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On Numerical Integration of the Differential Equation $y^{(n)}=f(x,y)$

By

Minoru URABE and Hiroki YANAGIHARA

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§ 1. Introduction.

The formulas usually used for numerical integration of the differential equation of the form

$$(E) \qquad y^{(n)}=f(x,y) \qquad (n\geqq 2)$$

are deduced in the following way:

first, to integrate Newton's backward interpolation formula over n intervals $[x_{-n+1}, x_0]$, $[x_{-n+2}, x_0]$, $\cdots$, $[x_{-1}, x_0]$, $[x_0, x_1]$ or $[x_{-n}, x_0]$, $[x_{-n+1}, x_0]$, $\cdots$ $[x_{-1}, x_0]$;

next, to make a linear combination of the above n integrated formulas so that the terms of the derivatives may not appear.

The formulas obtained in this way are more convenient for integration of the differential equation of the form (E) than the formulas containing the derivatives, but the former is inferior to the latter in accuracy since the expansion of the errors in each step of numerical integration is greater in the former than in the latter. [(1)]

In this paper, following the method of the paper (P), we make a general linear combination of the formulas obtained by integrating Newton's interporation formulas over several intervals, and we seek for the coefficients of the linear combination so that the obtained formula may not contain the terms of the derivatives and moreover be as accurate as possible.

We consider the equation of the second order and that of the third order. For the equations of the higher order, the similar reasonings will prevail.

§ 2. Integration of Newton's interpolation formula.

If we put

$$(2.1) \qquad U_\rho=\frac{u(u+1)\cdots\cdots(u+\rho-1)}{\rho\,!},$$

Newton's backward interpolation formula is written as follows:

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1) The reason for this phenomenon will be seen in the paper: M. Urabe and T. Tsushima, *On Numerical Integration of Ordinary Differential Equations*, this journal, 17(1953), 193-219. In the following, we denote this paper by (P).

(2.2) $$f(x)=\sum_{\rho=0}^{p} U_\rho \nabla^\rho f_0 + S_{p+1} ,$$

where $x=x_0+uh$, h being the breadth of an interval. The remainder S_{p+1} is estimated as follows:

(2.3) $$|S_{p+1}| \leqq |U_{p+1}| h^{p+1} |f^{(p+1)}|_{\max} .$$

Now given the differential equation as follows:

(E) $$y^{(n)}=f(x, y) .$$

If we integrate $y^{(n)}$ n-times over the interval $[x_0, x_N]$, then we have:

$$\int_{x_0}^{x_N}\int_{x_0}^{x}\cdots\cdots\int_{x_0}^{x} y^{(n)} \underbrace{dx\cdots\cdots dx}_{n-\text{times}} = y_N - \sum_{\nu=0}^{n-1}\frac{(x_N-x_0)^\nu}{\nu\,!} y_0^{(\nu)} .$$

Consequently, if we put $f(x)=y^{(n)}$ in (2.2) and integrate the result n-times over the interval $[x_0, x_N]$, we have:

(2.4) $$y_N=\sum_{\nu=0}^{n-1}\frac{N^\nu h^\nu}{\nu\,!} y_0^{(\nu)} + h^n \sum_{\rho=0}^{p} \xi_{n,\rho}^{N} \nabla^\rho f_0 + R_{n,\,p+1}^{N} ,$$

where

$$\xi_{n,\rho}^{N}=\int_0^N\int_0^u\cdots\cdots\int_0^u U_\rho \underbrace{du\cdots\cdots du}_{n-\text{times}} .$$

From (2.3), the remainder $R_{n,\,p+1}^{N}$ is estimated as follows:

(2.5) $$|R_{n,\,p+1}^{N}| \leqq h^{n+p+1} \left|\int_0^N\right| \left|\int_0^u\right| \cdots\cdots \left|\int_0^u |U_{p+1}| \underbrace{|du|\cdots|du|}_{n-\text{times}}\right| \cdot |f^{(p+1)}|_{\max} .$$

When $N=1$, (2.4) is an ordinary extrapolation formula, which is written by means of the symbols of the paper (P) as follows:

(2.6) $$y_1=\sum_{\nu=0}^{n-1}\frac{h^\nu}{\nu\,!} y_0^{(\nu)} + h^n \sum_{\rho=0}^{p} \alpha_{n,\,\rho}^{1} \nabla^\rho f_0 + R_{n,\,p+1}^{1} .$$

When $N=-s$ $(s>0)$, we write (2.4) as follows:

(2.7) $$y_{-s}=\sum_{\nu=0}^{n-1}(-1)^\nu\frac{s^\nu h^\nu}{\nu\,!} y_0^{(\nu)} + h^n \sum_{\rho=0}^{p} \gamma_{n,\,\rho}^{s} \nabla^\rho f_0 + \overset{*}{R}{}_{n,\,p+1}^{s} ,$$

where

(2.8) $$\gamma_{n,\,\rho}^{s}=\xi_{n,\,\rho}^{-s}=\int_0^{-s}\int_0^u\cdots\cdots\int_0^u U_\rho \underbrace{du\cdots\cdots du}_{n-\text{times}} .$$

Then, from (2.5), the remainder $\overset{*}{R}{}_{n,\,p+1}^{s}$ is estimated as follows:

(2.9) $$|\overset{*}{R}{}_{n,\,p+1}^{s}| \leqq h^{n+p+1} \overset{*}{\gamma}{}_{n,\,p+1}^{s} |f^{(p+1)}|_{\max} ,$$

where

$$\overset{*}{\gamma}{}^{s}_{n,\rho}=\left|\int_0^{-s}\right|\int_0^{u}\Big|\cdots\cdots\Big|\int_0^{u}|U_\rho|\underbrace{du|\cdots\cdots|du}_{n-\text{times}}\Big| . \tag{2.10}$$

§ 3. Calculation of the numbers $\gamma^{s}_{n,\rho}$ and $\overset{*}{\gamma}{}^{s}_{n,\rho}$.

First we compute $\gamma^{s}_{n,\rho}$ by direct calculation. The results are shown in Table 1. Next, we seek for $\overset{*}{\gamma}{}^{s}_{n,\rho}$. They are written as follows:

$$\overset{*}{\gamma}{}^{s}_{n,\rho}=\int_{-s}^{0}\int_{u}^{0}\cdots\cdots\int_{u}^{0}|U_\rho|\underbrace{du\cdots\cdots du}_{n-\text{times}} . \tag{3.1}$$

Dividing the intervals of integration, we transform $\overset{*}{\gamma}{}^{s}_{n,\rho}$ in the following way:

$$\begin{aligned}\overset{*}{\gamma}{}^{s}_{n,\rho}&=\int_{-s}^{-(s-1)}\int_{u}^{0}\cdots\cdots\int_{u}^{0}|U_\rho|du\cdots\cdots du+\overset{*}{\gamma}{}^{s-1}_{n,\rho}\\&=\int_{-s}^{-(s-1)}\left[\int_{u}^{-(s-1)}+\int_{-(s-1)}^{0}\right]\int_{u}^{0}\cdots\cdots\int_{u}^{0}|U_\rho|du\cdots\cdots du+\overset{*}{\gamma}{}^{s-1}_{n,\rho}\\&=\int_{-s}^{-(s-1)}\int_{u}^{-(s-1)}\int_{u}^{0}\cdots\cdots\int_{u}^{0}|U_\rho|du\cdots\cdots du+\int_{-s}^{-(s-1)}du\cdot\overset{*}{\gamma}{}^{s-1}_{n-1,\rho}+\overset{*}{\gamma}{}^{s-1}_{n,\rho} .\end{aligned}$$

Continuing this process, ultimately we have:

$$\begin{aligned}\overset{*}{\gamma}{}^{s}_{n,\rho}=&\int_{-s}^{-(s-1)}\int_{u}^{-(s-1)}\cdots\cdots\int_{u}^{-(s-1)}|U_\rho|du\cdots du+\int_{-s}^{-(s-1)}\int_{u}^{-(s-1)}\cdots\cdots\int_{u}^{-(s-1)}\underbrace{du\cdots du}_{(n-1)\text{-times}}\cdot\overset{*}{\gamma}{}^{s-1}_{1,\rho}\\&+\cdots\cdots+\int_{-s}^{-(s-1)}\int_{u}^{-(s-1)}du\,du\cdot\overset{*}{\gamma}{}^{s-1}_{n-2,\rho}+\int_{-s}^{-(s-1)}du\cdot\overset{*}{\gamma}{}^{s-1}_{n-1,\rho}+\overset{*}{\gamma}{}^{s-1}_{n,\rho} .\end{aligned}\tag{3.2}$$

