrak{A}_{(0)}$ *such that* $e^{x^{(0)}} = e^x$; *and then is a polynomial of* e^x *(here its coefficients may depend on* e^x*),* $x_{(0)} x_{(p)} = x_{(p)} x_{(0)}$ *and* $e^{x^{(p)}} = 1$. $x_{(p)}$ *is expressed as* $x_{(p)} = s^{-1} f s$, *where* s *is any matrix such that* $s x_{(0)} = x_{(0)} s$; *and* f *has the following form:*

$$f = q^{-1} \left(\sum_{i=1}^{p} \oplus f_i\right) q, \quad f_i = 2\pi\sqrt{-1} \sum_{\alpha=1}^{l_i} \oplus f_{i\alpha}\, 1_{n_i},$$

corresponding to

$$x_{(0)} = q^{-1} \left(\sum_{i=1}^{p} \oplus x_{(0)i}\right) q, \quad x_{(0)i} = \sum_{\alpha=1}^{p_i} x_{(0)i\alpha}, \quad x_{(0)i\alpha} = \begin{pmatrix} \lambda_i & 1 & & 0 \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{pmatrix},$$

where q *is a regular matrix,* $x_{(0)i}$ *and* $x_{(0)i\alpha}$ *are the matrices of order* n_i *and* $n_{i\alpha}$ *respectively,* f_i *and* $f_{i\alpha}$ *are arbitrary integers and* $\lambda_i \neq \lambda_j \, (i \neq j)$.

LEMMA 6. x *is an element of* $\mathfrak{A}_0$, *if and only if* $f_{i\alpha}=f_{i\beta}$ *for all* $\alpha, \beta = 1, \cdots, p_i$, *in Lemma 5; that is, if and only if* x *is written as a polynomial of* e^x.

LEMMA 7. $e^xe^y=e^ye^x$ *implies* $xy=yx$, *if and only if* $x, y \in \mathfrak{A}_0$. *The general solutions of* $e^xe^y=e^ye^x$ *are given by* $x=x_{(0)}+x_{(p)}$ *and* $y=y_{(0)}+y_{(p)}$ *such that* $x_{(0)}y_{(0)}=y_{(0)}x_{(0)}$.

PROOF. If $x, y \in \mathfrak{A}_0$, then x and y are written as the polynomials of e^x and e^y respectively; therefore $e^xe^y=e^ye^x$ implies $xy=yx$. If $x \notin \mathfrak{A}_0$, then it can not be asserted that $e^xe^y=e^ye^x$ implies $xy=yx$; for example if we take $x=\begin{pmatrix} 2\pi\sqrt{-1} & 0 \\ 0 & 0 \end{pmatrix}$ and $y=\begin{pmatrix} \mu & 1 \\ 0 & \nu \end{pmatrix}$, then $e^xe^y=e^ye^x$ but $xy \neq yx$. The last half is clear, because $\mathfrak{A}_{(0)} \subset \mathfrak{A}_0$.

LEMMA 8. *If* $x \in \mathfrak{A}_0$, *then* $e^x=e^y$ *implies* $xy=yx$.

PROOF. $e^x=e^y$ implies $ye^x=e^xy$, and since $x \in \mathfrak{A}_0$, x is a polynomial of e^x, therefore $xy=yx$.

We shall call x the matrix of degree n if x has the minimal polynomial of degree n. Now we shall prove

THEOREM 6. *If* x *and* y *are the elements of* $\mathfrak{A}$ *such that either* $x_{(0)}$, $y_{(0)}$ *or* $(x+y)_{(0)}$ *has the degree* n, *then* $e^xe^y=e^ye^x=e^{x+y}$ *implies* $xy=yx$.

PROOF. $e^xe^y=e^ye^x$ is equivalent to $e^{x(0)}e^{y(0)}=e^{y(0)}e^{x(0)}$, and by Lemma 7, this is equivalent to

$$x_{(0)}y_{(0)}=y_{(0)}x_{(0)}\,; \tag{6.1}$$

and hence we have

$$e^{x(0)}e^{y(0)}=e^{x(0)+y(0)}, \tag{6.2}$$

on the other hand, we have

$$e^{x(0)}e^{y(0)}=e^xe^y=e^{x+y}=e^{x(0)+y(0)+x(p)+y(p)}, \tag{6.3}$$

therefore we have

$$e^{x(0)+y(0)}=e^{x(0)+y(0)+x(p)+y(p)}. \tag{6.4}$$

Since the degree of $x_{(0)}$ is n, $x_{(0)}$ is transformed to its canonical form

$$p^{-1}x_{(0)}p=\sum_{i=1}^{r}\oplus\begin{pmatrix} \lambda_i & 1 & & 0 \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{pmatrix}, \qquad \mu_i \neq \mu_j \text{ for } i \neq j, \tag{6.5}$$

by means of $x_{(0)}y_{(0)}=y_{(0)}x_{(0)}$, we have

$$(6.6)\qquad p^{-1}y_{(0)}p=\sum_{i=1}^{r}\oplus\begin{pmatrix}\overset{1}{\beta_i} & {}_1\overset{2}{\beta_i} & \ddots & \overset{n_i}{\beta_i}\\ & \beta_i & \ddots & \overset{2}{\beta_i}\\ & & \ddots & \overset{1}{\beta_i}\\ 0 & & & \beta_i\end{pmatrix},$$

And, by Lemma 5, we hahe

$$(6.7)\qquad x_{(0)}x_{(p)}=x_{(p)}x_{(0)},$$

hence we have

$$(6.8)\qquad p^{-1}x_{(p)}p=\sum_{i=1}^{r}\oplus\begin{pmatrix}\overset{1}{\gamma_i} & \overset{2}{\gamma_i} & \ddots & \overset{n_i}{\gamma_i}\\ & \overset{1}{\gamma_i} & \ddots & \overset{2}{\gamma_i}\\ 0 & & \ddots & \overset{1}{\gamma_i}\end{pmatrix},$$

therefore, by comparing the form of (6.6) and (6.8), we have

$$(6.9)\qquad y_{(0)}x_{(p)}=x_{(p)}y_{(0)}\ ;$$

moreover, we have

$$(6.10)\qquad y_{(0)}y_{(p)}=y_{(p)}y_{(0)}.$$

By these results (6.1), (6.9) and (6.10), it follows from (6.4) that

$$(6.11)\qquad e^{x(0)}=e^{x(0)+x(p)+y(p)}.$$

From this, by Lemma 8, we have

$$x_{(0)}\cdot(x_{(p)}+y_{(p)})=(x_{(p)}+y_{(p)})\,x_{(0)},$$

i. e.,

$$(6.12)\qquad x_{(0)}\cdot y_{(p)}=y_{(p)}\cdot x_{(0)}.$$

By the same argument as used in order to obtain (6.9), it follows from (6.7) and (6.12) that

$$(6.13)\qquad x_{(p)}y_{(p)}=y_{(p)}x_{(p)}\ .$$

Therefore we have $xy=yx$. Also by exchanging the role of $x_{(0)}$ by $y_{(0)}$ or $(x+y)_{(0)}$, we arrive at the same result $xy=yx$; because from $e^xe^y=e^ye^x=e^{x+y}$, we have $e^{-x}e^{x+y}=e^{x+y}e^{-x}=e^y$. Thus our assertion is proved.

