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Infinitesimal Deformation of Cycles

By

Minoru URABE

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§ 1. Introduction.

Given the system of differential equations

$$\frac{dx}{dt}=X(x, y), \quad \frac{dy}{dt}=Y(x, y). \tag{1.1}$$

To the system (1.1) corresponds a vector field $F=(X, Y)$ in the phase plane of the variables x and y. Then, how does the cycle[1] behave when the field F is deformed infinitesimally ?

T. Uno[2] and G. F. D. Duff[3] have studied the case where the deformation of the field F is a rotation. In this paper, assuming the analyticity of the function $X(x, y)$ and $Y(x, y)$, we have studied the general case and obtained the result that, *in the neighborhood of the stable or unstable cycle C for the field F, there always exists at least one cycle of the same stability as C for any infinitesimally deformed field of F.* For the semi-stable cycle, we have obtained a conclusion analogous to that which Duff has obtained for the deformation of rotation. Besides, for the continuum of cycles, we have obtained the results more general than those of Kryloff and Begoliuboff.[4]

In the end, we apply these results to the field depending on one parameter.

§ 2. Deformation of characteristics.

For simplicity, we assume that $X(x, y)$ and $Y(x, y)$ are integral. Let the characteristic C of (1.1) be $x=x(t)$, $y=y(t)$. Let the direction cosines of the tangent and the normal of C be (α, β) and (l, m) respectively, then, if we determine their positive senses so that the tangent and the normal have a positive orientation, we have:

$$\begin{cases} \alpha=X/\sqrt{X^2+Y^2}, & \beta=Y/\sqrt{X^2+Y^2}; \\ l=-Y/\sqrt{X^2+Y^2}, & m=X/\sqrt{X^2+Y^2}. \end{cases} \tag{2.1}$$

1) For simplicity, in this paper, we call the closed characteristics the cycles.

2) Toshio Uno, *On Some Systematic Method for Finding Limit Cycles*, Proc. 1st Japan Nat. Congress f. Appl. Mech. (1951), 513-516.
do., *On the Curves Defined by Some Differential Equations*, Math. Japonicae vol. II, No. 3 (1952), 119-126.

3) G. F. D. Duff, *Limit-cycles and roted vector fields*, Annals of Math., Vol. 57 (1953), 15-31.

4) N. Kryloff and N. Bogoliuboff, *Introduction to non-linear mechanics* (1949).

The infinitesimally deformed field $F_1=(X_1, Y_1)$ *of* $F=(X, Y)$ is written as follows:

$$
(2.2) \qquad \begin{cases} X_1=X+\varepsilon H(x, y, \varepsilon)=X+\varepsilon H_1(x, y)+\varepsilon^2 H_2(x, y)+\cdots\cdots, \\ Y_1=Y+\varepsilon K(x, y, \varepsilon)=Y+\varepsilon K_1(x, y)+\varepsilon^2 K_2(x, y)+\cdots\cdots. \end{cases}
$$

Here we assume that *$H(x, y, \varepsilon)$ and $K(x, y, \varepsilon)$ are integral with regard to x and y, and are analytic with regard to ε for sufficiently small* $|\varepsilon|$. Let the characteristic for F_1 lying in the neighborhood of C be C_1: $x=x_1(\tau)$, $y=y_1(\tau)$. Then the coordinates of any point of C_1 are expressed as follows:

$$
(2.3) \qquad x_1=x+\rho l, \qquad y_1=y+\rho m,
$$

where ρ is a normal distance from the point (x, y) of C. By (2.3), to a point (x, y) at time t of C, corresponds the point (x_1, y_1) at time τ of C_1. Let the relation between the corresponding times be

$$
(2.4) \qquad \tau=\tau(t).
$$

For C_1, it is valid that

$$
(2.5) \qquad \frac{dx_1}{d\tau}=X_1(x_1, y_1), \qquad \frac{dy_1}{d\tau}=Y_1(x_1, y_1).
$$

Now, from (2.1), it is easily seen that

$$
(2.6) \qquad \frac{dl}{dt}=\kappa X, \qquad \frac{dm}{dt}=\kappa Y,
$$

where

$$
(2.7) \qquad \kappa=\frac{Y\dfrac{dX}{dt}-X\dfrac{dY}{dt}}{(X^2+Y^2)^{3/2}}.
$$

Here the operation d/dt means the differentiation along C, namely $X\dfrac{\partial}{\partial x}+Y\dfrac{\partial}{\partial y}$. Then, differentiating (2.3) with regard to t, from (1.1), (2.4) and (2.5), we have:

$$
(2.8) \qquad \begin{cases} X+\left(\dfrac{d\rho}{dt}\, l+\rho\kappa X\right)=X_1\dfrac{d\tau}{dt}, \\ Y+\left(\dfrac{d\rho}{dt}\, m+\rho\kappa Y\right)=Y_1\dfrac{d\tau}{dt}. \end{cases}
$$

Eliminating $d\tau/dt$, we have:

$$
(2.9) \qquad \frac{d\rho}{dt}\,\frac{1}{\sqrt{X^2+Y^2}}\,(XX_1+YY_1)=(1+\rho\kappa)\,(XY_1-YX_1).
$$

Now, put $D\equiv l\dfrac{\partial}{\partial x}+m\dfrac{\partial}{\partial y}$, then, from (2.3), it follows that

$$
(2.10) \qquad X_1=X+\rho DX+\varepsilon H_1+\cdots\cdots, \quad Y_1=Y+\rho DY+\varepsilon K_1+\cdots\cdots,
$$

where the unwritten terms are those of the second and higher orders with regard to ρ and ε. Substituting (2.10) into (2.9), we have:

$$(2.11)\qquad \frac{1}{\sqrt{X^2+Y^2}}\frac{d\rho}{dt}\left[(X^2+Y^2)+\rho(XDX+YDY)+\varepsilon(XH_1+YK_1)+\cdots\cdots\right]$$
$$=(1+\rho\kappa)\left[\rho(XDY-YDX)+\varepsilon(XK_1-YH_1)+\cdots\cdots\right].$$

From the forms of both sides, we see that the solution $\rho(t;c,\varepsilon)$ of (2.11) such that $\rho(t_0;c,\varepsilon)=c$ is analytic with regard to c and ε for sufficiently small $|c|$ and $|\varepsilon|$. Consequently $\rho(t;c,\varepsilon)$ can be expanded as follows:

$$\rho=\rho(t;c,\varepsilon)=\rho_0(t)+\rho_1(t;c,\varepsilon)+\rho_2(t;c,\varepsilon)+\cdots+\rho_m(t;c,\varepsilon)+\cdots,$$

where $\rho_m(t;c,\varepsilon)$ is the homogeneous polynomial of m-th degree with regard to c and ε. However, from the form of (2.11), it is evident that $\rho(t;0,0)=0$, consequently $\rho_0(t)=0$. Thus the solution $\rho(t;c,\varepsilon)$ of (2.11) is expressed as follows:

$$(2.12)\qquad \rho=\rho(t;c,\varepsilon)=\rho_1(t;c,\varepsilon)+\rho_2(t;c,\varepsilon)+\cdots+\rho_m(t;c,\varepsilon)+\cdots.$$