Now, it is easily seen that

$$\int_{u}^{-(s-1)}\cdots\cdots\int_{u}^{-(s-1)}\underbrace{du\cdots\cdots du}_{r-\text{times}}=(-1)^r\frac{\{u+(s-1)\}^r}{r!} .$$

Consequently it follows that

$$\int_{-s}^{-(s-1)}\int_{u}^{-(s-1)}\cdots\cdots\int_{u}^{-(s-1)}\underbrace{du\cdots\cdots du}_{r-\text{times}}=1/r! .$$

Substituting this into (3.2), we have:

$$\overset{*}{\gamma}{}^{s}_{n,\rho}=\int_{-s}^{-(s-1)}\int_{u}^{-(s-1)}\cdots\cdots\int_{u}^{-(s-1)}|U_\rho|du\cdots\cdots du+\sum_{r=0}^{n-1}\frac{1}{r!}\overset{*}{\gamma}{}^{s-1}_{n-r,\rho} . \tag{3.3}$$

If we write $\gamma^{s}_{n,\rho}$ as follows:

$$\gamma^{s}_{n,\rho}=(-1)^n\int_{-s}^{0}\int_{u}^{0}\cdots\cdots\int_{u}^{0}U_\rho\underbrace{du\cdots\cdots du}_{n-\text{times}} ,$$

the similar reasonings prevail and ultimately we have:

$$(3.4)\qquad (-1)^n\gamma^s_{n,\rho}=\int_{-s}^{-(s-1)}\int_{u}^{-(s-1)}\cdots\cdots\int_{u}^{-(s-1)}U_\rho\,\underbrace{du\cdots du}_{n-\text{times}}+\sum_{r=0}^{n-1}\frac{(-1)^{n-r}}{r!}\gamma^{s-1}_{n-r,\rho}.$$

Now, from (2.1), it is evident that, when $\rho\geqq 1$,

$$\begin{cases}\text{for } u \text{ such that } -\sigma\leqq u\leqq -(\sigma-1) \text{ where } 1\leqq\sigma\leqq\rho, \\ \qquad\qquad |U_\rho|=(-1)^\sigma U_\rho; \\ \text{for } u \text{ such that } u\leqq-\rho, \qquad |U_\rho|=(-1)^\rho U_\rho.\end{cases}$$

Consequently it follows that

$$(3.5)\qquad \int_{-s}^{-(s-1)}\int_{u}^{-(s-1)}\cdots\cdots\int_{u}^{-(s-1)}|U_\rho|\,du\cdots\cdots du$$
$$=(-1)^\sigma\int_{-s}^{-(s-1)}\int_{u}^{-(s-1)}\cdots\cdots\int_{u}^{-(s-1)}U_\rho\,du\cdots\cdots du,$$

where $\sigma=s$ or ρ acccording as $s\leqq\rho$ or $s\geqq\rho$. Then, comparing (3.3) with (3.4), we have:

$$(3.6)\qquad \overset{*}{\gamma}{}^s_{n,\rho}-(-1)^{\sigma+n}\gamma^s_{n,\rho}=\sum_{r=0}^{n-1}\frac{1}{r!}\{\overset{*}{\gamma}{}^{s-1}_{n-r,\rho}-(-1)^{\sigma+n-r}\gamma^{s-1}_{n-r,\rho}\}.$$

By this formula, we can compute $\overset{*}{\gamma}{}^s_{n,\rho}$ successively from $\gamma^s_{n,\rho}$ already computed. The results are shown in Table 2.

The explicit formulas for $\overset{*}{\gamma}{}^s_{n,\rho}$ in terms of $\gamma^s_{n,\rho}$ can be obtained from (3.6). They are expressed as follows:

$$(3.7)\qquad \begin{cases}\text{(i) for } s=1,2,\cdots\cdots\rho, \\ \quad \overset{*}{\gamma}{}^s_{n,\rho}=(-1)^{n+s}\left[\gamma^s_{n,\rho}+2\sum_{t=1}^{s-1}\sum_{r=0}^{n-1}(-1)^{t+r}\frac{t^r}{r!}\gamma^{s-t}_{n-r,\rho}\right]; \\ \text{(ii) for } s=\rho,\rho+1,\cdots\cdots, \\ \quad \overset{*}{\gamma}{}^s_{n,\rho}=(-1)^{n+\rho}\left[\gamma^s_{n,\rho}+2\sum_{t=1}^{\rho-1}\sum_{r=0}^{n-1}(-1)^{t+r}\frac{(t+s-\rho)^r}{r!}\gamma^{\rho-t}_{n-r,\rho}\right].\end{cases}$$

These are easily proved by mathematical induction. Of course, $\overset{*}{\gamma}{}^s_{n,\rho}$ can be computed from $\gamma^s_{n,\rho}$ by means of these formulas, but the actual computation is rather laborious by (3.7) than by (3.6).

In the paper (P), we have calculated the numbers $\beta^s_{n,\rho}$ and $\overset{*}{\beta}{}^s_{n,\rho}$ which are defined as follows:

$$
(3.8)\quad \begin{cases} \beta^s_{n,\rho}=\displaystyle\int_{-s}^{0}\int_{-s}^{u}\cdots\cdots\int_{-s}^{u} U_\rho \ \underbrace{du\cdots\cdots du}_{n-\text{times}}, \\ \overset{*}{\beta}{}^s_{n,\rho}=\displaystyle\int_{-s}^{0}\int_{-s}^{u}\cdots\cdots\int_{-s}^{u} |U_\rho| \ \underbrace{du\cdots\cdots du}_{n-\text{times}}. \end{cases}
$$

Let us find the relations of these numbers to $\gamma^s_{n,\rho}$ and $\overset{*}{\gamma}{}^s_{n,\rho}$. Dividing the intervals of integration of $\gamma^s_{n,\rho}$, we transform it as follows:

$$
\gamma^s_{n,\rho}=-\int_{-s}^{0}\left[\int_{0}^{-s}+\int_{-s}^{u}\right]\int_{0}^{u}\cdots\cdots\int_{0}^{u}U_\rho\ du\cdots\cdots du
$$

$$
=-\left[\beta^s_{1,0}\ \gamma^s_{n-1,\rho}+\int_{-s}^{0}\int_{-s}^{u}\int_{0}^{u}\cdots\cdots\int_{0}^{u}U_\rho\ du\cdots\cdots du\right].
$$

Continuing this process, the following relations are obtained:

$$
(3.9)\qquad \gamma^s_{n,\rho}=-(\beta^s_{1,0}\,\gamma^s_{n-1,\rho}+\beta^s_{2,0}\,\gamma^s_{n-2,\rho}+\cdots\cdots+\beta^s_{n-1,0}\,\gamma^s_{1,\rho}+\beta^s_{n,\rho}).
$$

Now, by direct calculation, it is easily seen that

$$
\beta^s_{1,0}=s,\ \beta^s_{2,0}=\frac{s^2}{2},\ \beta^s_{3,0}=\frac{s^3}{6},\ \cdots\cdots.
$$

Then, if we solve (3.9) successively with regard to $\gamma^s_{n,\rho}$ $(n=1, 2, \cdots\cdots)$ and substitute the values of $\beta^s_{1,0}$, $\beta^s_{2,0}$ $\cdots\cdots$ just found, then we have:

$$
(3.10)\quad \begin{cases} \gamma^s_{1,\rho}=-\beta^s_{1,\rho}, \\ \gamma^s_{2,\rho}=-\beta^s_{2,\rho}+s\,\beta^s_{1,\rho}, \\ \gamma^s_{3,\rho}=-\beta^s_{3,\rho}+s\,\beta^s_{2,\rho}-\dfrac{s^2}{2}\,\beta^s_{1,\rho}, \\ \gamma^s_{4,\rho}=-\beta^s_{4,\rho}+s\,\beta^s_{3,\rho}-\dfrac{s^2}{2}\beta^s_{2,\rho}+\dfrac{1}{6}s^3\,\beta^s_{1,\rho}, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots. \end{cases}
$$

For $\overset{*}{\gamma}{}^s_{n,\rho}$, if we write it as (3.1), the similar reasonings prevail and ultimately the following relations are obtained:

$$
(3.11)\quad \begin{cases} \overset{*}{\gamma}{}^s_{1,\rho}=\overset{*}{\beta}{}^s_{1,\rho}, \\ \overset{*}{\gamma}{}^s_{2,\rho}=-\overset{*}{\beta}{}^s_{2,\rho}+s\,\overset{*}{\beta}{}^s_{1,\rho}, \\ \overset{*}{\gamma}{}^s_{3,\rho}=\overset{*}{\beta}{}^s_{3,\rho}-s\,\overset{*}{\beta}{}^s_{2,\rho}+\dfrac{s^2}{2}\overset{*}{\beta}{}^s_{1,\rho}, \\ \overset{*}{\gamma}{}^s_{4,\rho}=-\overset{*}{\beta}{}^s_{4,\rho}+s\,\overset{*}{\beta}{}^s_{3,\rho}-\dfrac{s^2}{2}\overset{*}{\beta}{}^s_{2,\rho}+\dfrac{1}{6}s^3\overset{*}{\beta}{}^s_{1,\rho}, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots. \end{cases}
$$