REMARK. It is easily seen that the degree of $x_{(0)}$ is n, if and only if $x\in\mathfrak{A}_0$ and the degree of x is n.

7. Some cases where $e^xe^y=e^{x+y}$ implies $e^xe^y=e^ye^x$. In this section we shall consider some cases where $e^xe^y=e^{x+y}$ implies $e^xe^y=e^ye^x$.

1°. Let x and y be the matrices of order n which are simultaneously transformed to the hermitian matrices, and if $e^x e^y$ is transformed to the hermitian matrix by the above transformation, then $e^x e^y = e^y e^x$.

For, x and y may be considered to be hermitian, i. e., ${}^t\bar{x} = x$ and ${}^t\bar{y} = y$ where ${}^t x$ denotes the transposed matrix of x, and $\bar{x}$ the complex conjugate matrix of x; then we have

$$(7.1) \qquad {}^{t}\overline{(e^x)} = e^{{}^{t}\bar{x}} = e^x, \quad {}^{t}\overline{(e^y)} = e^y.$$

Since $e^x e^y$ is hermitian, i. e.,

$$(7.2) \qquad {}^{t}\overline{(e^x e^y)} = e^x e^y,$$

we obtain

$$(7.3) \qquad e^y e^x = e^{{}^t\bar{y}} e^{{}^t\bar{x}} = e^x e^y.$$

As a corollary of this fact, if x and y are simultaneously transformed to the hermitian matrices, then $e^x e^y = e^{x+y}$ implies $e^x e^y = e^y e^x$.

Similarly, if x and y are simultaneously transformed to the symmetric matrices such that $e^x e^y$ is simultaneously transformed to be symmetric, then $e^x e^y = e^y e^x$; from this it follows that if x and y are simultaneously transformed to be symmetric, then $e^x e^y = e^{x+y}$ implies $e^x e^y = e^y e^x$. And moreover, if x and y are simultaneously transformed to be pure imaginary such that $e^x e^y$ is simaltaneously transformed to satisfy $\bar{z} \cdot z = 1$, then $e^x e^y = e^y e^x$; from this it follows that if x and y are simultaneously transformed to be pure imaginary, then $e^x e^y = e^{x+y}$ implies $e^x e^y = e^y e^x$.

Remark. e^x is hermitian, if and only if $x_{(0)}$ is hermitian; and if x and e^x are hermitian, then $x = x_{(0)}$.

For, since $e^x = e^{x(0)}$, if $x_{(0)}$ is hermitian, then clearly e^x is hermitian; conversely if e^x is hermitian, then ${}^{t}\overline{(e^{x(0)})} = e^{{}^t\bar{x}(0)} = e^{x(0)}$, consequently $e^{x(0)} = e^{{}^{t}\overline{x(0)}}$, since here ${}^t\bar{x}_{(0)}$, $x_{(0)} \in \mathfrak{A}_{(0)}$, (the characteristic roots of $x_{(0)}$ are all real, since the characteristic roots of e^x are all real), we have $x_{(0)} = {}^t\bar{x}_{(0)}$, that is, $x_{(0)}$ is hermitian. If x and e^x are hermitian, then ${}^t\bar{x} = {}^t\bar{x}_{(0)} + {}^t\bar{x}_{(p)} = x_{(0)} + x_{(p)}$ and ${}^t\bar{x}_{(0)} = x_{(0)}$, consequently ${}^t\bar{x}_{(p)} = x_{(p)}$. Therefore all the characteristic values of $x_{(p)}$ are real, while on the other hand, these are pure imaginary (see Lemma 5); so we have $x_{(p)} = 0$, i. e., $x = x_{(0)}$.

Similarly, if and only if e^x is symmetric, then $x_{(0)}$ is symmetric; and, if and only if $x_{(0)}$ is skew hermitian, then e^x is unitary.

2°. As another examples of the case where $e^x e^y = e^{x+y}$ implies $e^x e^y = e^y e^x$, we shall consider the following cases. Let e_a and f_a, $a = 0, 1, 2, 3$ be two quaternion bases, i. e.,

$$(7.4) \qquad e_{(i}e_{j)} = -\delta_{ij}, \quad f_{(i}f_{j)} = -\delta_{ij},$$

$$e_0^2 = e_0, \quad f_0^2 = f_0, \quad e_0 e_i = e_i e_0 = e_i, \quad f_0 f_i = f_i f_0 = f_i, \quad i = 1, 2, 3.$$

Let $\mathfrak{Q}$ be the algebra of elements $\sum_{a,b=0}^{3} \alpha_{ab} e_a f_a$ where $e_a f_a = f_a e_a$, α_{ab} are complex numbers, and let

$$(7.5) \qquad x = \alpha_{1i} e_1 f_i + \alpha_{2i} e_2 f_i, \quad y = \beta_{1i} e_1 f_i + \beta_{2i} e_2 f_i .$$

Since it is easily seen that

$$(7.6) \qquad e_3^{-1} x e_3 = -x, \quad e_3^{-1} y e_3 = -y,$$

from $e^x e^y = e^{x+y}$, it follows that

$$e_3^{-1}(e^x e^y) e_3 = e_3^{-1} e^{x+y} e_3 \qquad \text{i. e.,} \ e^{-x} e^{-y} = e^{-x-y},$$

and hence we have $e^y e^x = e^{x+y}$, that is, $e^y e^x = e^x e^y$.

Similarly, if we take

$$x = \alpha_{10} e_1 + \alpha_{13} e_1 f_3 + \alpha_{20} e_2 + \alpha_{23} e_2 f_3,$$

$$y = \beta_{01} f_1 + \beta_{31} e_3 f_1 + \beta_{02} f_2 + \beta_{32} e_3 f_2,$$

then we have

$$(e_3 f_3)^{-1} x (e_3 f_3) = -x, \quad (e_3 f_3)^{-1} y (e_3 f_3) = -y,$$

and hence, by the same argument as the above, we see that $e^x e^y = e^{x+y}$ implies $e^x e^y = e^y e^x$.

Finally we note that for the real quaternion algebra, $e^x e^y = e^{x+y}$ implies $xy = yx$.[1]

8. Non-commutative solutions of $e^x e^y = e^y e^x = e^{x+y}$ for the total matric algebra of order three. We shall consider the non-commutative solutions of $e^x e^y = e^y e^x = e^{x+y}$ for the total matric algebra of order three. By Theorem 6, for the non-commutative solutions x and y of $e^x e^y = e^y e^x = e^{x+y}$ it is necessary that the degrees of $x_{(0)}$ and $y_{(0)}$ are less than three. ${}^t x$ and ${}^t y$ satisfy $e^x e^y = e^y e^x = e^{x+y}$, if and only if x and y satisfy it. And $e^x e^y = e^y e^x = e^{x+y}$ is equivalent to $e^{x'} e^{y'} = e^{y'} e^{x'} = e^{x'+y'}$ where

$$(8.1) \qquad x' = p^{-1} x p + \alpha \cdot 1, \quad y' = p^{-1} y p + \beta \cdot 1.$$

By making use of these facts we shall investigate the non-commutative solutions of

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[1] See 〔2〕, p. 354.

$e^xe^y=e^ye^x=e^{x+y}$.