Now, from the condition that $\rho(t_0;c,\varepsilon)=c$, it follows that

$$(2.13)\qquad \rho_1(t_0;c,\varepsilon)=c,\quad \rho_2(t_0;c,\varepsilon)=\cdots=\rho_m(t_0;c,\varepsilon)=\cdots=0.$$

Substitute (2.12) into (2.11), then, comparing the terms of the same degrees with regard to (c,ε), we have:

$$(2.14)\qquad \begin{cases} \sqrt{X^2+Y^2}\,\dfrac{d\rho_1}{dt}=(XDY-YDX)\rho_1+\varepsilon(XK_1-YH_1), \\ \sqrt{X^2+Y^2}\,\dfrac{d\rho_m}{dt}=(XDY-YDX)\rho_m+R_m(\rho_1,\rho_2,\cdots,\rho_{m-1};\varepsilon),\ (m\geqq 2) \end{cases}$$

where R_m is a polynomial of $\rho_1, \rho_2, \cdots\cdots, \rho_{m-1}$ and their derivatives. Now, from (2.1), it is easily seen that

$$XDY-YDX=\sqrt{X^2+Y^2}\left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}\right)-\frac{d}{dt}\sqrt{X^2+Y^2}.$$

Therefore, if we put

$$(2.15)\qquad h(t)=\int_{t_0}^{t}\left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}\right)dt,$$

then, integrating (2.14), by means of (2.13), we have:

$$(2.16)\qquad \begin{cases} \rho_1 = \dfrac{\varepsilon}{\sqrt{X^2+Y^2}} e^h \displaystyle\int_{t_0}^{t} e^{-h}(XK_1 - YH_1)dt + \dfrac{\sqrt{X_0^2+Y_0^2}}{\sqrt{X^2+Y^2}} e^h c, \\ \rho_m = \dfrac{1}{\sqrt{X^2+Y^2}} e^h \displaystyle\int_{t_0}^{t} e^{-h} R_m dt, \qquad (m \geqq 2) \end{cases}$$

where $X_0 = X\{x(t_0),\ y(t_0)\}$ and $Y_0 = Y\{x(t_0),\ y(t_0)\}$.

Next we seek for $d\tau/dt$. Adding the squares of both sides of (2.8), by means of (2.12), we have:

$$(X^2+Y^2)(1+2\kappa\rho_1+\cdots) = \left(\frac{d\tau}{dt}\right)^2 \left[(X^2+Y^2)+2\{\rho_1(XDX+YDY)+\varepsilon(XH_1+YK_1)\}+\cdots\right].$$

Consequently it follows that

$$\frac{d\tau}{dt} = 1 + \left\{\left(\kappa - \frac{XDX+YDY}{X^2+Y^2}\right)\rho_1 - \frac{XH_1+YK_1}{X^2+Y^2}\varepsilon\right\} + \cdots\cdots.$$

Substituting (2.1) and (2.7) into the right-hand side, we have:

$$(2.17)\qquad \frac{d\tau}{dt} = 1 + \left\{\frac{2XY\left(\dfrac{\partial X}{\partial x} - \dfrac{\partial Y}{\partial y}\right) + (Y^2 - X^2)\left(\dfrac{\partial X}{\partial y} + \dfrac{\partial Y}{\partial x}\right)}{(X^2+Y^2)^{3/2}}\rho_1 - \frac{XH_1+YK_1}{X^2+Y^2}\varepsilon\right\} + \cdots.$$

If we take τ so that $\tau(t_0) = t_0$, then, integrating (2.17), we have:

$$(2.18)\qquad \tau = t + \int_{t_0}^{t}\left[\left\{\frac{2XY\left(\dfrac{\partial X}{\partial x} - \dfrac{\partial Y}{\partial y}\right) + (Y^2 - X^2)\left(\dfrac{\partial X}{\partial y} + \dfrac{\partial Y}{\partial x}\right)}{(X^2+Y^2)^{3/2}}\rho_1 - \frac{XH_1+YK_1}{X^2+Y^2}\varepsilon\right\} + \cdots\cdots\right]dt.$$

§ 3. Stability of cycles.

In this paragraph, making use of the results of § 2, we investigate the stability of the cycle. Let the period of the cycle C for the field F be T. Then, for any characteristic for the field F lying in the neighborhood of C, from (2.12), it is valid that

$$(3.1)\qquad \rho \equiv \rho(t; c) = \rho(t; c, 0) = c\rho_1(t) + c^2\rho_2(t) + \cdots + c^m\rho_m(t) + \cdots,$$

where

$$(3.2)\qquad c^m\rho_m(t) = \rho_m(t; c, 0).$$

Consequently, from (2.13), it is valid that

$$(3.3)\qquad \rho_1(t_0) = 1, \quad \rho_2(t_0) = \cdots = \rho_m(t_0) = \cdots = 0.$$

From (2.16), it follows that

$$(3.4)\qquad \begin{cases} \rho_1(t)=\dfrac{\sqrt{X_0^2+Y_0^2}}{\sqrt{X^2+Y^2}}e^h, \\ \\ \rho_m(t)=\dfrac{1}{\sqrt{X^2+Y^2}}e^h\displaystyle\int_{t_0}^{t}e^{-h}R'_m dt, \qquad (m\geqq 2)\ . \end{cases}$$

where $R'_m=R_m(c\rho_1,\ c^2\rho_2\cdots\cdots,\ c^{m-1}\rho_{m-1}\,;0)/c^m$. Put

$$(3.5)\qquad h_0=h(t_0+T),$$

then, from (2.15), it is easily seen that

$$(3.6)\qquad h(t+T)=h(t)+h_0.$$

Now, so long as $c\neq 0$, from (3.1) and (3.3), it follows that

$$(3.7)\qquad \rho(t_0+T\,;c)/\rho(t_0\,;c)=\rho_1(t_0+T)+c\rho_2(t_0+T)+\cdots+c^{m-1}\rho_m(t_0+T)+\cdots.$$

From (3.4), by means of (3.5), it is valid that

$$(3.8)\qquad \rho_1(t_0+T)=e^{h(t_0+T)}=e^{h_0}.$$

Consequently, *if* $h_0\neq 0$, from (3.7), it follows that:

when $h_0<0$, $0<\rho(t_0+T\,;c)/\rho(t_0\,;c)<1$, *namely C is stable,*

when $h_0>0$, $1<\rho(t_0+T\,;c)/\rho(t_0\,;c)$, *namely C is unstable.*

These are the conditions of orbital stability of Poincaré.