By means of (3.10) and (3.11), we can compute $\gamma^s_{n,\rho}$ and $\overset{*}{\gamma}{}^s_{n,\rho}$ from $\beta^s_{n,\rho}$ and $\overset{*}{\beta}{}^s_{n,\rho}$, and conversely. The calculation by means of (3.10) and (3.11) is much easier than that by the first method. Therefore, in making Tables 1 and 2, for $n=1$ and 2, we have calculated $\gamma^s_{n,\rho}$ and $\overset{*}{\gamma}{}^s_{n,\rho}$ by means of (3.10) and (3.11) since $\beta^s_{n,\rho}$ and $\overset{*}{\beta}{}^s_{n,\rho}$ are already given in the paper (P). For $n=3$, we have calculated $\gamma^s_{3,\rho}$ and $\overset{*}{\gamma}{}^s_{3,\rho}$ by the first method because $\beta^s_{3,\rho}$ and $\overset{*}{\beta}{}^s_{3,\rho}$ are not given there.

§ 4. Integation formulas.

First, we seek for the extrapolation formulas. We write (2.6) and (2.7) in the forms as follows:

$$(4.1)\quad \begin{cases} \text{(i)}\quad y_{r+1}=\sum_{\nu=0}^{n-1}\frac{h^\nu}{\nu\,!}y_r^{(\nu)}+h^n\sum_{\rho=0}^{p}\alpha^1_{n,\rho}\nabla^\rho f_r+R^1_{n,\,p+1}\,, \\ \text{(ii)}\quad y_{r-s}=\sum_{\nu=0}^{n-1}(-1)^\nu\frac{s^\nu h^\nu}{\nu\,!}y_r^{(\nu)}+h^n\sum_{\rho=0}^{p}\gamma^s_{n,\rho}\nabla^\rho f_r+\overset{*}{R}{}^s_{n,\,p+1}\,. \end{cases}$$

$$(s=1,2,\cdots\cdots,N)$$

Multiplying $-l_s$ on both sides of (ii) and adding them for all s to (i), we have the extrapolation formula as follows:

$$(4.2)\qquad y_{r+1}=\sum_{s=0}^{N}l_s y_{r-s}+\sum_{\nu=1}^{n-1}\{1-(-1)^\nu\sum_{s=1}^{N}s^\nu l_s\}\frac{h^\nu}{\nu\,!}y_r^{(\nu)}+h^n\sum_{\rho=0}^{p}a^N_{n,\rho}\nabla^\rho f_r+R_{n,\,p+1}\,,$$

where

$$(4.3)\qquad \begin{cases} l_0=1-\sum_{s=1}^{N}l_s\,, \\ a^N_{n,\rho}=\alpha^1_{n,\rho}-\sum_{s=1}^{N}l_s\gamma^s_{n,\rho}\,. \end{cases}$$

Here the remainder $R_{n,\,p+1}$ is estimated from (2.9) as follows:

$$(4.4)\qquad |R_{n,\,p+1}|\leqq h^{n+p+1}A^N_{n,\,p+1}|f^{(p+1)}|_{\max}\,,$$

where

$$A^N_{n,\,p+1}=\alpha^1_{n,\,p+1}+\sum_{s=1}^{N}|l_s|\overset{*}{\gamma}{}^s_{n,\,p+1}\,.$$

Now, in order that (4.2) be suitable to numerical solution of the equation of the form (E), it is necessary that (4.2) does not contain the terms of the derivatives, namely that

$$(4.5)\qquad 1+(-1)^{\nu-1}\sum_{s=1}^{N}s^\nu l_s=0\,.\qquad(\nu=1,2,\cdots\cdots,n-1)$$

Thus the desired extrapolation formula is written as follows:

$$y_{r+1}=\sum_{s=0}^{N} l_s y_{r-s}+h^n \sum_{\rho=0}^{p} a^N_{n,\rho} \nabla^\rho f_r . \tag{4.6}$$

Now, let the error of the approximate value of y_i calculated by means of (4.6) be ε_i. Put

$$\alpha_{n,\sigma}=(-1)^\sigma \sum_{\rho=\sigma}^{p} a^N_{n,\rho} \binom{\rho}{\sigma} ,$$

then, as in the paper (P), from (4.6) and (4.4), it follows that

$$|\varepsilon_{r+1}| \leqq \sum_{s=0}^{N} |l_s| \; |\varepsilon_{r-s}| + h^n K \sum_{\sigma=0}^{p} |\alpha_{n,\sigma}| \; |\varepsilon_{r-\sigma}| + h^{n+p+1} A^N_{n,\,p+1} L , \tag{4.7}$$

where

$$K=\left|\frac{\partial f(x,y)}{\partial y}\right|_{\max} \quad \text{and} \quad L=|f^{(p+1)}|_{\max} .$$

Put $\max\limits_{r,s\geqq 0} (|\varepsilon_{r-s}|, |\varepsilon_{r-\sigma}|)=|\varepsilon|$, then, from (4.7), it follows that

$$|\varepsilon_{r+1}| \leqq \Big(\sum_{s=0}^{N} |l_s| + h^n K \sum_{\sigma=0}^{p} |\alpha_{n,\sigma}|\Big) |\varepsilon| + h^{n+p+1} A^N_{n,p+1} L . \tag{4.8}$$

Then, in order that the extrapolation formula (4.6) be accurate, the quantity $\big(\sum_{s=0}^{N} |l_s| + h^n K \sum_{\sigma=0}^{p} |\alpha_{n,\sigma}|\big)$ should be as small as possible. However, since $h \ll 1$, the quantity $\sum_{s=0}^{N} |l_s|$ should be as small as possible.

Thus the conditions that l_s's should satisfy for our purpose, are summarized from (4.3) and (4.5) as follows:

$$\left\{\begin{array}{ll} \text{(i)} \;\; \sum_{s=0}^{N} l_s=1 , & \\ \text{(ii)} \;\; 1+(-1)^{\nu-1} \sum_{s=1}^{N} s^\nu l_s=0 , & (\nu=1, 2, \cdots\cdots, n-1) \\ \text{(iii)} \;\; \sum_{s=0}^{N} |l_s|=\min . & \end{array}\right. \tag{4.9}$$

The numbers l_s's satisfying these conditions are determined by the method that was explained in the paper (P).

Next, we seek for the improving formulas. We write (2.7) as follows:

$$y_{r+1-s}=y_{r+1}+\sum_{\nu=1}^{n-1} (-1)^\nu \frac{s^\nu h^\nu}{\nu\,!} y^{(\nu)}_{r+1} + h^n \sum_{\rho=0}^{p} \gamma^s_{n,\rho} \nabla^\rho f_{r+1} + \overset{*}{R}{}^s_{n,\,p+1} . \tag{4.10}$$

$$(s=1, 2, \cdots\cdots, N)$$

Multiplying $-l_s$ on both sides of this formula and summing up them for all s, we have:

$$\Big(\sum_{s=1}^{N} l_s\Big) y_{r+1}=\sum_{s=1}^{N} l_s y_{r+1-s} + \sum_{\nu=1}^{n-1} \Big(\sum_{s=1}^{N} s^\nu l_s\Big) (-1)^{\nu-1} \frac{h^\nu}{\nu\,!} y^{(\nu)}_{r+1}$$

$$+h^n \sum_{\rho=0}^{p} b^N_{n,\rho} \nabla^\rho f_{r+1} - \sum_{s=1}^{N} l_s \overset{*}{R}{}^s_{n,\,p+1} ,$$

where

(4.11) $$b^N_{n,\rho}=-\sum_{s=1}^{N} l_s\gamma^s_{n,\rho}\,.$$

Normalizing l_s's so that

(4.12) $$\sum_{s=1}^{N} l_s=1\,,$$

and moreover imposing on l_s's the conditions that

(4.13) $$\sum_{s=1}^{N} s^\nu l_s=0\,, \qquad (\nu=1,2,\cdots\cdots,n-1)$$

we have the following improving formula suitable to numerical solution of the equation of the form (E):

(4.14) $$y_{r+1}=\sum_{s=1}^{N} l_s y_{r+1-s}+h^n\sum_{\rho=0}^{p} b^N_{n,\rho}\nabla^\rho f_{r+1}+R'_{n,\,p+1}\,.$$