The types of the canonical form of matrices of degree less than three under the transformation (8.1) are given as follows:

$$\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix},\quad \begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix} \text{ and } \begin{pmatrix}0&0&0\\0&0&0\\0&0&\lambda\end{pmatrix} \quad (\lambda \neq 0). \tag{8.2}$$

(I). The case where $x_{(0)}=\begin{pmatrix}0&0&0\\0&0&0\\0&0&\lambda\end{pmatrix}$, $(\lambda\neq 0)$: Since $x_{(0)}y_{(0)}=y_{(0)}x_{(0)}$ by Lemma 7, we have

$$y_{(0)}=\begin{pmatrix}0&0&0\\0&0&0\\0&0&\kappa\end{pmatrix},\quad \begin{pmatrix}\nu&0&0\\0&0&0\\0&0&0\end{pmatrix} \quad (\nu\neq 0) \text{ or } \begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}. \tag{8.3}$$

(I₁). $x_{(0)}=\begin{pmatrix}0&0&0\\0&0&0\\0&0&\lambda\end{pmatrix}$ $(\lambda\neq 0)$, $y_{(0)}=\begin{pmatrix}0&0&0\\0&0&0\\0&0&\kappa\end{pmatrix}$. If $y_{(0)}=\begin{pmatrix}0&0&0\\0&0&0\\0&0&\kappa\end{pmatrix}$ $(\kappa\neq 0)$, then by Lemma 5 we have

$$x=p^{-1}\begin{pmatrix}*&*&0\\ *&*&0\\0&0&*\end{pmatrix}p,\quad y=p^{-1}\begin{pmatrix}*&*&0\\ *&*&0\\0&0&*\end{pmatrix}p. \tag{8.4}$$

If $y_{(0)}=0$, then by Lemma 8, from $e^{x(0)}e^{y(0)}=e^{x(0)+y(0)+x(p)+y(p)}$ i. e., $e^{x(0)}=e^{x(0)+x(p)+y(p)}$, we have $x_{(0)}y_{(p)}=y_{(p)}x_{(0)}$; therefore we have

$$x=p^{-1}\begin{pmatrix}*&*&0\\ *&*&0\\0&0&*\end{pmatrix}p,\quad y=p^{-1}\begin{pmatrix}*&*&0\\ *&*&0\\0&0&*\end{pmatrix}p. \tag{8.5}$$

Thius, in this case, our problem is reduced to the case for the total matric algebra of order two. As for this, see our previous paper [**2**].

(I₂). $x_{(0)}=\begin{pmatrix}0&0&0\\0&0&0\\0&0&\lambda\end{pmatrix}$ $(\lambda\neq 0)$, $y_{(0)}=\begin{pmatrix}\nu&0&0\\0&0&0\\0&0&0\end{pmatrix}$ $(\nu\neq 0)$. From $e^{x(0)+y(0)}=e^{x(0)+y(0)+x(p)y(p)}$ we have

$$e^{\begin{pmatrix}\nu&0&0\\0&0&0\\0&0&\lambda\end{pmatrix}}=e^{\begin{pmatrix}\nu&0&0\\0&0&0\\0&0&\lambda\end{pmatrix}+x_{(p)}+y_{(p)}}. \tag{8.6}$$

But since $\begin{pmatrix}\nu&0&0\\0&0&0\\0&0&\lambda\end{pmatrix}\in\mathfrak{A}_{(0)}$, by Lemma 8 we see that $x_{(p)}+y_{(p)}$ is commutative with $\begin{pmatrix}\nu&0&0\\0&0&0\\0&0&\lambda\end{pmatrix}$ therefore $x_{(p)}+y_{(p)}$ has the form

$$x_{(p)} + y_{(p)} = \begin{pmatrix} * & 0 & * \\ 0 & * & 0 \\ * & 0 & * \end{pmatrix}, \tag{8.7}$$

on the other hand, $x_{(0)}x_{(p)} = x_{(p)}x_{(0)}$ and $y_{(0)}y_{(p)} = y_{(p)}y_{(0)}$ imply

$$x_{(p)} = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix}, \quad y_{(p)} = \begin{pmatrix} * & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}. \tag{8.8}$$

From (8.7) and (8.8) we have

$$x_{(p)} = \begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix}, \quad y_{(p)} = \begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix},$$

therefore, in this case, $xy = yx$.

(I₃). $x_{(0)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \lambda \end{pmatrix}$ $(\lambda \neq 0)$, $y_{(0)} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. In this case, $x_{(0)}x_{(p)} = x_{(p)}x_{(0)}$ and $y_{(0)}y_{(p)} = y_{(p)}y_{(0)}$ imply

$$x_{(p)} = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix}, \quad y_{(p)} = \begin{pmatrix} \tau & * & * \\ 0 & \tau & 0 \\ 0 & * & * \end{pmatrix}. \tag{8.9}$$

On the other hand, since $e^{x_{(0)}+y_{(0)}} = e^{x_{(0)}+y_{(0)}+x_{(p)}+y_{(p)}}$ and $x_{(0)} + y_{(0)} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \lambda \end{pmatrix} \in \mathfrak{A}_{(0)}$, by Lemma 8, we have

$$(x_{(0)} + y_{(0)})\,(x_{(p)} + y_{(p)}) = (x_{(p)} + y_{(p)})\,(x_{(0)} + y_{(0)})$$

therefore from this we have

$$x_{(p)} + y_{(p)} = \begin{pmatrix} \rho & * & 0 \\ 0 & \rho & 0 \\ 0 & 0 & * \end{pmatrix}. \tag{8.10}$$

By comparing this and (8.9) we have

$$x_{(p)} = \begin{pmatrix} \sigma & * & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & * \end{pmatrix}, \quad y_{(p)} = \begin{pmatrix} \tau & * & 0 \\ 0 & \tau & 0 \\ 0 & 0 & * \end{pmatrix}. \tag{8.11}$$

Thus we have

$$x_{(p)}y_{(p)} = y_{(p)}x_{(p)}, \quad x_{(0)}y_{(p)} = y_{(p)}x_{(0)} \text{ and } x_{(p)}y_{(0)} = y_{(0)}x_{(p)},$$

that is, $xy = yx$.

(II). The case where $x_{(0)} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and besides $y_{(0)}$ belongs to the type

$\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}$ or $\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$. In the case we have

$$x_{(0)}=\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix},\quad y_{(0)}{}^2=0. \tag{8.12}$$

From $x_{(0)}y_{(0)}=y_{(0)}x_{(0)}$ and $y_{(0)}{}^2=0$, it follows that

$$y_{(0)}=\begin{pmatrix}0&*&*\\0&0&0\\0&0&0\end{pmatrix} \text{ or } \begin{pmatrix}0&*&0\\0&0&0\\0&*&0\end{pmatrix}. \tag{8.13}$$

But, by means of $\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}$, the matrices $x_{(0)}=\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}$ and $y_{(0)}=\begin{pmatrix}0&*&0\\0&0&0\\0&*&0\end{pmatrix}$ are simultaneously transformed to $\begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix}$ and $\begin{pmatrix}0&0&0\\*&0&0\\*&0&0\end{pmatrix}$, which are the transposed matrices of $\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}$ and $\begin{pmatrix}0&*&*\\0&0&0\\0&0&0\end{pmatrix}$; so that we have only to consider the case:

$$x_{(0)}=\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix},\quad y_{(0)}=\begin{pmatrix}0&\mu&\omega\\0&0&0\\0&0&0\end{pmatrix}. \tag{8.14}$$