If $h_0=0$, from (3.8), $\rho_1(t_0+T)=1$.

When it is valid that

$$(3.9)\qquad \rho_2(t_0+T)=\cdots=\rho_{m-1}(t_0+T)=0,\quad \rho_m(t_0+T)\neq 0,$$

for $c\neq 0$, from (3.7), it follows that

$$\frac{\rho(t_0+T\,;c)}{\rho(t_0\,;c)}=1+c^{m-1}\rho_m(t_0+T)+\cdots\cdots.$$

Consequently we have the conditions of stability as follows:

m	*sign of* $\rho_m(t_0+T)$	*stability of* C	
odd	$-$	*stable*	
	$+$	*unstable*	
even	$-$	*stable for* $c>0$	*semi-stable*
		unstable for $c<0$	
	$+$	*unstable for* $c>0$	
		stable for $c<0$	

When it is valid that

$$\rho_2(t_0+T)=\cdots=\rho_m(t_0+T)=\cdots=0, \tag{3.10}$$

from (3.1), it follows that, for any c, $\rho(t_0+T\,;\,c)=\rho(t_0\,;\,c)$. In this case, in the neighborhood of C, *there exists a continuum of cycles.*

When (3.10) is valid, let the period of the cycle C_1 lying in the neighborhood of C be T_1. Then, from (2.18) and (3.4) it follows that

$$T_1=T+\int_{t_0}^{t_0+T}\left[c\frac{2XY\left(\dfrac{\partial X}{\partial x}-\dfrac{\partial Y}{\partial y}\right)+(Y^2-X^2)\left(\dfrac{\partial X}{\partial y}+\dfrac{\partial Y}{\partial x}\right)}{(X^2+Y^2)^2}\sqrt{X_0^2+Y_0^2}e^h+\cdots\right]dt. \tag{3.11}$$

From this, it is seen that the period T_1 considered as the function of c is analytic wtth regard to c.

§ 4. Deformation of cycles.

Let the cycle for the field F be C and its period be T. Then, for any characteristic C_1 for the deformed field F_1 lying in the neighborhood of C, from (2.12), it is vaild that

$$\rho(t_0+T\,;\,c,\,\varepsilon)=\rho_1(t_0+T\,;\,c,\,\varepsilon)+\cdots\cdots+\rho_m(t_0+T\,;\,c,\,\varepsilon)+\cdots\cdots, \tag{4.1}$$

where, from (2.16),

$$\rho_1(t_0+T\,;\,c,\,\varepsilon)=\frac{\varepsilon}{\sqrt{X_0^2+Y_0^2}}e^{h_0}\int_{t_0}^{t_0+T}e^{-h}(XK_1-YH_1)dt+e^{h_0}c. \tag{4.2}$$

We consider the expression as follows:

$$\begin{aligned}\Phi(c,\,\varepsilon)\equiv\rho(t_0+T\,;\,c,\,\varepsilon)-\rho(t_0\,;\,c,\,\varepsilon)=c(e^{h_0}-1)\\ +\frac{\varepsilon}{\sqrt{X_0^2+Y_0^2}}e^{h_0}I+\rho_2(t_0+T\,;\,c,\,\varepsilon)+\cdots+\rho_m(t_0+T\,;\,c,\,\varepsilon)+\cdots,\end{aligned} \tag{4.3}$$

where

$$I=\int_{t_0}^{t_0+T} e^{-h}(XK_1-YH_1)\,dt. \tag{4.4}$$

Then, whether or not there exists a closed C_1, namely, a cycle for F_1 lying in the neighborhood of C, is decided by whether or not there exists a real root c of small absolute value of the equation $\Phi\ (c, \varepsilon)=0$ for sufficiently small $|\varepsilon|$.

I. The case where $h_0 \neq 0$. In this case, the equation $\Phi\ (c, \varepsilon)=0$ has unique real solution which follows:

$$c=c_0=\varepsilon\,\frac{e^{h_0}}{1-e^{h_0}}\,\frac{1}{\sqrt{X_0^2+Y_0^2}}\,I+\cdots\cdots. \tag{4.5}$$

Let the charecterisitc corresponding to c_0 be C', then evidently C' is a cycle. Now, when $h_0<0$, $e^{h_0}-1<0$, consequently $\Phi\,(c, \varepsilon)$ is monotone decreasing with regard to c for sufficiently small $|c|$ and $|\varepsilon|$. Therefore, for $c<c_0$, $\rho(t_0+T\,;\,c, \varepsilon)>c$ and for $c>c_0$, $\rho(t_0+T\,;\,c, \varepsilon)<c$. Consequently C' is stable. In like manner, when $h_0>0$, C' is unstable. Then, by § 3, the stability of C' is the same as that of C. Thus we see that, *when* $h_0 \neq 0$, *for the sufficiently slightly deformed field* F_1, *there exists a unique cycle lying in the neighborhood of* C, *which has the same stability as that of* C.

II. The case where $h_0=0$ and $\rho_2(t_0+T)=\cdots\cdots=\rho_{m-1}(t_0+T)=0$, $\rho_m(t_0+T)\neq 0$. Put

$$a_m=\rho_m(t_0+T), \tag{4.6}$$

then, from (3.2), $\rho_m(t_0+T\,;\,c, 0)=a_m c^m$. Therefore, by means of Weierstrass's preparation theorem[1], for sufficiently small $|c|$ and $|\varepsilon|$, $\Phi(c,\varepsilon)$ is expressed as follows:

$$\Phi(c, \varepsilon)=a_m\{c^m+k(\varepsilon)c^{m-1}+\cdots+l(\varepsilon)\}\,\{1+\Psi(c, \varepsilon)\}, \tag{4.7}$$

where $k(\varepsilon)$, $\cdots\cdots$, $l(\varepsilon)$ are real analytic functions vanishing with ε and $\Psi\,(c, \varepsilon)$ is a real analytic function vanishing with c and ε. Then the equation $\Phi(c, \varepsilon)=0$ is equivalent to the equation as follows:

$$\Phi_1(c,\varepsilon)\equiv c^m+k(\varepsilon)\,c^{m-1}+\cdots+l(\varepsilon)=0. \tag{4.8}$$

1°. The case where m is odd. In this case, it is easily seen that there exists at least one real root c_0 of (4.8) such that, for c sufficiently near c_0, when $c<c_0$, $\Phi_1<0$ and when $c>c_0$, $\Phi_1>0$. Let the characteristic corresponding to c_0 de C'. Then, since $\Phi(c, \varepsilon)$ and $a_m\Phi_1(c, \varepsilon)$ are of the same sign for sufficiently small $|c|$ and $|\varepsilon|$, it is seen that, for sufficiently small $|\varepsilon|$, C' is stable or unstable according as $a_m<0$ or >0. Now, by § 3, C is stable or unstable according as $a_m<0$ or >0. Thus we see that, *in the*

1) W. F. Osgood, *Lehrbuch der Funktionentheorie*, II_1, (1929), 86-92.

case considered here, for the sufficiently slightly deformed field F_1, there exists at least one cycle lying in the neighborhood of C, which has the same stability as that of C.