Here the remainder $R'_{n,\,p+1}$ is estimated from (2.9) as follows:

(4.15) $$|R'_{n,\,p+1}|\leqq h^{n+p+1}B^N_{n,\,p+1}|f^{(p+1)}|_{\max}\,,$$

where

$$B^N_{n,\,p+1}=\sum_{s=1}^{N}|l_s|\overset{*}{\gamma}{}^s_{n,\,p+1}\,.$$

Now, let the error of the approximate value of y_i calculated by means of (4.14) be ε_i. Put

(4.16) $$\beta_{n,\sigma}=(-1)^\sigma\sum_{\rho=\sigma}^{p} b^N_{n,\rho}\binom{\rho}{\sigma}\,,$$

then, as in the case of the extrapolation formula, from (4.14) and (4.15), it follows that

$$|\varepsilon_{r+1}|\leqq\sum_{s=1}^{N}|l_s|\,|\varepsilon_{r+1-s}|+h^n K\sum_{\sigma=0}^{p}|\beta_{n,\sigma}|\,|\varepsilon_{r+1-\sigma}|+h^{n+p+1}B^N_{n,\,p+1}L\,.$$

Put $\max\limits_{s,\sigma\geqq 1}(|\varepsilon_{r+1-s}|,\ |\varepsilon_{r+1-\sigma}|)=|\varepsilon|$, then it follows that

$$(1-h^n K|\beta_{n,0}|)\,|\varepsilon_{r+1}|\leqq\Big(\sum_{s=1}^{N}|l_s|+h^n K\sum_{\sigma=1}^{p}|\beta_{n,\sigma}|\Big)\,|\varepsilon|+h^{n+p+1}B^N_{n,p+1}L\,.$$

Now $h\ll 1$, therefore, neglecting the higher orders with regard to h, we have:

(4.17) $$|\varepsilon_{r+1}|\leqq\Big[\sum_{s=1}^{N}|l_s|+h^n K\Big(|\beta_{n,0}|\sum_{s=1}^{N}|l_s|+\sum_{\sigma=1}^{p}|\beta_{n,\sigma}|\Big)\Big]|\varepsilon|+h^{n+p+1}B^N_{n,\,p+1}L\,.$$

Then, as in the case of the extrapolation formula, the quantity $\sum_{s=1}^{N}|l_s|$ should be as small as possible.

Thus the conditions that l_s's should satisfy are summarized from (4.12) and (4.13) as follows:

$$
(4.18)\qquad \begin{cases} \text{(i)} & \sum_{s=1}^{N} l_s = 1\,, \\ \text{(ii)} & \sum_{s=1}^{N} s^{\nu} l_s = 0\,, \qquad (\nu = 1, 2, \cdots\cdots, n-1) \\ \text{(iii)} & \sum_{s=1}^{N} |l_s| = \min\,. \end{cases}
$$

Determination of the numbers l_s's satisfying these conditions is carried out by the same method as in the case of (4.9).

Now, the improved value of y_{r+1} is found by the method of iteration. In this process, as seen from (4.14) and (4.16), the quantity $|\beta_{n,0}| = |\sum_{\rho=0}^{p} b^N_{n,\rho}|$ expresses the rapidity of convergence of iteration process. Hence, we have shown these quantities also in the tables of the improving formulas.

§ 5. Formulas for the equation of the second order.

For the differential equation of the second order, the conditions (4.9) are written as follows:

$$
(5.1)\qquad \begin{cases} \text{(i)} & l_0 + l_1 + l_2 + \cdots\cdots + l_s + \cdots\cdots + l_N = 1\,, \\ \text{(ii)} & 1 + l_1 + 2l_2 + \cdots\cdots + s\,l_s + \cdots\cdots + Nl_N = 0, \\ \text{(iii)} & |l_0| + |l_1| + |l_2| + \cdots\cdots + |l_s| + \cdots\cdots + |l_N| = \min\,. \end{cases}
$$

According to the method of the paper (P), we put arbitrary $N-1$ of l_s's zero and we seek for the remaining two so that the sum of their absolute values may become minimum.

If we take l_t and l_u $(0 \leqq t < u \leqq N)$ and put other l_s's zero, the conditions (i) and (ii) of (5.1) become

$$
\begin{cases} l_t + l_u = 1\,, \\ 1 + t\,l_t + u\,l_u = 0\,. \end{cases}
$$

The solution of these simultaneous equations is sought as follows:

$$
(5.2)\qquad l_t = \frac{u+1}{u-t}\,, \quad l_u = -\frac{t+1}{u-t}\,,
$$

consequently, for this solution, it becomes that

$$
(5,3)\qquad |l_t| + |l_u| = 1 + \frac{2\,(t+1)}{u-t}\,.
$$

Now, since $\dfrac{t+1}{u-t} \geqq \dfrac{1}{u}$ and $\dfrac{1}{u}$ decreases monotonely, it follows from (5.3) that

$$|l_0|+|l_N|\leqq|l_0|+|l_u|\leqq|l_t|+|l_u|\,.$$

Therefore $\sum_{s=0}^{N}|l_s|$ becomes minimum when $l_0, l_N \neq 0$ and other l_s's all vanish. Thus, from (5.2), we see thet l_s's satisfying all the conditions of (5.1) become

$$l_0=\frac{N+1}{N}\,,\quad l_N=-\frac{1}{N}\,,\quad l_s=0\,. \qquad (s=1, 2, \cdots\cdots, N-1) \tag{5.4}$$

For these values of l_s's, it is evident that

$$\sum_{s=0}^{N}|l_s| = 1+\frac{2}{N}\,. \tag{5.5}$$

Thus, by (4.6), the desired extrapolation formula is written as follows:

$$y_{r+1}=\frac{N+1}{N}y_r-\frac{1}{N}y_{r-N}+h^2\sum_{\rho=0}^{p}a^N_{2,\rho}\nabla^\rho f_r+R_{2,p+1}\,, \tag{5.6}$$

where

$$a^N_{2,\rho}=\alpha^1_{2,\rho}+\frac{1}{N}\gamma^N_{2,\rho}\,.$$

Here the remainder $R_{2,\,p+1}$ is estimated as follows:

$$|R_{2,\,p+1}|\leqq h^{p+3}A^N_{2,\,p+1}\,|f^{(p+1)}|_{\max}\,,$$

where

$$A^N_{2,\,p+1}=\alpha^1_{2,\,p+1}+\frac{1}{N}\overset{*}{\gamma}{}^N_{2,\,p+1}\,.$$

The extrapolation formulas of the form (5.6) are tabulated to $N=5$ in Table 3.

Next, we consider the conditions (4.18), which, for $n=2$, are written as follows:

$$\begin{cases} \text{(i)} & l_1+l_2+\cdots\cdots+l_s+\cdots\cdots+l_N=1\,, \\ \text{(ii)} & l_1+2l_2+\cdots\cdots+s\,l_s+\cdots\cdots+Nl_N=0\,, \\ \text{(iii)} & |l_1|+|l_2|+\cdots\cdots+|l_s|+\cdots\cdots+|l_N|=\min\,. \end{cases} \tag{5.7}$$

As in the case of the conditions (5.1), we take l_t and l_u $(1\leqq t<u\leqq N)$ and put other l_s's zero, then the conditions (i) and (ii) of (5.7) become

$$\begin{cases} l_t+l_u=1\,, \\ t\,l_t+u\,l_u=0\,. \end{cases}$$

The solution of these simultaneous equations is sought as follows:

$$l_t=\frac{u}{u-t}\,,\quad l_u=\frac{-t}{u-t}\,, \tag{5.8}$$

consequently, for this solution, it becomes that

$$|l_t|+|l_u|=1+\frac{2t}{u-t}. \tag{5.9}$$

Now, for fixed t, $\frac{t}{u-t}$ is monotone decreasing with regard to u. Therefore, for any $u\leqq N$, it is valid that

$$\frac{t}{u-t}\geqq\frac{t}{N-t}.$$

Now, $\frac{t}{N-t}$ is monotone increasing with regard to t. Therefore, for any $t\geqq 1$, it is valid that

$$\frac{t}{N-t}\geqq\frac{1}{N-1}.$$

Thus it follows that

$$\frac{t}{u-t}\geqq\frac{1}{N-1},$$

namely, that

$$|l_t|+|l_u|\geqq|l_1|+|l_N|.$$

Thus, from (5.8), we see that l_s's satisfying all the conditions of (5.7) become

$$l_1=\frac{N}{N-1},\quad l_N=-\frac{1}{N-1},\quad l_s=0. \quad (s=2, 3, \cdots, N-1) \tag{5.10}$$

For these values of l_s's, it is evident that

$$\sum_{s=1}^{N}|l_s|=1+\frac{2}{N-1}.$$

Thus, by (4.14), the desired improving formula is written as follows:

$$y_{r+1}=\frac{N}{N-1}y_r-\frac{1}{N-1}y_{r+1-N}+h^2\sum_{\rho=0}^{p}b^N_{2,\rho}\nabla^\rho f_{r+1}+R'_{2,\,p+1}, \tag{5.11}$$

where

$$b^N_{2,\rho}=-\frac{N}{N-1}\gamma^1_{2,\rho}+\frac{1}{N-1}\gamma^N_{2,\rho}.$$

Here the remainder $R'_{2,\,p+1}$ is estimated as follows:

$$|R'_{2,\,p+1}|\leqq h^{p+3}B^N_{2,p+1}\ L,$$

where

$$B^N_{2,\,p+1}=\frac{N}{N-1}\overset{*}{\gamma}{}^1_{2,\rho}+\frac{1}{N-1}\overset{*}{\gamma}{}^N_{2,\rho}.$$

The improving formulas of the form (5.11) are tabulated to $N=5$ in Table 4.