Then, by means of Lemma 5, we may take

$$x_{(p)}=\begin{pmatrix}0&0&0\\0&0&0\\0&0&2l\pi\sqrt{-1}\end{pmatrix}. \tag{8.15}$$

Since $x_{(0)}+y_{(0)}=\begin{pmatrix}0&1+\mu&\omega\\0&0&0\\0&0&0\end{pmatrix}\in\mathfrak{A}_{(0)}$, by Lemma 8, from $e^{x_{(0)}+y_{(0)}}=e^{x_{(0)}+y_{(0)}+x_{(p)}+y_{(p)}}$, we have

$$(x_{(0)}+y_{(0)})\,(x_{(p)}+y_{(p)})=(x_{(p)}+y_{(p)})\,(x_{(0)}+y_{(0)}). \tag{8.16}$$

From this and $y_{(0)}y_{(p)}=y_{(p)}y_{(0)}$, it follows that

$$y_{(0)}=\begin{pmatrix}0&\mu&0\\0&0&0\\0&0&0\end{pmatrix},\quad y_{(p)}=\begin{pmatrix}\tau&\tau_1&\tau_2\\0&\tau&0\\0&\tau_3&\sigma\end{pmatrix}, \tag{8.17}$$

or

$$y_{(0)}=\begin{pmatrix}0&\mu&\omega\\0&0&0\\0&0&0\end{pmatrix}\ (\omega\neq 0),\quad y_{(p)}=\begin{pmatrix}\rho&\rho_1&\rho_2\\0&\rho&\rho_3\\0&0&\kappa\end{pmatrix}, \tag{8.18}$$

where $\rho_3 = -2l\pi\sqrt{-1} \cdot \omega$, $\kappa - \rho = 2l\pi\sqrt{-1} \cdot \mu$; if $l = 0$, then $xy = yx$. By (8.16), from $e^{x(0)+y(0)} = e^{x(0)+y(0)+x(p)+y(p)}$ we have $e^{x(p)+y(p)} = 1$, so that we have only to determine $y_{(p)}$ to satisfy

$$e^{y(p)} = e^{x(p)+y(p)} = 1. \tag{8.19}$$

But from (8.17) we have

$$e^{y(p)} = e^{\tau}. \begin{pmatrix} 1 & \tau_1 + \tau_2\tau_3 h(\sigma-\tau) & \tau_2 e(\sigma-\tau) \\ 0 & 1 & 0 \\ 0 & \tau_3 e(\sigma-\tau) & e^{\sigma-\tau} \end{pmatrix}, \tag{8.20}$$

$$e^{x(p)+y(p)} = e^{\tau} \cdot \begin{pmatrix} 1 & \tau_1 + \tau_2\tau_3 h(\sigma-\tau+2l\pi\sqrt{-1}) & \tau_2 e(\sigma-\tau+2l\pi\sqrt{-1}) \\ 0 & 1 & 0 \\ 0 & \tau_3 e(\sigma-\tau+2l\pi\sqrt{-1}) & e^{\sigma-\tau+2l\pi\sqrt{-1}} \end{pmatrix},$$

where $h(\xi) = \frac{1}{\xi^2}(e^{\xi} - 1 - \xi)$ $(\xi \neq 0)$, $h(0) = \frac{1}{2}$; therefore (8.19) is equivalent to

$$\begin{cases} \tau = 2m_1\pi\sqrt{-1},\ \sigma = 2m_2\pi\sqrt{-1},\ \tau_2 e(\sigma-\tau) = \tau_3 e(\sigma-\tau) = \tau_1 + \tau_2\tau_3 h(\sigma-\tau) = 0 \\ \text{and } \tau_2 e(\sigma-\tau+2l\pi\sqrt{-1}) = \tau_3 e(\sigma-\tau+2l\pi\sqrt{-1}) = \tau_1 + \tau_2\tau_3 h(\sigma-\tau+ \\ 2l\pi\sqrt{-1}) = 0. \end{cases} \tag{8.21}$$

Here if $\tau = \sigma$, or $\tau - \sigma = 2l\pi\sqrt{-1}$, then $\tau_1 = \tau_2 = \tau_3 = 0$; from which we have $xy = yx$. If $\tau \neq \sigma$ and $\tau - \sigma \neq 2l\pi\sqrt{-1}$, then we have $\tau_2\tau_3 = (\sigma-\tau)\tau_1 = (\sigma-\tau+2l\pi\sqrt{-1})\tau_1$; since $l \neq 0$ for $xy \neq yx$,

$$\tau_1 = 0, \quad \tau_2\tau_3 = 0, \quad m_1 - m_2 \neq 0, l. \tag{8.22}$$

And then $xy \neq yx$ if and only if $\tau_2 \neq 0$, so that, for x and y such that $xy \neq yx$, $\tau_2 \neq 0$ and $\tau_3 = 0$. Thus we have

$$x = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2l\pi\sqrt{-1} \end{pmatrix}, \quad y = \begin{pmatrix} 2m_1\pi\sqrt{-1} & \mu & \tau_2 \\ 0 & 2m_1\pi\sqrt{-1} & 0 \\ 0 & 0 & 2m_2\pi\sqrt{-1} \end{pmatrix}, \tag{8.23}$$

where l, m_1 and m_2 are arbitrary integers such that $m_1 - m_2 \neq 0, l$; τ_2 is an arbitrary complex number.

Similarly from (8.18) we have

$$e^{y(p)} = e^{\rho} \cdot \begin{pmatrix} 1 & \rho_1 & \rho_1\rho_3 h(\kappa-\rho) + \rho_2 e(\kappa-\rho) \\ 0 & 1 & \rho_3 e(\kappa-\rho) \\ 0 & 0 & e^{\kappa-\rho} \end{pmatrix}, \tag{8.24}$$

$$e^{x(p)+y(p)} = e^{\rho} \cdot \begin{pmatrix} 1 & \rho_1 & \rho_1\rho_3 h(\kappa-\rho+2l\pi\sqrt{-1}) + \rho_2 e(\kappa-\rho+2l\pi\sqrt{-1} \\ 0 & 1 & \rho_3 e(\kappa-\rho+2l\pi\sqrt{-1}) \\ 0 & 0 & e^{\kappa-\rho+2l\pi\sqrt{-1}} \end{pmatrix},$$

so that (8.19) is equivalent to

$$(8.25)\qquad \begin{gathered}\rho=2m_1\pi\sqrt{-1},\ \kappa=2m_2\pi\sqrt{-1},\ \rho_1=0,\ \rho_2e(\kappa-\rho)=\rho_3e(\kappa-\rho)=0,\\ \text{and } \rho_2e(\kappa-\rho+2l\pi\sqrt{-1})=\rho_3e(\kappa-\rho+2l\pi\sqrt{-1})=0.\end{gathered}$$

Here if $\rho=\kappa$ or $\rho-\kappa=2l\pi\sqrt{-1}$, then we have $\rho_2=\rho_3=0$, and hence from $\rho_3=-2l\pi\sqrt{-1}\cdot\omega$ we have $l=0$; so that $xy=yx$. If $\rho\neq\kappa$ and $\rho-\kappa\neq 2l\pi\sqrt{-1}$, then we have from (8.25) and (8.18)

$$(8.26)\qquad \begin{gathered}\rho=2m_1\pi\sqrt{-1},\ \kappa=2m_2\pi\sqrt{-1},\ \rho_1=0,\ \rho_3=-2l\pi\sqrt{-1}\,\omega,\\ m_2-m_1=\mu\cdot 2l\pi\sqrt{-1}.\end{gathered}$$

Thus we have

$$(8.27)\qquad x=\begin{pmatrix}0&1&0\\0&0&0\\0&0&2l\pi\sqrt{-1}\end{pmatrix},\quad y=\begin{pmatrix}2m_1\pi\sqrt{-1}&\mu&\theta\\0&2m_1\pi\sqrt{-1}&\nu\\0&0&2m_2\pi\sqrt{-1}\end{pmatrix},$$

where $m_2-m_1=\mu l$, $\mu\neq 0,\ -1$, $l\neq 0$, $\nu\neq 0$ and θ are arbitrary complex numbers.