2°. The case where m is even. Comparing (4.7) with (4.3), we have:

$$l(\varepsilon)=\frac{I}{a_m\sqrt{X_0^2+Y_0^2}}\varepsilon+\cdots\cdots, \tag{4.9}$$

where the unwritten terms are those of the second and higher orders with regard to ε.

First we consider the case where $I\neq 0$. If we put

$$h_1(t)=\int_{t_1}^{t}\left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}\right)dt, \tag{4.10}$$

from (2.15), $h_1(t)=h(t)-h(t_1)$. Then it follows that the integral I_1 of (4.4) obtained by substituting t_1 for t_0 becomes

$$I_1=e^{h(t_1)}\int_{t_1}^{t_1+T} e^{-h}(XK_1-YH_1)dt. \tag{4.11}$$

Now, from (3.6), when $h_0=0$, $h(t)$ is periodic. Then, in our case, from the periodicity of the integrand of the above integral, we have:

$$I_1=e^{h(t_1)}I. \tag{4.12}$$

From this, it is seen that, if $I\neq 0$ for any one point of C, then $I\neq 0$ for all points of C.

Put

$$\sigma=-\frac{\varepsilon}{a_m\sqrt{X_0^2+Y_0^2}}I. \tag{4.13}$$

then, $\Phi_1(c, \varepsilon)$ is written as follows:

$$\Phi_1=c^m+k_1(\sigma)c^{m-1}+\cdots+l_1(\sigma), \tag{4.14}$$

where $k_1(\sigma)=k(\varepsilon),\cdots\cdots$, $l_1(\sigma)=l(\varepsilon)$. By (4.9) and (4.13), $l_1(\sigma)$ is of the form as follows:

$$l_1(\sigma)=-\sigma+\cdots\cdots. \tag{4.15}$$

Then, by Newton's polygon method [1], it is seen that the roots of $\Phi_1=0$ are of the forms as follows:

$$c=\omega\sigma^{1/m}[1+v(\omega\sigma^{1/m})], \tag{4.16}$$

1) G. A. Bliss, *Algebraic Functions* (1932), 35–40.

where ω is any m-th root of unity and v is a real analytic function vanishing with the argument. For sufficiently small $|\sigma|$, in order that the value of c be real, it is necessary and sufficient that $\omega\sigma^{1/m}$ is real. Thus, from (4.13), we see that, when and only when $\varepsilon I/a_m<0$, the equation $\Phi_1=0$ has real roots. In this case, the number of the real roots is two, and one of the roots is positive —— which we denote by c_+ —— and the other is negative —— which we denote by c_-. Let the characteristics corresponding to c_+ and c_- be C'_+ and C'_- respectively.

If C'_+ intersects with C in a point P for $t=t_1$, then, in P, the equation $\Phi(c, \varepsilon)=0$ must have a zero root. Namely, from (4.3), making use of the notations given by (4.10) and (4.11), it must be that

$$(4.17) \qquad \frac{\varepsilon}{\sqrt{X_1^2+Y_1^2}}I_1+\rho'_2(t_1+T\,;0,\varepsilon)+\cdots+\rho'_m(t_1+T\,;0,\varepsilon)+\cdots=0.$$ [1]

Here $X_1=X\{x(t_1), y(t_1)\}$ and $Y_1=Y\{x(t_1), y(t_1)\}$, and $\rho'_2, \cdots, \rho'_m, \cdots\cdots$ are the functions $\rho_2, \cdots, \rho_m, \cdots\cdots$ obtained by substituting t_1 for t_0. When $|\varepsilon|$ is sufficiently small, in order that (4.17) be valid, it is necessary that $|I_1|$ is arbitrarily small. This contradicts the assumption that $I\neq 0$. Thus we see that, for sufficiently small $|\varepsilon|$, C'_+ can not intersect with C. In the same manner we see that C'_- also can not intersect with C.

From (4.14), $\Phi_1=(c-c_-)(c-c_+)\Phi_0(c, \varepsilon)$ and, for sufficiently small $|c|$ and $|\varepsilon|$, $\Phi_0(c,\varepsilon)>0$ except for $c=\varepsilon=0$. Consequently it follows that,

when $c<c_-$, $\quad \Phi_1>0$;
when $c_-<c<c_+$, $\quad \Phi_1<0$;
when $c>c_+$, $\quad \Phi_1>0$.

Therefore, from (4.7), it is seen that, when $a_m<0$, C'_- is unstable and C'_+ is stable, and when $a_m>0$, the stability is reversed. Comparing with the stability of C by means of the results of § 3, we see that C'_+ and C'_- be respectively in the opposite sides of C and moreover have the same stability as that of the sides in which they lie.

Thus, summarizing the results, we see that, *for the sufficiently slightly deformed field F_1, when $\varepsilon I/a_m>0$, in the neighborhood of C, there exists no cycle and when $\varepsilon I/a_m<0$, in the neighborhood of C, there exists one and only one cycle in each side of C having the same stability as that of the side containing that cycle.*

In addition, from the above discussions, we see that, *when m is odd and $I\neq 0$*, $\Phi_1=0$ has only one real root c, consequently, *in the neighborhood of the cycle C, there exists only one cycle which lies in one side of C and has the same stability as that of C.*

Next we consider the case where $I=0$. In this case, it may happen that there exists no real root of the equation (4.8). In this case, in the neighborhood of the cycle C, there exists no cycle for the deformed field F_1. When there exist the real roots of the equation (4.8), let them be $c_1, c_2, \cdots c_k$. Since m is even, k must be even. From (4.7),

1) For, $h_1(t_1+T)=h(t_1+T)-h(t_1)=h_0$ because of (3.6).

it follows that

$$\Phi(c, \varepsilon)=a_m(c-c_1)(c-c_2)\cdots(c-c_k)\,\Phi_0(c, \varepsilon)(1+\Psi), \tag{4.18}$$

where $\Phi_0(c, \varepsilon)>0$ for sufficiently small $|c|$ and $|\varepsilon|$ except for $c=\varepsilon=0$. From the form of (4.18), we see that, when $c_1, c_2, \cdots\cdots, c_k$ are all distinct, there exist k cycles and these k cycles are arranged according to the magnitude of $c_1, c_2, \cdots\cdots, c_k$ and have the alternating absolute stability.[1] When some of the roots coincide with each other, the corresponding cycle has the absolute or half stability according as the number of coincident roots is odd or even.