§ 6. An example for the equation of the second order.

As an example, we shall find correctly to four decimal places the solution of the equation

$$y'' = 1 + y, \tag{6.1}$$

with the initial condition that $y(0)=y'(0)=0$.

In order to obtain the solution correct to four decimal places, we take $h=0.1$ and adopt the formulas for $p=3$. Now, for numerical computation, it is desirable that the errors in the extrapolation formulas be of the same magnitude as in the improving formulas. For this reason, we adopt the extrapolation formula for $N=4$ and the improving formula for $N=5$. Thus the formulas of which we make use in this example are written as follows:

(6.2) for extrapolation,

$$y_{r+1}=\frac{1}{4}(5y_r-y_{r-4})+\frac{0.01}{24}(60f_r-60\nabla f_r+35\nabla^2 f_r-6\nabla^3 f_r);$$

for improving,

$$y_{r+1}=\frac{1}{4}(5y_r-y_{r-4})+\frac{0.01}{24}(60f_{r+1}-120\nabla f_{r+1}+95\nabla^2 f_{r+1}-41\nabla^3 f_{r+1}).$$

In order to find the starting values, we make use of Taylor's series. From (6.1), it follows that

$$y^{(n+2)} = y^{(n)}. \qquad (n=1, 2, \cdots\cdots)$$

Therefore, making use of the initial conditions, we have:

$$y(0)=0,\ y'(0)=0,\ y''(0)=1,\ y^{(3)}(0)=0,\ y^{(4)}(0)=1, \cdots\cdots.$$

By means of Taylor's series determined by these values, we compute the starting values of y correct to four decimal places for $x=-0.2,\ -0.1,\ 0,\ 0.1$ and 0.2.

Then, by means of (6.2), we compute successively the values of y correct to four decimal places. These results are tabulated in the following table as y_1.

Now, the equation (6.1) is easily integrated and the solution satisfying the given initial conditions, becomes

$$y=\frac{1}{2}(e^x+e^{-x})-1=\cosh x-1. \tag{6.3}$$

For comparison, the true values of y computed from this function are also tabulated in the table.

Solution of the equation $y''=1+y$ *with the initial condition that* $y(0)=y'(0)=0$

x	y				
	y_1 values	y_1 errors	true values	y_2 values	y_2 errors
−0.2	0.0201		0.0201		
−0.1	0.0050		0.0050	0.0050	
0	0.0000		0.0000	0.0000	
0.1	0.0050		0.0050	0.0050	
0.2	0.0201		0.0201	0.0201	
0.3	0.0454	+ 1	0.0453	0.0454	+ 1
0.4	0.0812	+ 1	0.0811	0.0812	+ 1
0.5	0.1278	+ 2	0.1276	0.1278	+ 2
0.6	0.1857	+ 2	0.1855	0.1857	+ 2
0.7	0.2555	+ 3	0.2552	0.2555	+ 3
0.8	0.3378	+ 4	0.3374	0.3379	+ 5
0.9	0.4336	+ 5	0.4331	0.4337	+ 6
1.0	0.5436	+ 5	0.5431	0.5438	+ 7
1.1	0.6691	+ 6	0.6685	0.6694	+ 9
1.2	0.8113	+ 6	0.8107	0.8117	+10
1.3	0.9717	+ 8	0.9709	0.9721	+12
1.4	1.1517	+ 8	1.1509	1.1522	+13
1.5	1.3533	+ 9	1.3524	1.3538	+14
1.6	1.5785	+10	1.5775	1.5790	+15
1.7	1.8295	+12	1.8283	1.8300	+17
1.8	2.1088	+13	2.1075	2.1093	+18
1.9	2.4192	+15	2.4177	2.4197	+20
2.0	2.7639	+17	2.7622	2.7643	+21

For comparison, we have computed the values of y also by means of the customary difference formulas as follows:

(6.4) for extrapolation,

$$y_{r+1}=2y_r-y_{r-1}+\frac{0.01}{12}(12f_r+\nabla^2 f_r+\nabla^3 f_r),$$

for improving,

$$y_{r+1}=2y_r-y_{r-1}+\frac{0.01}{12}(12f_{r+1}-12\nabla f_{r+1}+\nabla^2 f_{r+1}).$$

These values are tabulated in the table as y_2.

As seen from our example, the new formulas are more accurate than the customary ones.

§ 7. Formulas for the equation of the third order.

For the differential equation of the third order, the conditions (4.9) are written as follows:

$$
(7.1)\qquad \begin{cases} \text{(i)} & l_0+l_1+l_2+\cdots\cdots+l_s+\cdots\cdots+l_N=1\,, \\ \text{(ii)} & 1+l_1+2l_2+\cdots\cdots+s\,l_s+\cdots\cdots+Nl_N=0\,, \\ \text{(iii)} & 1-l_1-4l_2-\cdots\cdots-s^2l_s-\cdots\cdots-N^2l_N=0\,, \\ \text{(iv)} & |l_0|+|l_1|+|l_2|+\cdots\cdots+|l_s|+\cdots\cdots+|l_N|=\min\,. \end{cases}
$$

According to the method of the paper (P), we put arbitrary $N-2$ of l_s's zero and we seek for the remaining three so that the sum of their absolute values may become minimum.

If we take l_t, l_u and l_v $(0\leqq t<u<v\leqq N)$ and put the other l_s's zero, the conditions (i), (ii) and (iii) of (7.1) become

$$
\begin{cases} l_t+l_u+l_v=1\,, \\ 1+t\,l_t+u\,l_u+v\,l_v=0\,, \\ 1-t^2l_t-u^2l_u-v^2l_v=0\,. \end{cases}
$$

Solving these simultaneous equations, we find that

$$
(7.2)\qquad l_t=\frac{(v+1)\,(u+1)}{(v-t)\,(u-t)}\,,\quad l_u=-\frac{(v+1)\,(t+1)}{(v-u)\,(u-t)}\,,\quad l_v=\frac{(t+1)\,(u+1)}{(v-u)\,(v-t)}\,,
$$

consequently, for this solution, it becomes that

$$
(7.3)\qquad |l_t|+|l_u|+|l_v|=1+2\cdot\frac{(v+1)\,(t+1)}{(v-u)\,(u-t)}\,.
$$

Now, it is evident that

$$
\frac{(v+1)\,(t+1)}{(v-u)\,(u-t)}\geqq\frac{v+1}{v-u}\cdot\frac{1}{u}=\frac{1}{u}\left(1+\frac{u+1}{v-u}\right)
$$

$$
\geqq\frac{1}{u}\left(1+\frac{u+1}{N-u}\right)=\frac{N+1}{u(N-u)}\,,
$$

consequently, from (7.3), it follows that

$$
|l_0|+|l_u|+|l_N|\leqq|l_0|+|l_u|+|l_v|\leqq|l_t|+|l_u|+|l_v|\,.
$$

Now, since $u+(N-u)=N$, evidently $\dfrac{N+1}{u(N-u)}$ becomes minimum when $u=\dfrac{N}{2}$ for even N and when $u=\dfrac{N\pm1}{2}$ for odd N. Thus, from (7.2), it is seen that l_s's satisfying all the conditions of (7.1) become

$$(7.4)\qquad l_0=\frac{(u+1)(N+1)}{uN},\quad l_u=-\frac{N+1}{u(N-u)},\quad l_N=\frac{u+1}{N(N-u)},$$
$$l_s=0,\ (s=1, 2, \cdots, u-1, u+1, \cdots, N-1)$$

where

$$u=\begin{cases}\dfrac{N}{2} & \text{when } N \text{ is even,}\\[2mm] \dfrac{N\pm1}{2} & \text{when } N \text{ is odd.}\end{cases}$$

For these values of l_s's, from (7.3), it is seen that

$$\sum_{s=0}^{N}|l_s|=1+2\cdot\frac{N+1}{u(N-u)}=\begin{cases}1+8\left(\dfrac{1}{N}+\dfrac{1}{N^2}\right) & \text{when } N \text{ is even,}\\[2mm] 1+8\cdot\dfrac{1}{N-1} & \text{when } N \text{ is odd.}\end{cases}$$

From this equality, it is easily seen that, as in the case of the equation of the second order, the quantity $\sum_{s=0}^{N}|l_s|$ decreases monotonely as N increases.