(III). The case where $x_{(0)}=0$, $y_{(0)}=0$, i. e., $e^x=e^y=e^{x+y}=1$. In this case we have from Lemma 5

$$x=x_{(p)}=s_1^{-1}\begin{pmatrix}2l_1\pi\sqrt{-1}&0&0\\0&2l_2\pi\sqrt{-1}&0\\0&0&2l_3\pi\sqrt{-1}\end{pmatrix}s_1,$$

$$y=y_{(p)}=s_2^{-1}\begin{pmatrix}2m_1\pi\sqrt{-1}&0&0\\0&2m_2\pi\sqrt{-1}&0\\0&0&2m_3\pi\sqrt{-1}\end{pmatrix}s_2,$$

and

$$x+y=s_3^{-1}\begin{pmatrix}2n_1\pi\sqrt{-1}&0&0\\0&2n_2\pi\sqrt{-1}&0\\0&0&2n_3\pi\sqrt{-1}\end{pmatrix}s_3,$$

where s_1, s_2 and s_3 are arbitrary regular matrices, l_i, m_i and n_i $(i=1,2,3)$ are arbitrary integers. Therefore $e^x=e^y=e^{x+y}=1$ is equivalent to

$$(8.28)\qquad s_1^{-1}\begin{pmatrix}l_1&0&0\\0&l_2&0\\0&0&l_3\end{pmatrix}s_1+s_2^{-1}\begin{pmatrix}m_1&0&0\\0&m_2&0\\0&0&m_3\end{pmatrix}s_2=s_3^{-1}\begin{pmatrix}n_1&0&0\\0&n_2&0\\0&0&n_3\end{pmatrix}s_3.\ ^{1)}$$

By taking account of the transformation: $x'=x+\alpha$, $y'=y+\beta$ we may take $l_3=0$ and $n_3=0$. Thus we have to solve the following equation

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1) This fact is true for the case where x and y are the matrices of any order n.

$$s^{-1}\begin{pmatrix} l_1 & 0 & 0 \\ 0 & l_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} s + \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} = t^{-1}\begin{pmatrix} n_1 & 0 & 0 \\ 0 & n_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} t, \tag{8.29}$$

here we shall put

$$a = \| a_{ij} \| = s^{-1}\begin{pmatrix} l_1 & 0 & 0 \\ 0 & l_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} s, \quad b = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} \quad \text{and } c = t^{-1}\begin{pmatrix} n_1 & 0 & 0 \\ 0 & n_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} t.$$

(III$_1$). $l_1 n_1 \neq 0$, $l_2 = 0$ and $n_2 = 0$: In this case a is characterized by

$$a = \| a_i b_j \|, \quad \sum_{i=1}^{3} a_i b_i = l_1, \tag{8.30}$$

and then (8.29) is equivalent to that the rank of $\| a_i b_j \| + \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix}$ is equal to 1 and $\sum_{i=1}^{3} a_i b_i + \sum_{i=1}^{3} m_i = n_1$, i. e.,

$$\sum_{i=1}^{3} a_i b_i = l_1 = n_1 - \sum_{i=1}^{3} m_i. \tag{8.31}$$

By these conditions the non-commutative of (8.29) are obtained as follows:

(i)

$$a = \begin{pmatrix} l_1 & \alpha & \beta \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} l_1 & 0 & 0 \\ \gamma & 0 & 0 \\ \delta & 0 & 0 \end{pmatrix},$$

$$b = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $l_1 m_1 (l_1 + m_1) \neq 0$; α and β are not both zero, γ and δ are not both zero.

(ii)

$$a = \begin{pmatrix} \dfrac{m_1(l_1+m_2)}{m_1-m_2} & \sigma & 0 \\ \tau & -\dfrac{m_2(l_1+m_1)}{m_1-m_2} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$b = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $\sigma\tau = -\dfrac{m_1 m_2 (l_1+m_1)(l_1+m_2)}{(m_1-m_2)^2}$, $l_1 m_1 m_2 (m_1-m_2)(l_1+m_1+m_2) \neq 0$, and, σ and τ are not both zero. This result coincides with the result of the case (I$_1$).

(III$_2$). $l_2 = 0$, and $l_1 n_1 n_2 (n_1 - n_2) \neq 0$: In this case (8.29) is equivalent to that $a = \| a_i b_j \|$, $\sum_{i=1}^{3} a_i b_i = l_1$ and the characteristic roots of $a+b$ are n_1, n_2 and 0.

So we have

$$(8.32)\quad \begin{cases} a_{11}+a_{22}+a_{33}=l_1=n_1+n_2-(m_1+m_2+m_3), \\ (m_2+m_3)a_{11}+(m_3+m_1)a_{22}+(m_1+m_2)a_{33}=-(m_1m_2+m_2m_3+m_1m_3)+n_1n_2, \\ m_2m_3a_{11}+m_1m_3a_{22}+m_1m_2a_{33}=-m_1m_2m_3. \end{cases}$$

(i). The case where m_1, m_2 and m_3 are distinct: From (8.32) we have

$$(8.33)\quad \begin{cases} a_{11}=\dfrac{m_1(m_1-n_1)(m_1-n_2)}{(m_1-m_2)(m_3-m_1)}, \\ a_{22}=\dfrac{m_2(m_2-n_1)(m_2-n_2)}{(m_1-m_2)(m_2-m_3)}, \\ a_{33}=\dfrac{m_3(m_3-n_1)(m_3-n_2)}{(m_2-m_3)(m_3-m_1)}. \end{cases}$$