III. The case where $h_0=0$ and $\rho_2(t_0+T)=\cdots\cdots=\rho_m(t_0+T)=\cdots\cdots=0$. In this case, from (4.3), $\Phi(c, \varepsilon)$ is written as follows:

$$\Phi(c, \varepsilon)=\varepsilon\left[\frac{1}{\sqrt{X_0^2+Y_0^2}}I+\sigma_1(t_0+T\,;c, \varepsilon)+\cdots\cdots+\sigma_{m-1}(t_0+T\,;c, \varepsilon)+\cdots\right], \tag{4.19}$$

where

$$\sigma_{m-1}(t_0+T\,;c, \varepsilon)=\rho_m(t_0+T\,;c, \varepsilon)/\varepsilon. \qquad (m\geqq 2) \tag{4.20}$$

Then, when $I\neq 0$, for sufficiently small $|c|$ and $|\varepsilon|$, $\Phi(c, \varepsilon)\neq 0$, namely in the neighborhood of C, there exists no cycle for the deformed field F_1. When $I=0$, the equation $\Phi(c,\varepsilon)=0$ is reduced to the equation as follows:

$$\sigma_1(t_0+T\,;c, \varepsilon)+\cdots\cdots+\sigma_{m-1}(t_0+T\,;c, \varepsilon)+\cdots\cdots=0. \tag{4.21}$$

Therefore, foı the cycle such that $I=0$, whether or not there exists a cycle for the deformed field F_1 is decided by whether or not the equation (4.21) has a real root.

Now, in our case, by § 3, for the field F, in the neighborhood of C, there exists a continuum of cycles. Making use of this fact, in the next paragraph, we shall seek for cycles for the deformed field F_1 by the method somewhat different from this paragraph.

We shall summarize the above results:

For sufficiently slightly deformed field F_1, in the neighborhood of the cycle C for the field F,

(i) *in the case where C has the absolute stability, there exists at least one cycle C' having the same stability as that of C; moreover C' is unique when $h_0\neq 0$ or $I\neq 0$, and specially when $I\neq 0$, C' does not intersect with C;*

(ii) *in the case where C has the half stability,*

(a) *when $\varepsilon I/a_m<0$, there exists one and only one cycle lying in each side of C having the same stability as that of the side containing that cycle;*

(b) *when $\varepsilon I/a_m>0$, there exists no cycle;*

(c) *when $I=0$, the existence of the cycles is decided by that of the real roots*

1) We say that a cycle C has the absolute stability when C is stable or unstable, and that C has the half stability when C is semi-stable.

of the equation (4.8) ;

(iii) *in the case where there exists a continuum of cycles in the neighborhood of* C,

(a) *when* $I \neq 0$, *there exists no cycle ;*

(b) *when* $I = 0$, *the existence of the cycles is decided by that of the real roots of the equation* (4.21).

In the case where the deformation of the field is a rotation as T. Uno[1] and G. F. D. Duff[2] have assumed, $XK_1 - YH_1$ is of the definite sign, consequently $I \neq 0$. Then, from the above general results, it is evident that all the results of G. F. D. Duff are valid.

Remark. The above results are deduced under the assumption that the functions $X(x, y)$, $Y(x, y)$, $X_1(x, y, \varepsilon)$ and $Y_1(x, y, \varepsilon)$ are analytic with regard to x, y and ε. If we dismiss the assumption of the analyticity and merely assume the continuity of the above functions and the uniqueness of the solutions of the equations (1.1) and (2.5), then, in what case does the cycle C' for F_1 exist in the neighborhood of C for F? In the case (i), namely when C has the absolute stability, it can be easily proved that there exists certainly at least one cycle C'[3]. However, in the other cases, the problem to determine the case where the cycle C' exists seems to be considerably difficult. As to the stability of C', even in the case (i), as seen from the proof of the existence of C', the result under the assumption of the analyticity does not prevail. When C' is unique, it is readily seen that C' has the same absolute stability as that of C but, in the general case, C' has not necessarily the absolute stability.

§ 5. Deformation of continuum of cycles.

We assume that, for the field F, in the neighborhood of the cycle C, there exists a continuum of cycles. Then, by (2.3), the cycles lying in the neighborhood of C are given by the equations as follows ;

$$x_1 = x + \rho(t\,;c)l, \quad y_1 = y + \rho(t\,;c)m. \tag{5.1}$$

Making use of the letter a instead of c, we express the equations (5.1) as follows :

$$x = x(t\,;a), \quad y = y(t\,;a), \tag{5.2}$$

and by $C(a)$ we denote the cycle defined by (5.2). Then, evidently, for sufficiently small $|a|$, $x(t\,;a)$ and $y(t\,;a)$ are analytic with regard to a. Let the period of $C(a)$ be $T(a)$, then, from (3.11), $T(a)$ is also analytic with regard to a. Consequently, if we write the

1) T. Uno, ibid.

2) G. F. D. Duff, ibid.

3) Take a small segment L crossing C. Then any characteristic for the field F passing through any point on L crosses L monotonely approaching C when the characteristic is followed as t increases or decreases according as C is stable or unstable. For sufficiently small $|\varepsilon|$, in the neighborhood of C, the characteristics for the field F_1 behave in the same manner as those for the field F. Then, from the continuity of the solution, it is seen that there exists at least one cycle for F_1 in the neighborhood of C.

function $h(t)$ for $C(a)$ as $h(t\,;a)$, $h(t\,;a)$ becomes analytic with regard to a. If we write the function $\rho(t\,;c,\varepsilon)$ for $C(a)$ as $\rho(t\,;c,\varepsilon,a)$, then, from the analyticity of the equation (2.9), it is evident that $\rho(t\,;c,\varepsilon,a)$ is also analytic with regard to a. Let the integral (4.4) for $C(a)$ be $I(a)$, then it is readily seen that $I(a)$ is also analytic with regard to a.

Now, by § 4, for the deformed field F_1, in order that there exist a cycle in the neighborhood of $C(a)$, it is necessary that $I(a)=0$. Let any one of the real roots of the equation $I(a)=0$ be a_0. Put $a=a'+a_0$, then $x(t\,;a)$ and $y(t\,;a)$ become the analytic functions of a'. If we express the above various functions of a as the functions of a', then they become the analytic functions of a' and moreover $I(a')=0$ for $a'=0$. Thus, without loss of generality, we may assume that $a_0=0$, namely $I(0)=0$. In the following, we assume this.