By means of (7.4), from (4.6), the desired extrapolation formula is written as follows:

$$(7.5)\qquad y_{r+1}=\frac{(u+1)(N+1)}{uN}y_r-\frac{N+1}{u(N-u)}y_{r-u}+\frac{u+1}{N(N-u)}y_{r-N}$$
$$+h^3\sum_{\rho=0}^{p}a_{3,\rho}^{N}\nabla^{\rho}f_r+R_{3,\,p+1},$$

where

$$a_{3,\rho}^{N}=\alpha_{3,\rho}^{1}+\frac{N+1}{u(N-u)}\gamma_{3,\rho}^{u}-\frac{u+1}{N(N-u)}\gamma_{3,\rho}^{N},$$

and

$$u=\begin{cases}\dfrac{N}{2} & \text{when } N \text{ is even,}\\[2mm] \dfrac{N\pm1}{2} & \text{when } N \text{ is odd.}\end{cases}$$

Here the remainder $R_{3,\,p+1}$ is estimated as follows:

$$|R_{3,\,p+1}|\leqq h^{p+4}A_{3,\,p+1}^{N}\,|f^{(p+1)}|\max,$$

where

$$A_{3,\,p+1}^{N}=\alpha_{3,\,p+1}^{1}+\frac{N+1}{u(N-u)}\overset{*}{\gamma}{}_{3,\rho}^{u}+\frac{u+1}{N(N-u)}\overset{*}{\gamma}{}_{3,\rho}^{N}.$$

Now, when N in odd, $A_{3,\,p+1}^{N}$ becomes

$$\alpha^1_{3,\,p+1}+\frac{4}{N-1}\overset{*}{\gamma}{}^{(N-1)/2}_{3,p}+\frac{1}{N}\overset{*}{\gamma}{}^{N}_{3,p} \qquad \text{for} \quad u=(N-1)/2$$

and

$$\alpha^1_{3,\,p+1}+\frac{4}{N-1}\overset{*}{\gamma}{}^{(N+1)/2}_{3,p}+\frac{1}{N}\left(1+\frac{4}{N-1}\right)\overset{*}{\gamma}{}^{N}_{3,p} \quad \text{for } u=(N+1)/2\,.$$

Since $\overset{*}{\gamma}{}^{(N+1)/2}_{3,p} > \overset{*}{\gamma}{}^{(N-1)/2}_{3,p}$ form (3.3), $A^N_{3,\,p+1}$ is less in the case where $u=(N-1)/2$ than in the case where $u=(N+1)/2$, namely the formula for $u=(N-1)/2$ is more accurate than that for $u=(N+1)/2$. Thus, when N is odd, the desired extrapolation formula is given by (7.5) in which $u=(N-1)/2$. These extrapolation formulas are tabulated to $N=5$ in Table 5.

Next, we consider the conditions (4.18), which, for $n=3$, are written as follows:

(7.6)
$$\begin{cases} \text{(i)} & l_1+l_2+\cdots\cdots+l_s+\cdots\cdots+l_N=1\,, \\ \text{(ii)} & l_1+2l_2+\cdots\cdots+s\,l_s+\cdots\cdots+Nl_N=0\,, \\ \text{(iii)} & l_1+4l_2+\cdots\cdots+s^2l_s+\cdots\cdots+N^2l_N=0\,, \\ \text{(iv)} & |l_1|+|l_2|+\cdots\cdots+|l_s|\cdots\cdots+|l_N|=\min\,. \end{cases}$$

In the same manner as in the case of the conditions (7.1), it is seen that l_s's satisfying all the conditions of (7.6) become

(7,7)
$$l_1=\frac{Nu}{(N-1)(u-1)}\,,\quad l_u=-\frac{N}{(N-u)(u-1)}\,,\quad l_N=\frac{u}{(N-1)(N-u)}\,,$$
$$l_s=0,\ (s=2,3,\cdots,u-1,u+1,\cdots.\,N-1)$$

where

$$u=\begin{cases} \dfrac{N+1}{2} & \text{when } N \text{ is odd,} \\[2ex] \dfrac{N}{2} \text{ or } \dfrac{N+2}{2} & \text{when } N \text{ is even.} \end{cases}$$

For these values of l_s's, from (7.7), it is seen that

$$\sum_{s=1}^{N}|l_s|=1+2\cdot\frac{N}{(N-u)(u-1)}=\begin{cases} 1+8\left(\dfrac{1}{N-1}+\dfrac{1}{(N-1)^2}\right) & \text{when } N \text{ is odd,} \\[2ex] 1+8\cdot\dfrac{1}{N-2} & \text{when } N \text{ is even.} \end{cases}$$

Therefore, as in the case of (7.4), the quantity $\sum_{s=1}^{N}|l_s|$ decreases monotonely as N increases.

By means of (7.7), from (4.14), the desired improving formula is written as follows:

$$(7.8)\quad y_{r+1}=\frac{Nu}{(N-1)(u-1)}y_r-\frac{N}{(N-u)(u-1)}y_{r+1-u}+\frac{u}{(N-1)(N-u)}y_{r+1-N}$$
$$+h^3\sum_{\rho=0}^{p}b^N_{3,\rho}\nabla^\rho f_{r+1}+R'_{3,p+1},$$

where

$$b^N_{3,\rho}=-\frac{Nu}{(N-1)(u-1)}\gamma^1_{3,\rho}+\frac{N}{(N-u)(u-1)}\gamma^u_{3,\rho}-\frac{u}{(N-1)(N-u)}\gamma^N_{3,\rho},$$

and

$$u=\begin{cases}\dfrac{N+1}{2} & \text{when } N \text{ is odd,}\\[2mm] \dfrac{N}{2} \text{ or } \dfrac{N+2}{2} & \text{when } N \text{ is even.}\end{cases}$$

Here the remainder $R'_{3,p+1}$ is estimated as follows:

$$|R'_{3,p+1}|\leqq h^{p+4}B^N_{3,p+1}|f^{(p+1)}|_{\max},$$

where

$$B^N_{3,p+1}=\frac{Nu}{(N-1)(u-1)}\overset{*}{\gamma}{}^1_{3,\rho}+\frac{N}{(N-u)(u-1)}\overset{*}{\gamma}{}^u_{3,\rho}+\frac{u}{(N-1)(N-u)}\overset{*}{\gamma}{}^N_{3,\rho}.$$

Now, when N is even, $B^N_{3,p+1}$ becomes

$$\frac{N^2}{(N-1)(N-2)}\overset{*}{\gamma}{}^1_{3,p}+\frac{4}{N-2}\overset{*}{\gamma}{}^{N/2}_{3,\rho}+\frac{1}{N-1}\overset{*}{\gamma}{}^N_{3,\rho}\quad\text{for}\quad u=N/2$$

and

$$\frac{N+2}{N-1}\overset{*}{\gamma}{}^1_{3,\rho}+\frac{4}{N-2}\overset{*}{\gamma}{}^{(N/2)+1}_{3,\rho}+\frac{1}{N-1}\left(1+\frac{4}{N-2}\right)\overset{*}{\gamma}{}^N_{3,\rho}\quad\text{for}\quad u=(N/2)+1.$$

Then, the difference of both values of $B^N_{3,p+1}$ becomes

$$\frac{4}{(N-1)(N-2)}(\overset{*}{\gamma}{}^N_{3,\rho}-\overset{*}{\gamma}{}^1_{3,\rho})+\frac{4}{N-2}(\overset{*}{\gamma}{}^{(N/2)+1}_{3,\rho}-\overset{*}{\gamma}{}^{N/2}_{3,\rho}),$$

which is positive by (3.3). Consequently the formula for $u=N/2$ is more accurate than that for $u=(N/2)+1$. Thus, when N is even, the desired improving formula is given by (7.8) in which $u=N/2$. These improving formulas are tabulated to $N=5$ in Table 6.