Since $a=\|a_ib_j\|$, we have $a_{12}a_{21}=a_{11}a_{22}$, $a_{13}a_{31}=a_{11}a_{33}$ and $a_{23}a_{32}=a_{22}a_{33}$. For the case where $a_{11}a_{22}a_{33}\neq 0$, we have $a_{12}a_{21}\neq 0$, $a_{13}a_{31}\neq 0$ and $a_{23}a_{32}\neq 0$; so by means of the matrix $\begin{pmatrix}1&0&0\\0&p&0\\0&0&q\end{pmatrix}$ leaving invariant $b=\begin{pmatrix}m_1&0&0\\0&m_2&0\\0&0&m_3\end{pmatrix}$, $a=\|a_{ij}\|$ is transformed to $\mathring{a}=\|\mathring{a}_{ij}\|$ such that $\mathring{a}_{ij}=\mathring{a}_{ji}$, and then $a_{11}, a_{22}, a_{33}, a_{23}a_{32}$, $a_{13}a_{31}$ and $a_{12}a_{21}$ are invariant under this transformation. So we have

$$(8.34)\quad \mathring{a}=\begin{pmatrix} a_{11} & \sqrt{a_{11}a_{22}} & \sqrt{a_{11}a_{33}} \\ \sqrt{a_{11}a_{22}} & a_{22} & \sqrt{a_{22}a_{33}} \\ \sqrt{a_{11}a_{33}} & \sqrt{a_{22}a_{33}} & a_{33} \end{pmatrix}, \quad a_{11}a_{22}a_{33}\neq 0,$$

where a_{11}, a_{22} and a_{33} are given by (8.33). For the case where $a_{11}a_{22}a_{33}=0$, by using of the matrix $\begin{pmatrix}1&0&0\\0&p&0\\0&0&q\end{pmatrix}$ leaving invariant $b=\begin{pmatrix}m_1&0&0\\0&m_2&0\\0&0&m_3\end{pmatrix}$, $a=\|a_{ij}\|$ is transformed to either of the following forms:

$$(8.35)\quad \begin{cases} \mathring{a}=\begin{pmatrix} a_{11} & \sqrt{a_{11}a_{22}} & \sqrt{a_{11}a_{22}} \\ \sqrt{a_{11}a_{22}} & a_{22} & a_{22} \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} a_{11} & \sqrt{a_{11}a_{22}} & 0 \\ \sqrt{a_{11}a_{22}} & a_{22} & 0 \\ \sqrt{a_{11}a_{22}} & a_{22} & 0 \end{pmatrix} \text{ or } \begin{pmatrix} a_{11} & \sqrt{a_{11}a_{22}} & 0 \\ \sqrt{a_{11}a_{22}} & a_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \text{where } a_{11}=-\dfrac{(m_1-n_1)(m_1-n_2)}{m_1-m_2}\neq 0 \text{ and } a_{22}=\dfrac{(m_2-n_1)(m_2-n_2)}{m_1-m_2}\neq 0; \\ \mathring{a}=\begin{pmatrix} l_1&1&1\\0&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} l_1&0&0\\1&0&0\\1&0&0 \end{pmatrix}, \begin{pmatrix} l_1&1&0\\0&0&0\\l_1&1&0 \end{pmatrix}, \begin{pmatrix} l_1&0&1\\l_1&0&1\\0&0&0 \end{pmatrix} \text{ or } \begin{pmatrix} l_1&0&0\\0&0&0\\0&0&0 \end{pmatrix} \\ \text{where } l_1 \text{ is a non-zero integer.} \end{cases}$$

(ii). The case where $m_1 \neq m_2 = m_3$. From (8.32) we have

$$m_3 (m_3 - n_1)(m_3 - n_2) = 0, \tag{8.36}$$

that is, $m_3 = 0$, n_1 or n_2. By taking account of the transformation: $z' = p^{-1}zp + \gamma$, we may assume $m_3 = 0$. And then among the equations (8.32), the first two are linearly independent and the third is a linear combination of these two; from these two we have

$$\begin{cases} a_{11} = -\dfrac{1}{m_1}(m_1 - n_1)(m_1 - n_2), \\ a_{22} = -\alpha + \dfrac{n_1 n_2}{m_1}, \qquad\qquad (m_1 \neq 0) \\ a_{33} = \alpha, \end{cases} \tag{8.37}$$

where m_1, n_1 and n_2 are integers, and α is an arbitrary number. Thus we have

$$a = \| a_i b_j \|, \quad b = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix},$$

where

$$\begin{cases} a_1 b_1 = -\dfrac{1}{m_1}(m_1 - n_1)(m_1 - n_2), \\ a_1 b_2 = -\alpha + \dfrac{n_1 n_2}{m_1}, \\ a_3 b_3 = \alpha. \end{cases}$$

(III$_3$). $n_1 l_1 l_2 (l_1 - l_2) \neq 0$ and $n_2 = 0$: Since $e^x = e^y = e^{x+y} = 1$ is equivalent to $e^{x'} = e^{y'} = e^{x'+y'} = 1$ where $x' = x + y$ and $y' = -y$, this case is reduced to (III$_2$), by putting as follows: $l_2' = n_2 = 0$, $n_1' = l_1$, $n_2' = l_2$, $l_1' = n_1$ and $m_i' = -m_i \, (i = 1, 2, 3)$. And from the result of the case (III$_2$) we have the following result:

(i). The case where m_1, m_2 and m_3 are distinct:

$$a = \| a_{ij} \|, \quad b = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix}$$

where

$$\begin{cases} a_{ij} = a_i b_j \qquad (i \neq j), \\ a_{11} = a_1 b_1 - m_1, \\ a_{22} = a_2 b_2 - m_2, \\ a_{33} = a_3 b_3 - m_3, \end{cases}$$

and

$$\begin{cases} a_1 b_1 = \dfrac{-m_1(m_1+l_1)(m_1+l_2)}{(m_1-m_2)(m_3-m_1)}, \\[2ex] a_2 b_2 = \dfrac{-m_2(m_2+l_1)(m_2+l_2)}{(m_2-m_3)(m_2-m_1)}, \\[2ex] a_2 b_3 = \dfrac{-m_3(m_3+l_1)(m_3+l_2)}{(m_3-m_1)(m_3-m_2)}. \end{cases}$$

(ii). The case where $m_1 \neq m_2 = m_3$:

$$a = \| a_{ij} \|, \quad b = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where

$$\begin{cases} a_{ij} = a_i b_j \qquad (i \neq j), \\ a_{11} = a_1 b_1 - m_1, \\ a_{22} = a_2 b_2, \\ a_{33} = a_3 b_3, \end{cases}$$

and

$$\begin{cases} a_1 b_1 = \dfrac{1}{m_1}(m_1+l_1)(m_2+l_2), \\[2ex] a_2 b_2 = -\alpha - \dfrac{l_1 l_2}{m_1}, \\[2ex] a_3 b_3 = \alpha. \end{cases}$$

(III$_4$). $l_1 l_2 (l_1 - l_2) \neq 0$ and $n_1 n_2 (n_1 - n_2) \neq 0$: In this case the condition (8.29) is equivalent to that a and $a+b$ have the distinct characteristic root and $l_1, l_2, 0$, and $n_1, n_2, 0$, respectively; that is,

$$(8.38) \quad \begin{cases} a_{11} + a_{22} + a_{33} = l_1 + l_2 = n_1 + n_2 - (m_1 + m_2 + m_3), \\ a_{(12)} + a_{(23)} + a_{(31)} = l_1 l_2, \quad (a_{(ij)} = a_{ii} a_{jj} - a_{ij} a_{ij}), \\ \det a = 0, \\ (m_2 + m_3) a_{11} + (m_3 + m_1) a_{22} + (m_1 + m_2) a_{33} \\ \qquad\qquad = n_1 n_2 - l_1 l_2 - (m_1 m_2 + m_2 m_3 + m_3 m_1), \\ m_2 m_3 a_{11} + m_1 m_3 a_{22} + m_1 m_2 a_{33} + m_1 a_{(23)} + m_2 a_{(13)} + m_3 a_{(12)} = -m_1 m_2 m_3, \\ m_2 m_3 a_{11} + m_1 m_3 a_{22} + m_1 m_2 a_{33} + m_1 a_{(23)} + m_2 a_{(13)} + m_3 a_{(12)} = -m_1 m_2 m_3. \end{cases}$$