By (2.12), the function $\rho(t\,;c,\varepsilon,a)$ is expanded as follows:

$$\rho(t\,;c,\varepsilon,a)=\rho_1(t\,;c,\varepsilon,a)+\rho_2(t\,;c,\varepsilon,a)+\cdots\cdots+\rho_m(t\,;c,\varepsilon,a)+\cdots. \tag{5.3}$$

Here evidently $\rho_m(t\,;c,\varepsilon,a)$ is analytic with regard to a. Now, from our assumption, for any a, in the neighborhood of $C(a)$, there exists a continuum of cycles, consequently for any $C(a)$, it is valid that

$$\left\{\begin{array}{l} h_0(a)\equiv h(t_0+T(a)\,;a)=0, \\ \rho_2(t_0+T(a)\,;c,0,a)=\cdots=\rho_m(t_0+T(a)\,;c,0,a)=\cdots=0. \end{array}\right. \tag{5.4}$$

Put

$$J(a)=\frac{1}{\sqrt{X_0^2+Y_0^2}}\,I(a), \tag{5.5}$$

where $X_0=X\{x(t_0\,;a),\ y(t_0\,;a)\}$ and $Y_0=Y\{x(t_0\,;a),\ y(t_0\,;a)\}$. Then, from (4.3) and (4.19), we have:

$$\begin{aligned} \Phi(c,\varepsilon,a)&\equiv\rho(t_0+T(a)\,;c,\varepsilon,a)-c \\ &=\varepsilon J(a)+\rho_2(t_0+T(a)\,;c,\varepsilon,a)+\cdots+\rho_m(t_0+T(a)\,;c,\varepsilon,a)+\cdots \\ &=\varepsilon[J(a)+\sigma_1(t_0+T(a)\,;c,\varepsilon,a)+\cdots+\sigma_{m-1}(t_0+T(a)\,;c,\varepsilon,a)+\cdots]. \end{aligned} \tag{5.6}$$

If there exists a cycle C' for the field F_1 in the neighborhood of $C(0)$, let the point of intersection of C' with the normal of $C(0)$ at the point corresponding to $t=t_0$ be Q. Then, by our assumption, there exists a cycle $C(a_1)$ for the field F passing through Q. With regard to $C(a_1)$, $c=0$ and $\rho(t_0+T(a_1)\,;0,\varepsilon,a_1)=0$, namely $\Phi(0,\varepsilon,a_1)=0$. Conversely, if there exists a number a_1 of small absolute value such that $\Phi(0,\varepsilon,a_1)=0$, then, corresponding to $C(a_1)$, there exists a cycle C' for the field F_1 passing through the point of intersection of $C(a_1)$ with the normal of $C(0)$ at the point corresponding to $t=t_0$. Thus, from (5.6), we see that the existence of the cycles for the field F_1 in the neighborhood of $C(0)$ is decided by the existence of the real roots of the equations as follows:

(5.7) $$\Phi(0, \varepsilon, a)=\varepsilon[J(a)+\varepsilon\pi_1(a)+\cdots+\varepsilon^{m-1}\pi_{m-1}(a)+\cdots]=0,$$

where, for $m\geqq 2$,

(5.8) $$\pi_{m-1}(a)=\sigma_{m-1}(t_0+T(a)\,;\,0,\,\varepsilon,\,a)/\varepsilon^{m-1}=\rho_m(t_0+T(a)\,;\,0,\,\varepsilon,\,a)/\varepsilon^m.$$

Here, from (5.5), $J(0)=0$.

I. The case where $J(a)\not\equiv 0$.

From the analyticity of $I(a)$ results the analyticity of $J(a)$. Therefore we assume that

$$J(0)=J'(0)=\cdots=J^{(p-1)}(0)=0,\quad J^{(p)}(0)\neq 0.$$

As in § 4, we have:

1° *when p is odd, there exists at least one cycle C' for the deformed field F_1 which is stable or unstable according as $\varepsilon J^{(p)}(0)<0$ or >0; specially when $p=1$ or $\pi_1(0)\neq 0$, the cycle C' is unique;*

2° *when p is even,*

(a) *when $\varepsilon\pi_1(0)/J^{(p)}(0)<0$, there exist two and only two cycles for the field F_1 having the opposite absolute stability;*

(b) *when $\varepsilon\pi_1(0)/J^{(p)}(0)>0$, there exists no cycle for the field F_1;*

(c) *when $\pi_1(0)=0$, the existence of the cycles for the field F_1 is decided by the existence of the real roots of the equation (5.7); when the cycles for the field F_1 exist, their stability is of the same character as in the case* II, 2° *of* § 4 *in which $I=0$.*

II. The case where $J(a)\equiv 0$.

First we consider the the case where $\pi_1(a)\equiv\cdots\equiv\pi_{m-1}(a)\equiv\cdots\cdots\equiv 0$. In this case, from (5.7), $\Phi(0, \varepsilon, a)\equiv 0$. Therefore there exists a continuum of cycles also for the deformed field F_1.

Next we consider the case where $\pi_1(a)\equiv\cdots\cdots\equiv\pi_{p-1}(a)\equiv 0$, $\pi_p(a)\not\equiv 0$. In this case, the equation (5.7) becomes

(5.9) $$\Phi(0, \varepsilon, a)\equiv\varepsilon^{p+1}\left[\pi_p(a)+\varepsilon\pi_{p+1}(a)+\cdots\cdots\right]=0.$$

This is of the same form as (5.7). Therefore, as in § 4, we have:

(i) when $\pi_p(0)\neq 0$, there exists no cycle for the field F_1;

(ii) when $\pi_p(0)=\pi_p'(0)=\cdots=\pi_p^{(q-1)}(0)=0$, $\pi_p^{(q)}(0)\neq 0$, then we have:

1° *when q is odd, there exists at least one cycle C' for the field F_1 which is stable or unstable according as $\varepsilon^{p+1}\pi_p^{(q)}(0)<0$ or >0; specially when $q=1$ or $\pi_{p+1}(0)\neq 0$, the cycle C is uniqne;*

2° *when q is even,*

(a) *when $\varepsilon\pi_{p+1}(0)/\pi_p^{(q)}(0)<0$, there exist two and only two cycles for* *the field* F_1 *having the opposite absolute stability ;*

(b) *when* $\varepsilon\pi_{p+1}(0)/\pi_p^{(q)}(0)>0$, *there exists no cycle for the field* F_1 *;*

(c) *when* $\pi_{p+1}(0)=0$, *the existence of the cycles for the field* F_1 *is decided by the existence of the real roots of the equation* (5.9) *; when the cycles for the field* F_1 *exist, their stability is of the same character as in the case* II, 2° *of* §4 *in which* $I=0$.