Faculty of Science, Hiroshima University
and Fukuoka University of Liberal Arts.

Table 1. *The numbers* $\gamma^s_{n,\rho}$.

$n=1$

	$s \backslash \rho$	0	1	2	3	4	5	6
$\gamma^s_{1,\rho}$	1	-1	$\frac{1}{2}$	$\frac{1}{12}$	$\frac{1}{24}$	$\frac{19}{720}$	$\frac{27}{1440}$	$\frac{863}{60480}$
	2	-2	$\frac{4}{2}$	$-\frac{4}{12}$	0	$\frac{8}{720}$	$\frac{16}{1440}$	$\frac{592}{60480}$
	3	-3	$\frac{9}{2}$	$-\frac{27}{12}$	$\frac{9}{24}$	$\frac{27}{720}$	$\frac{27}{1440}$	$\frac{783}{60480}$
	4	-4	$\frac{16}{2}$	$-\frac{80}{12}$	$\frac{64}{24}$	$-\frac{224}{720}$	0	$\frac{512}{60480}$
	5	-5	$\frac{25}{2}$	$-\frac{175}{12}$	$\frac{225}{24}$	$-\frac{2125}{720}$	$\frac{475}{1440}$	$\frac{1375}{60480}$
$\alpha^1_{1,\rho}$		1	$\frac{1}{2}$	$\frac{5}{12}$	$\frac{9}{24}$	$\frac{251}{720}$	$\frac{475}{1440}$	$\frac{19087}{60480}$

$n=2$

	$s \backslash \rho$	0	1	2	3	4	5	6
$\gamma^s_{2,\rho}$	1	$\frac{1}{2}$	$-\frac{1}{6}$	$-\frac{1}{24}$	$-\frac{8}{360}$	$-\frac{21}{1440}$	$-\frac{107}{10080}$	$-\frac{995}{120960}$
	2	$\frac{4}{2}$	$-\frac{8}{6}$	0	$-\frac{16}{360}$	$-\frac{48}{1440}$	$-\frac{256}{10080}$	$-\frac{2432}{120960}$
	3	$\frac{9}{2}$	$-\frac{27}{6}$	$\frac{27}{24}$	$-\frac{54}{360}$	$-\frac{81}{1440}$	$-\frac{405}{10080}$	$-\frac{3807}{120960}$
	4	$\frac{16}{2}$	$-\frac{64}{6}$	$\frac{128}{24}$	$-\frac{512}{360}$	0	$-\frac{512}{10080}$	$-\frac{5120}{120960}$
	5	$\frac{25}{2}$	$-\frac{125}{6}$	$\frac{375}{24}$	$-\frac{2500}{360}$	$\frac{1875}{1440}$	$-\frac{1375}{10080}$	$-\frac{6875}{120960}$
$\alpha^1_{2,\rho}$		$\frac{1}{2}$	$\frac{1}{6}$	$\frac{3}{24}$	$\frac{38}{360}$	$\frac{135}{1440}$	$\frac{863}{10080}$	$\frac{9625}{120960}$

$n=3$

	$s \backslash \rho$	0	1	2	3	4	5	6
$\gamma^s_{3,\rho}$	1	$-\frac{1}{6}$	$\frac{1}{24}$	$\frac{3}{240}$	$\frac{5}{720}$	$\frac{94}{20160}$	$\frac{695}{201600}$	$\frac{9809}{3628800}$
	2	$-\frac{8}{6}$	$\frac{16}{24}$	$\frac{16}{240}$	$\frac{32}{720}$	$\frac{608}{20160}$	$\frac{4480}{201600}$	$\frac{62848}{3628800}$
	3	$-\frac{27}{6}$	$\frac{81}{24}$	$-\frac{81}{240}$	$\frac{81}{720}$	$\frac{1458}{20160}$	$\frac{10935}{201600}$	$\frac{155277}{3628800}$
	4	$-\frac{64}{6}$	$\frac{256}{24}$	$-\frac{768}{240}$	$\frac{512}{720}$	$\frac{2560}{20160}$	$\frac{20480}{201600}$	$\frac{290816}{3628800}$
	5	$-\frac{125}{6}$	$\frac{625}{24}$	$-\frac{3125}{240}$	$\frac{3125}{720}$	$-\frac{6250}{20160}$	$\frac{34375}{201600}$	$\frac{465625}{3628800}$
$\alpha^1_{3,\rho}$		$\frac{1}{6}$	$\frac{1}{24}$	$\frac{7}{240}$	$\frac{17}{720}$	$\frac{410}{20160}$	$\frac{3655}{201600}$	$\frac{59941}{3628800}$

Table 2. *The numbers* $\overset{*}{\gamma}{}^{s}_{n,\rho}$.

$n=1$

	$s \backslash \rho$	0	1	2	3	4	5	6
$\overset{*}{\gamma}{}^{s}_{1,\rho}$	1	1	$\frac{1}{2}$	$\frac{1}{12}$	$\frac{1}{24}$	$\frac{19}{720}$	$\frac{27}{1440}$	$\frac{863}{60480}$
	2	2	$\frac{4}{2}$	$\frac{6}{12}$	$\frac{2}{24}$	$\frac{30}{720}$	$\frac{38}{1440}$	$\frac{1134}{60480}$
	3	3	$\frac{9}{2}$	$\frac{29}{12}$	$\frac{11}{24}$	$\frac{49}{720}$	$\frac{49}{1440}$	$\frac{1325}{60480}$
	4	4	$\frac{16}{2}$	$\frac{82}{12}$	$\frac{66}{24}$	$\frac{300}{720}$	$\frac{76}{1440}$	$\frac{1596}{60480}$
	5	5	$\frac{25}{2}$	$\frac{177}{12}$	$\frac{227}{24}$	$\frac{2201}{720}$	$\frac{551}{1440}$	$\frac{2459}{60480}$

$n=2$

	$s \backslash \rho$	0	1	2	3	4	5	6
$\overset{*}{\gamma}{}^{s}_{2,\rho}$	1	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{24}$	$\frac{8}{360}$	$\frac{21}{1440}$	$\frac{107}{10080}$	$\frac{995}{120960}$
	2	$\frac{4}{2}$	$\frac{8}{6}$	$\frac{6}{24}$	$\frac{30}{360}$	$\frac{70}{1440}$	$\frac{336}{10080}$	$\frac{3010}{120960}$
	3	$\frac{9}{2}$	$\frac{27}{6}$	$\frac{37}{24}$	$\frac{98}{360}$	$\frac{147}{1440}$	$\frac{639}{10080}$	$\frac{5469}{120960}$
	4	$\frac{16}{2}$	$\frac{64}{6}$	$\frac{142}{24}$	$\frac{586}{360}$	$\frac{380}{1440}$	$\frac{1064}{10080}$	$\frac{8372}{120960}$
	5	$\frac{25}{2}$	$\frac{125}{6}$	$\frac{393}{24}$	$\frac{2604}{360}$	$\frac{2407}{1440}$	$\frac{2459}{10080}$	$\frac{12295}{120960}$

$n=3$

	$s \backslash \rho$	0	1	2	3	4	5	6
$\overset{*}{\gamma}{}^{s}_{3,\rho}$	1	$\frac{1}{6}$	$\frac{1}{24}$	$\frac{3}{240}$	$\frac{5}{720}$	$\frac{94}{20160}$	$\frac{695}{201600}$	$\frac{9809}{3628800}$
	2	$\frac{8}{6}$	$\frac{16}{24}$	$\frac{30}{240}$	$\frac{40}{720}$	$\frac{700}{20160}$	$\frac{4970}{201600}$	$\frac{68250}{3628800}$
	3	$\frac{27}{6}$	$\frac{81}{24}$	$\frac{207}{240}$	$\frac{147}{720}$	$\frac{2166}{20160}$	$\frac{14565}{201600}$	$\frac{194279}{3628800}$
	4	$\frac{64}{6}$	$\frac{256}{24}$	$\frac{1014}{240}$	$\frac{696}{720}$	$\frac{5320}{20160}$	$\frac{31220}{201600}$	$\frac{400260}{3628800}$
	5	$\frac{125}{6}$	$\frac{625}{24}$	$\frac{3531}{240}$	$\frac{3487}{720}$	$\frac{20514}{20160}$	$\frac{61475}{201600}$	$\frac{705149}{3628800}$

Table 3. *Extrapolation formulas for the equation of the second order.*

$$y_{r+1}=\sum_{s=0}^{N} l_s y_{r-s}+h^2\sum_{\rho=0}^{P} a_{2,\rho}^{N}\nabla^{\rho}f_r$$