Since $ab \neq ba$, we may assume without loss of generality that $m_1 \neq m_2$, and then we have from (8.38)

$$(8.39)\qquad \begin{cases} a_{11} = \rho\,(m_2 - m_3) + \mathring{a}_{11}, \\ a_{22} = \rho\,(m_3 - m_1) + \mathring{a}_{22}, \\ a_{33} = \rho\,(m_1 - m_2) + \mathring{a}_{33}, \end{cases}$$

and

$$(8.40)\qquad \begin{cases} a_{(23)} = \rho\,(m_2^2 - m_3^2) + \sigma\,(m_2 - m_3) + \mathring{k}_{23}, \\ a_{(31)} = \rho\,(m_3^2 - m_1^2) + \sigma(m_3 - m_1) + \mathring{k}_{31}, \\ a_{(12)} = \rho\,(m_1^2 - m_2^2) + \sigma(m_1 - m_2) + \mathring{k}_{12}, \end{cases}$$

where ρ and σ are arbitrary complex numbers, $\mathring{a}_{11}$, $\mathring{a}_{22}$ and $\mathring{a}_{33}$ is a special solution of

$$a_{11} + a_{22} + a_{33} = l_1 + l_2,$$

$$(m_2+m_3)\,a_{11} + (m_3+m_1)\,a_{22} + (m_1+m_2)a_{33} = n_1n_2 - l_1l_2 - (m_1m_2+m_2m_3+m_3m_1),$$

and is taken, for example, as follows:

$$\begin{cases} \mathring{a}_{11} = \dfrac{1}{m_1-m_2}\Big[(l_1+l_2)\,(m_1+m_3) + l_1l_2 + (m_1m_2 + m_2m_3+m_3m_1) - n_1n_2\Big], \\ \mathring{a}_{22} = \dfrac{-1}{m_1-m_2}\Big[(l_1+l_2)\,(m_2+m_3) + l_1l_2 + (m_1m_2 + m_2m_3+m_3m_1) - n_1n_2\Big], \\ \mathring{a}_{33} = 0, \end{cases}$$

and (for these $\mathring{a}_{11}$, $\mathring{a}_{22}$, $\mathring{a}_{33}$,) $\mathring{k}_{23}$, $\mathring{k}_{31}$ and $\mathring{k}_{12}$ is a special solution of

$$k_{23} + k_{31} + k_{12} = l_1l_2,$$

$$m_1k_{23} + m_2k_{31} + m_3k_{12} = m_3\{l_1l_2 - (m_3 - n_1)\,(m_3 - n_2)\},$$

and is taken, for example, as follows:

$$\begin{cases} \mathring{k}_{23} = \dfrac{-1}{m_1-m_2}\Big[(m_2 - m_3)\,l_1l_2 + m_3\,(m_3 - n_1)\,(m_3 - n_2)\Big], \\ \mathring{k}_{31} = \dfrac{1}{m_1-m_2}\Big[(m_1 - m_3)\,l_1l_2 + m_3\,(m_3 - n_1)\,(m_3 - n_2)\Big], \\ \mathring{k}_{12} = 0. \end{cases}$$

Since

$$\det\,a = a_{(12)}a_{33} + a_{(23)}a_{11} + a_{(31)}a_{22} - 2a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{32}a_{21},$$

by (8.39) and (8.40) we have from $\det\,a = 0$,

$$(8.41)\quad \begin{cases} a_{12}a_{23}a_{31} + a_{13}a_{32}a_{21} = G, \\ G = 2\rho^3 (m_1 - m_2)(m_2 - m_3)(m_3 - m_1) \\ \quad + \rho^2 [2\{\mathring{a}_{11}(m_3 - m_1)(m_1 - m_2) + \mathring{a}_{22}(m_1 - m_2)(m_2 - m_3)\} \\ \quad - \{(m_1{}^2 - m_2{}^2)(m_1 - m_2) + (m_2{}^2 - m_3{}^2)(m_2 - m_3) + (m_3{}^2 - m_1{}^2)(m_3 - m_1)\}] \\ \quad - \rho\sigma[(m_1 - m_2)^2 + (m_2 - m_3)^2 + (m_3 - m_1)^2] \\ \quad + \rho[2\mathring{a}_{11}\mathring{a}_{22}(m_1 - m_2) - \{\mathring{a}_{11}(m_2{}^2 - m_3{}^2) + \mathring{a}_{22}(m_3{}^2 - m_1{}^2) \\ \qquad\qquad + \mathring{k}_{23}(m_2 - m_3) + \mathring{k}_{31}(m_3 - m_1)\}] \\ \quad - \sigma[\mathring{a}_{11}(m_2 - m_3) + \mathring{a}_{22}(m_3 - m_1)] \\ \quad - (\mathring{a}_{11}\mathring{k}_{23} + \mathring{a}_{22}\mathring{k}_{31}). \end{cases}$$

And moreover from (8.39) and (8.40) we have

$$(8.42)\quad \begin{cases} a_{23}a_{32} = F_1 = \rho^2 (m_1 - m_2)(m_3 - m_1) + \rho\{(m_1 - m_2)\mathring{a}_{22} - (m_2{}^2 - m_3{}^2)\} - \sigma(m_2 - m_3) \\ \qquad + \dfrac{1}{m_1 - m_2}\{(m_2 - m_3) l_1 l_2 + m_3(m_3 - n_1)(m_3 - n_2)\}, \\ a_{13}a_{31} = F_2 = \rho^2 (m_2 - m_3)(m_1 - m_2) + \rho\{(m_1 - m_2)\mathring{a}_{11} - (m_3{}^2 - m_1{}^2) - \sigma(m_3 - m_1)\} \\ \qquad - \dfrac{1}{m_1 - m_2}\{(m_1 - m_3) l_1 l_2 + m_3(m_3 - n_1)(m_3 - n_2)\}, \\ a_{12}a_{21} = F_3 = \rho^2 (m_2 - m_3)(m_3 - m_1) + \rho\{(m_2 - m_3)\mathring{a}_{22} + (m_3 - m_1)\mathring{a}_{11} - (m_1{}^2 - m_2{}^2)\} \\ \qquad - \sigma(m_1 - m_2) + \mathring{a}_{11}\mathring{a}_{22}. \end{cases}$$

Since $a_{12}a_{23}a_{31} + a_{13}a_{32}a_{21} = G$, $(a_{12}a_{23}a_{31})(a_{13}a_{32}a_{21}) = F_1 F_2 F_3$, we have

$$(8.43)\qquad a_{12}a_{23}a_{31} = H_1, \quad a_{13}a_{32}a_{21} = H_2,$$

where $H_1, H_2 = \frac{1}{2}(G \pm \sqrt{G^2 - 4F_1F_2F_3})$.

Thus the matrix a is determined by means of (8.39), (8.42) and (8.43).