Our results contain the results of Kryloff and Bogoliuboff[1] as a special case. Their results are concerned with the continuum of circles in the case I, 1° in which $p=1$. Our method is entirely different from theirs and our results are more general than theirs.

§ 6. Motion of the cycles for the field varying with one parameter.

In this paragraph, we consider the field $F(\alpha)=\{X(x, y, \alpha), Y(x, y, \alpha)\}$ depending on the parameter α. For simplicity, we assume that $X(x, y, \alpha)$ and $Y(x, y, \alpha)$ are integral with regard to x, y and α. We denote the cycle for the field $F(\alpha)$ by $C(\alpha)$. If there exists a cycle $C(\alpha_0)$, then, for the field $F(\alpha)=F(\alpha_0+\delta\alpha)$, the discussions of §4 and §5 are applied by putting $\delta\alpha=\varepsilon$ and $\partial X/\partial\alpha_0=H_1$, $\partial Y/\partial\alpha_0=K_1,\cdots\cdots$.

When, in the neighborhood of $C(\alpha_0)$, there exist the continuums of cycles for both fields $F(\alpha_0)$ and $F(\alpha_0+\delta\alpha)$, by §5, it is valid that $\Phi(0, \varepsilon, a)\equiv\rho(t_0+T(a)\,;0, \varepsilon, a)=0$ for any a and ε. By §4, this means that, for $C(\alpha_0)$, $\rho(t_0+T\,;c, \varepsilon)=c$ for any c and ε. If we adopt the cycles corresponding to $c=0$, then $C(\alpha_0+\varepsilon)$ varies continuously from $C(\alpha_0)$ to $C(\alpha_0+\delta\alpha)$ because of analyticity of $\rho(t\,;0, \varepsilon)$. Thus we have

Theorem 1. *When, in the neighborhood of* $C(\alpha_0)$, *there exist the continuums of cycles for both fields* $F(\alpha_0)$ *and* $F(\alpha_0+\delta\alpha)$, *each cycle* $C(\alpha_0)$ *varies continuously forming a cycle from the initial position to the cycle for the field* $F(\alpha_0+\delta\alpha)$.

If we exclude the above case, the cycle for the field $F(\alpha_0+\varepsilon)$ lying in the neighborhood of $C(\alpha_0)$ is determined corresponding to the real root of the equation of the form as follows:

$$\Phi_1(c, \varepsilon)\equiv c^m+k(\varepsilon)c^{m-1}+\cdots\cdots+l(\varepsilon)=0, \tag{6.1}$$

where $k(\varepsilon),\cdots\cdots, l(\varepsilon)$ are analytic functions vanishing with ε. Now the roots of (6.1) are expressed as follows:

$$c=\alpha_1\xi^{N_1}+\alpha_2\xi^{N_2}+\cdots\cdots, \tag{6.2}$$

where $N_1, N_2,\cdots\cdots$ are posititive integers such that $N_1<N_2<\cdots\cdots$, and $\xi=\varepsilon^{1/M}$, M being a positive integer. Put $\varepsilon=re^{i\theta}$ where $\theta=0$ or π according as $\varepsilon>0$ or <0. Then

$$\xi=r^{1/M}e^{i\theta/M}\omega^R, \tag{6.3}$$

where $\omega=e^{2\pi i/M}$ and R is a non-negative integer. Substituting (6.3) into (6.2), we have

1) N. Kryloff and N. Bogoliuboff, ibid.

(6.4) $$c=\alpha_1' r^{N_1/M}+\alpha_2' r^{N_2/M}+\cdots\cdots,$$

where $\alpha_1'=\alpha_1 e^{iN_1\theta/M}\omega^{RN_1}$, $\alpha_2'=\alpha_2 e^{iN_2\theta/M}\omega^{RN_2},\cdots\cdots$. If c is real for sufficiently small $|\varepsilon|=r$, then, from (6.4), α_1', $\alpha_2'\cdots\cdots$ must be real. Thus we see that, if c is real, c can be expressed as (6.4) which has the real coefficients. Then, if the root $c=c(\varepsilon)$ is real for $\varepsilon=\delta\alpha$, then $c(\varepsilon)$ is also real for any ε such that $0\leqq\varepsilon\leqq\delta\alpha$ or $0\geqq\varepsilon\geqq\delta\alpha$. Since $\rho(t\,;c,\varepsilon)=\rho(t\,;c(\varepsilon),\varepsilon)$ is continuous with regard to ε, the cycle $C(\alpha_0+\varepsilon)$ corresponding to $c=c(\varepsilon)$ for the field $F(\alpha_0+\varepsilon)$ varies continuously from $C(\alpha_0)$ to $C(\alpha_0+\delta\alpha)$ as ε varies from 0 to $\delta\alpha$.

Let the real roots of (6.1) be $c_1, c_1, \cdots\cdots, c_k$, then, by (6.4,) they are expressed as follows;

$$c_i=\alpha_i r^{M_i}+\beta_i r^{L_i}+\cdots\cdots,\qquad (i=1, 2,\cdots\cdots, k)$$

where $r=|\varepsilon|$, $0<M_i<L_i<\cdots\cdots$, and α_i, $\beta_i\cdots\cdots$ are real. Consequently the order after magnitudes of $c_1, c_1,\cdots\cdots, c_k$ is fixed for sufficiently small $|\varepsilon|$ if the sign of ε is fixed. From (6.1),

$$\Phi_1(c,\varepsilon)=(c-c_1)^{m_1}(c-c_2)^{m_2}\cdots\cdots(c-c_k)^{m_k}\Phi_0(c,\varepsilon),$$

where $\Phi_0(c,\varepsilon)>0$ for sufficiently small $|c|$ and $|\varepsilon|$ except for $c=\varepsilon=0$. Then the signs of $\Phi_1(c,\varepsilon)$ in the neighborhood of the roots $c_1, c_2,\cdots\cdots, c_k$ are fixed for sufficiently small $|\varepsilon|$ if the sign of ε is fixed. Thus, for sufficiently small $|\varepsilon|$, the number of the cycles $C(\alpha_0+\varepsilon)$'s and the stability of each cycle $C(\alpha_0+\varepsilon)$ are fixed if the sign of ε is fixed.