N	$\sum l_s y_{r-s}$	ρ	0	1	2	3	4	5	6
1	$2y_r-y_{r-1}$	$a_{2,\rho}^{1}$	1	0	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{19}{240}$	$\frac{3}{40}$	$\frac{863}{12096}$
		$A_{2,\rho}^{1}$	1.00000	0.33333	0.16667	0.12778	0.10833	0.09623	0.08780
2	$\frac{1}{2}(3y_r-y_{r-2})$	$a_{2,\rho}^{2}$	$\frac{3}{2}$	$-\frac{1}{2}$	$\frac{1}{8}$	$\frac{1}{12}$	$\frac{37}{480}$	$\frac{7}{96}$	$\frac{2803}{40320}$
		$A_{2,\rho}^{2}$	1.50000	0.83333	0.25000	0.14722	0.11806	0.10228	0.09201
3	$\frac{1}{3}(4y_r-y_{r-3})$	$a_{2,\rho}^{3}$	2	$-\frac{4}{3}$	$\frac{1}{2}$	$\frac{1}{18}$	$\frac{3}{40}$	$\frac{13}{180}$	$\frac{2089}{30240}$
		$A_{2,\rho}^{3}$	2.00000	1.66667	0.63889	0.19630	0.12778	0.10675	0.09464
4	$\frac{1}{4}(5y_r-y_{r-4})$	$a_{2,\rho}^{4}$	$\frac{5}{2}$	$-\frac{5}{2}$	$\frac{35}{24}$	$-\frac{1}{4}$	$\frac{3}{32}$	$\frac{7}{96}$	$\frac{1669}{24192}$
		$A_{2,\rho}^{4}$	2.50000	2.83333	1.60417	0.51250	0.15972	0.11200	0.09688
5	$\frac{1}{5}(6y_r-y_{r-5})$	$a_{2,\rho}^{5}$	3	-4	$\frac{13}{4}$	$-\frac{77}{60}$	$\frac{17}{48}$	$\frac{7}{120}$	$\frac{275}{4032}$
		$A_{2,\rho}^{5}$	3.00000	4.33333	3.40000	1.55222	0.42806	0.13440	0.09990

Table 4. *Improving formulas for the equation of the second order.*

$$y_{r+1}=\sum_{s=1}^{N} l_s y_{r+1-s}+h^2\sum_{\rho=0}^{P} b_{2,\rho}^{N}\nabla^{\rho}f_{r+1}$$

N	$\sum l_s y_{r+1-s}$	ρ	0	1	2	3	4	5	6
2	$2y_r-y_{r-1}$	$b_{2,\rho}^{2}$	1	-1	$\frac{1}{12}$	0	$-\frac{1}{240}$	$-\frac{1}{240}$	$-\frac{221}{60480}$
		$B_{2,\rho}^{2}$	3.00000	1.66667	0.33333	0.12778	0.07778	0.05456	0.04134
		$\beta_{2,0}$	1.00000	0.00000	0.08333	0.08333	0.07916	0.07499	0.07134
3	$\frac{1}{2}(3y_r-y_{r-2})$	$b_{2,\rho}^{3}$	$\frac{3}{2}$	-2	$\frac{5}{8}$	$-\frac{1}{24}$	$-\frac{1}{160}$	$-\frac{1}{240}$	$-\frac{137}{40320}$
		$B_{2,\rho}^{3}$	3.00000	2.50000	0.83333	0.16944	0.07292	0.04762	0.03495
		$\beta_{2,0}$	1.50000	-0.50000	0.12500	0.08333	0.07708	0.07291	0.06951
4	$\frac{1}{3}(4y_r-y_{r-3})$	$b_{2,\rho}^{4}$	2	$-\frac{10}{3}$	$\frac{11}{6}$	$-\frac{4}{9}$	$\frac{7}{360}$	$-\frac{1}{360}$	$-\frac{19}{6048}$
		$B_{2,\rho}^{4}$	3.33333	3.77778	2.02778	0.57222	0.10741	0.04934	0.03404
		$\beta_{2,0}$	2.00000	-1.33333	0.50000	0.05556	0.07500	0.07222	0.06908
5	$\frac{1}{4}(5y_r-y_{r-4})$	$b_{2,\rho}^{5}$	$\frac{5}{2}$	-5	$\frac{95}{24}$	$-\frac{41}{24}$	$\frac{11}{32}$	$-\frac{1}{48}$	$-\frac{95}{24192}$
		$B_{2,\rho}^{5}$	3.75000	5.41667	4.14583	1.83611	0.43611	0.07426	0.03569
		$\beta_{2,0}$	2.50000	-2.50000	1.45833	-0.25000	0.09375	0.07292	0.06899

Table 5. *Extrapolation formulas for the equation of the third order.*

$$y_{r+1}=\sum_{s=0}^{N} l_s y_{r-s}+h^3 \sum_{\rho=0}^{P} a_{3,\rho}^{N} \nabla^{\rho} f_r$$

N	$\sum l_s y_{r-s}$	ρ	0	1	2	3	4	5	6
2	$3y_r-3y_{r-1}+y_{r-2}$	$a_{3,\rho}^{2}$	1	$-\frac{1}{2}$	0	0	$\frac{1}{240}$	$\frac{1}{160}$	$\frac{221}{30240}$
		$A_{3,\rho}^{2}$	*	0.83333	0.19167	0.10000	0.06905	0.05312	0.04344
3	$\frac{1}{3}(8y_r-6y_{r-1}+y_{r-3})$	$a_{3,\rho}^{3}$	$\frac{4}{3}$	-1	$\frac{1}{6}$	0	$\frac{1}{180}$	$\frac{1}{144}$	$\frac{139}{18144}$
		$A_{3,\rho}^{3}$	*	1.25000	0.34167	0.10556	0.06548	0.04911	0.03977
4	$\frac{1}{8}(15y_r-10y_{r-2}+3y_{r-4})$	$a_{3,\rho}^{4}$	$\frac{5}{2}$	$-\frac{25}{8}$	$\frac{21}{16}$	$-\frac{3}{16}$	$\frac{1}{96}$	$\frac{1}{128}$	$\frac{1963}{241920}$
		$A_{3,\rho}^{4}$	*	4.87500	1.76979	0.45556	0.16270	0.10702	0.08139
5	$\frac{1}{5}(9y_r-5y_{r-2}+y_{r-5})$	$a_{3,\rho}^{5}$	3	$-\frac{9}{2}$	$\frac{27}{10}$	$-\frac{4}{5}$	$\frac{9}{80}$	$\frac{1}{160}$	$\frac{103}{12600}$
		$A_{3,\rho}^{5}$	*	5.91667	3.09667	1.04778	0.25857	0.10377	0.07419

Table 6. *Improving formulas for the equation of the third order.*

$$y_{r+1}=\sum_{s=1}^{N} l_s y_{r+1-s}+h^3\sum_{\rho=0}^{P} b_{3,\rho}^{N}\nabla^{\rho} f_{r+1}$$

N	$\sum l_s y_{r+1-s}$	ρ	0	1	2	3	4	5	6
3	$3y_r-3y_{r-1}+y_{r-2}$	$b_{3,\rho}^{3}$	1	$-\frac{3}{2}$	$\frac{13}{20}$	0	$\frac{1}{240}$	$\frac{1}{480}$	$\frac{1}{945}$
		$B_{3,\rho}^{3}$	*	5.50000	1.27500	0.39167	0.22560	0.15655	0.11807
		$\beta_{3,0}$	1.00000	−0.50000	0.15000	0.15000	0.15417	0.15625	0.15731
4	$\frac{1}{3}(8y_r-6y_{r-1}+y_{r-3})$	$b_{3,\rho}^{4}$	$\frac{4}{3}$	$-\frac{7}{3}$	$\frac{7}{6}$	$-\frac{1}{6}$	$\frac{1}{180}$	$\frac{1}{720}$	$\frac{13}{18144}$
		$B_{3,\rho}^{4}$	*	5.00000	1.69167	0.45185	0.16984	0.11012	0.08159
		$\beta_{3,0}$	1.33333	−1.00000	0.16667	0.00000	0.00556	0.00695	0.00767
5	$\frac{1}{8}(15y_r-10y_{r-2}+3y_{r-4})$	$b_{3,\rho}^{5}$	$\frac{5}{2}$	$-\frac{45}{8}$	$\frac{71}{16}$	$-\frac{3}{2}$	$\frac{19}{96}$	$-\frac{1}{384}$	$\frac{73}{241920}$
		$B_{3,\rho}^{5}$	*	14.06250	6.61875	2.08438	0.52463	0.21112	0.14486
		$\beta_{3,0}$	2.50000	−3.12500	1.31250	−0.18750	0.01042	0.00782	0.00812