By summarizing the above results we have

THEOREM 7. *All the non-commutative solutions of* $e^x e^y = e^y e^x = e^{x+y}$ *for the complex total matric algebra of order three are given by*

$$x = p^{-1}\mathring{x}p + \alpha, \quad y = p^{-1}\mathring{y}p + \beta,$$

where p is an arbitrary regular matrix, α and β are arbitrary complex numbers, and $\mathring{x}$ and $\mathring{y}$ are given as follows: (l's, m's and n's are integers)

$$\text{(I).}\qquad \mathring{x} = \begin{pmatrix} x_1 & 0 \\ 0 & \lambda \end{pmatrix}, \quad \mathring{y} = \begin{pmatrix} y_1 & 0 \\ 0 & \mu \end{pmatrix},$$

where x_1 *and* y_1 *are the non-commutative solutions of* $e^{x_1}e^{y_1}=e^{y_1}e^{x_1}=e^{x_1+y_1}$ *for the complex total matric algebra of order two*[1], *and* λ *and* μ *are arbitrary complex numbers.*

$$\text{(II).}\quad \text{(i)}\ \mathring{x}=\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 2l\pi\sqrt{-1}\end{pmatrix},\quad \mathring{y}=\begin{pmatrix}2m_1\pi\sqrt{-1} & \mu & \gamma\\ 0 & 2m_1\pi\sqrt{-1} & 0\\ 0 & 0 & 2m_2\pi\sqrt{-1}\end{pmatrix},$$

$$(m_1\neq m_2,\quad \gamma\neq 0),$$

and the transposed matrices ${}^t\mathring{x}$ *and* ${}^t\mathring{y}$ *of these* $\mathring{x}$ *and* $\mathring{y}$;

$$\text{(ii)}\quad \mathring{x}=\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 2l\pi\sqrt{-1}\end{pmatrix},\quad \mathring{y}=\begin{pmatrix}2m_1\pi\sqrt{-1} & \nu & \omega+\delta\\ 0 & 2m_1\pi\sqrt{-1} & -2l\pi\sqrt{-1}\,\omega\\ 0 & 0 & 2m_2\pi\sqrt{-1}\end{pmatrix},$$

$$(m_2-m_1=\nu l,\quad \nu\neq 0,\ -1),$$

and the transposed matrices ${}^t\mathring{x}$ *and* ${}^t\mathring{y}$ *of these* $\mathring{x}$ *and* $\mathring{y}$.

$$\text{(III}_1\text{).}\qquad \mathring{x}=2\pi\sqrt{-1}\begin{pmatrix}l & \gamma & \delta\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},\quad \mathring{y}=2\pi\sqrt{-1}\begin{pmatrix}m & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},\quad lm\neq 0,$$

γ *and* δ *being not both zero.*

$$\text{(III}_2\text{).}\ \text{(i)}\qquad \mathring{x}=2\pi\sqrt{-1}\,\|a_ib_j\|,\quad \mathring{y}=2\pi\sqrt{-1}\begin{pmatrix}m_1 & 0 & 0\\ 0 & m_2 & 0\\ 0 & 0 & m_3\end{pmatrix},$$

where

$$\begin{cases} a_1b_1=\dfrac{m_1(m_1-n_1)(m_1-n_2)}{(m_1-m_2)(m_3-m_1)}, \\[2ex] a_2b_2=\dfrac{m_2(m_2-n_1)(m_2-n_2)}{(m_1-m_2)(m_2-m_3)}, \\[2ex] a_3b_3=\dfrac{m_3(m_3-n_1)(m_3-n_2)}{(m_2-m_3)(m_3-m_1)}; \end{cases}\qquad \begin{aligned}&(m_1-m_2)(m_2-m_3)(m_3-m_1)\neq 0,\\ &m_1+m_2+m_3-n_1-n_2\neq 0,\\ &\textit{(See (8.34) and (8.35)).}\end{aligned}$$

$$\text{(ii)}\qquad \mathring{x}=2\pi\sqrt{-1}\,\|a_ib_j\|,\quad \mathring{y}=2\pi\sqrt{-1}\begin{pmatrix}m & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},$$

where

$$\begin{cases} a_1b_1=-\dfrac{1}{m}(m-n_1)(m-n_2), \\[2ex] a_2b_2=-\tau+\dfrac{1}{m}n_1n_2, \\[2ex] a_3b_3=\tau. \end{cases}\qquad \begin{aligned}&m\neq 0,\\ &m-n_1-n_2\neq 0,\end{aligned}$$

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[1] See [2], p. 356.

$$(\mathrm{III}_3).\quad \text{(i)}\ \mathring{x} = 2\pi\sqrt{-1}\left\{\|a_i b_j\| - \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix}\right\},\quad \mathring{y} = 2\pi\sqrt{-1}\begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix}.$$

where

$$\left\{\begin{aligned} a_1 b_1 &= \frac{-m_1(m_1+l_1)(m_1+l_2)}{(m_1-m_2)(m_3-m_1)}, \\ a_2 b_2 &= \frac{-m_2(m_2+l_1)(m_2+l_2)}{(m_2-m_3)(m_2-m_1)}, \\ a_3 b_3 &= \frac{-m_3(m_3+l_1)(m_3+l_2)}{(m_3-m_1)(m_3-m_2)}, \end{aligned}\right. \qquad \begin{aligned} &(m_1-m_2)(m_2-m_3)(m_3-m_1) \neq 0, \\ &m_1+m_2+m_3+l_1+l_2 \neq 0\,; \end{aligned}$$

$$\text{(ii)}\quad \mathring{x} = 2\pi\sqrt{-1}\left\{\|a_i b_j\| - \begin{pmatrix} m & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right\},\quad \mathring{y} = 2\pi\sqrt{-1}\begin{pmatrix} m & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where

$$\left\{\begin{aligned} a_1 b_1 &= \frac{1}{m}(m+l_1)(m+l_2), && m \neq 0, \\ a_2 b_2 &= -\tau - \frac{1}{m} l_1 l_2, && m+l_1+l_2 \neq 0, \\ a_3 b_3 &= \tau. \end{aligned}\right.$$

$$(\mathrm{III}_4).\qquad \mathring{x} = 2\pi\sqrt{-1}\,\|a_{ij}\|,\quad \mathring{y} = 2\pi\sqrt{-1}\begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix},$$

where

$$\left\{\begin{aligned} &a_{11} = (m_2-m_3) + \frac{1}{m_1-m_2}\Big[(l_1+l_2)(m_1+m_3) + l_1 l_2 + (m_1 m_2 + m_2 m_3 + m_3 m_1) - n_1 n_2\Big], \\ &a_{22} = (m_3-m_1) - \frac{1}{m_1-m_2}\Big[(l_1+l_2)(m_2+m_3) + l_1 l_2 + (m_1 m_2 + m_2 m_3 + m_3 m_1) - n_1 n_2\Big], \\ &a_{33} = (m_1-m_2), \\ &a_{23}a_{32} = F_1,\quad a_{31}a_{13} = F_2,\quad a_{12}a_{21} = F_3, \\ &a_{12}a_{23}a_{31} = H_1,\quad a_{13}a_{32}a_{21} = H_2\,;\quad H_1,\ H_2 = \frac{G \pm \sqrt{G^2 - 4F_1F_2F_3}}{2}, \\ &l_1 + l_2 + m_1 + m_2 + m_3 = n_1 + n_2, \end{aligned}\right.$$

and G, F_1, F_2 *adn* F_3 *are given by (8.41) and (8.42).*

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