Thus we have

Theorem 2. *If there exist k cycles $C(\alpha_0+\delta\alpha)$ lying in the neighborhood of $C(\alpha_0)$, then k cycles $C(\alpha_0+\varepsilon)$ varies continuously keeping the same stability from $C(\alpha_0)$ to $C(\alpha_0+\delta\alpha)$ as ε varies monotonely.*

If $C(\alpha_0+\delta\alpha)$ intersect with $C(\alpha_0)$ in a point P for $t=t_1$, then, as in §4, it is readily seen that, for sufficiently small $|\delta\alpha|$, the absolute value of the integral I_1 given by (4.11) is arbitrarily small. Therefore, when $I_1\neq 0$ for any point of $C(\alpha_0)$, $C(\alpha_0+\delta\alpha)$ cannot intersect with $C(\alpha_0)$ for sufficiently small $|\delta\alpha|$. Consequently, *in this case, the cycle $C(\alpha_0+\varepsilon)$ varies continuously expanding or contracting from $C(\alpha_0)$ to $C(\alpha_0+\delta\alpha)$ as ε varies monotonely*, because $I_1\neq 0$ for any point of $C(\alpha_0+\varepsilon)$ from continuity of I_1.

When $h_0\neq 0$ for $C(\alpha_0)$, from (4.5), it follows that

(6.5) $$\lim_{\varepsilon\to 0}\frac{c}{\varepsilon}=\frac{e^{h_0}}{1-e^{h_0}}\frac{1}{\sqrt{X_0^2+Y_0^2}}I.$$

We call the left-hand side of the above formula the expanding coefficient in the point corresponding to $t=t_0$. When $h_0=0$ for $C(\alpha_0)$, if there exists a cycle $C(\alpha_0+\delta\alpha)$, the expanding coefficient becomes $0, \pm\infty$, or finite. Specially when $I\neq 0$, the expanding coefficient becomes infinite.

As noticed in §4, in the case studied by Uno[1] and Duff,[2] $I \neq 0$ for any point of the cycle, therefore, when the parameter varies monotonely, the cycles vary continuously expanding or contracting as Duff has proved.

We assume that, for $\delta\alpha$ of the fixed sign of the sufficiently small absolute value, there exists always at least one cycle $C(\alpha+\delta\alpha)$ in the neighborhood of the cycle $C(\alpha)$. We call this condition the condition of positive or negative continuation according as $\delta\alpha>0$ or <0. In the following, for example, we assume the condition of positive continuation.

Increasing α from α_0, we assume that, for any α such that $\alpha_0 \leqq \alpha < \alpha' (\alpha' < \infty)$, there exists a cycle $C(\alpha)$. We assume that $\{C(\alpha)\}$ is uniformly bounded and $C(\alpha)$ does not tend to critical points as $\alpha \to \alpha'$.

Let any point be P to which $C(\alpha)$ tends as $\alpha \to \alpha'$. Then there exist the number α_n and the point P_n on $C(\alpha_n)$ such that $\alpha_0 \leqq \alpha_n < \alpha'$ and $P_n \to P$ as $\alpha_n \to \alpha'$. From uniform boundedness, P lies at the finite distance, and moreover, by our assumption, P is an ordinary point for the field $F(\alpha')$. By $A(\alpha, R, t)$, we denote the point reached in time t along the characteristic $L(\alpha)$ from the point R. Then, from the continuity of the solutions of the differential equations, it follows that

$$A(\alpha_n, P_n, t) \to A(\alpha', P, t) \qquad \text{as } n \to \infty. \tag{6.6}$$

Let any point be Q, which belongs to the positive limiting set of the characteristic $L(\alpha', P)$ for the field $F(\alpha')$ passing through P. Then there exists a sequence $\{t_n\}$ such that $A(\alpha', P, t_n) \to Q$ as $t_n \to \infty$. In other words, for any given positive number η, it is valid that

$$\overline{A(\alpha', P, t_n)Q} < \eta/2 \text{ [3]}$$

for sufficiently large t_n. From (6.6), for sufficiently large m_n, it is valid that

$$\overline{A(\alpha_{m_n}, P_{m_n}, t_n)\, A(\alpha', P, t_n)} < \eta/2.$$

Consequently it follows that

$$\overline{A(\alpha_{m_n}, P_{m_n}, t_n)Q} < \eta. \tag{6.7}$$

Put $Q_n = A(\alpha_{m_n}, P_{m_n}, t_n)$, then (6.7) means that $Q_n \to Q$ as $n \to \infty$. Now $Q_n \in C(\alpha_{m_n})$, consequently, from our assumption, it is seen that Q lies at the finite distance and is an ordinary point for $F(\alpha')$. In other words, the positive limiting set of $L(\alpha', P)$ contains neither points at infinity nor critical points. Then this limiting set is a cycle which we denote by $C(\alpha')$. Since $\alpha_{m_n} \to \alpha'$ and $Q_n \to Q$ as $n \to \infty$, $C(\alpha_{m_n}) \to C(\alpha')$ as $n \to \infty$. Therefore $P \in C(\alpha')$, consequently $L(\alpha', P)$ coincides with $C(\alpha')$, namely $L(\alpha', P)$ becomes a cycle $C(\alpha')$. Thus we have obtained the cycle $C(\alpha')$ for the field $F(\alpha')$ to which $C(\alpha)$

1) T. Uno, ibid.
2) G. F. D. Duff, ibid.
3) The upper bar denotes the distance.

tends as $\alpha \to \alpha'$. Then, by the condition of positive continuation, for sufficiently small positive $\delta\alpha'$, there exists at least one cycle $C(\alpha' + \delta\alpha')$.

When the condition of negative continuation is assumed, the similar results are still valid. Thus we have

Theorem 3. *When the condition of positive continuation is valid, if* $\alpha'(\alpha' < \infty)$ *is a least upper bound of* α *such that* $C(\alpha)$ *exists, then, as* $\alpha \to \alpha'$, $C(\alpha)$ *approaches indefinitely either critical points or points at infinity. When the condition of negative continuation is valid, the conclusion is also valid if* $\alpha'(\alpha' > -\infty)$ *is a greatest lower bound of* α *such that* $C(\alpha)$ *exists.*

This theorem says that, when the condition of continuation is valid, the parameter α is monotonely increased or decreased starting from the certain value of α till the corresponding $C(\alpha)$ reaches either critical points or points at infinity.

For example, we consider the case where there exists a continuum of cycles in the neighborhood of $C(\alpha_0)$. If $J(a_0) = 0$ and $J'(a_0) \neq 0$ for certain $a = a_0$, then, by §5, in the neighborhood of $C(\alpha_0)$, there exists a unique cycle $C(\alpha_0 + \delta\alpha)$ having the absolute stability. Now, by §4, for the cycle having the absolute stability, the condition of both continuations is valid. Therefore, when α varies monotonely from α_0, the cycle $C(\alpha)$ varies contiuuously from certain $C(\alpha_0)$ forming a cycle of the fixed absolute stability till it reaches either critical points or points at infinity. Consequently, in general, for a considerably wide range of α containing α_0, there exists a cycle of the fixed absolute stability.

Department of Mathematics,
Faculty of Science,
Hiroshima University.