JOURNAL OF SCIENCE OF THE HIROSHIMA UNIVERSITY, SER. A, VOL. 19, No. 3, JANUARY, 1956

Spin Representation of the New Fundamental Group of Transformations in Relativistic Quantum Mechanics

by

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(Received Aug. 22, 1955)

§ 1. Introduction

Up to the present, spin representation of Lorentz transformations is investigated. Specially, the actual forms of spin transformations corresponding to special Lorentz transformations, space rotations and reflection are obtained. On the other hand, in the previous papers [**1**, **2**] †), we have proposed to replace the special Lorentz transformations by the new fundamental group of transformations, as representing the relations between the coordinates in two inertial systems one of which moves with uniform velocity to the other. The actual form of the transformations of the new group is obtained in the previous papers [**1**, **2**]. We see that the new group is a *3*-parameter sub-group of the *proper* Lorentz group. *We have an intention to develope physical laws based on the new group.* In this paper, for our future research, we shall investigate spin representation of the new group, obtaining the actual forms of spin transformations corresponding to the new group.

§ 2. Spin transformations corresponding to the new fundamental group of transformations

We take the Minkowski space, R_4, in which the interval is given by the form:

$$-(dx^1)^2-(dx^2)^2-(dx^3)^2+(dx^4)^2=g_{\mu\nu}\,dx^\mu\,dx^\nu \tag{2.1}$$

For the transformations of the new group, we adopt the tensor-expression in R_4. The actual tensor-form of the transformations of the new group is obtained in the previous paper [**3**] (the equations (3,4), (3.5) in [**3**]). Namely, in the coordinates in which the interval is expressed as (2.1), the transformations of the new group are given by

†) Numbers in brackets refer to the list of references at the end of the paper.

$$x'^{\lambda} = A^{\lambda}{}_{\mu}\, x^{\mu} \qquad (\lambda,\ \mu = 1,\ \cdots,\ 4) \tag{2.2}$$

with

$$A^{\lambda}{}_{\mu} = \delta^{\lambda}{}_{\mu} + k^{\lambda}\frac{U_{\mu}-U'_{\mu}}{(kU')} + \frac{U'^{\lambda}-U^{\lambda}}{(kU)}k_{\mu} + \frac{(UU')-(UU)}{(kU)\,(kU')}k^{\lambda}k_{\mu} \tag{2.3}$$

where k^{λ} is a fixed null-vector defined by

$$k^{\lambda} = (d^1,\ d^2,\ d^3,\ 1), \qquad g_{\mu\nu}\,k^{\mu}\,k^{\nu} = 0 \tag{2.4}$$

U^{λ} is four-velocity corresponding to ordinary three-dimensional velocity $u^h\,(h=1,2,3)$:

$$U^{\lambda} = (u^h/\sqrt{1-u^2/c^2},\quad c/\sqrt{1-u^2/c^2}) \text{ and } U'^{\lambda} = (0,\ 0,\ 0,\ c) \tag{2.5}$$

Further,

$$(kU) = g_{\lambda\mu}k^{\lambda}U^{\mu}, \quad (UU) = g_{\lambda\mu}U^{\lambda}U^{\mu} = c^2. \tag{2.6}$$

We see that the transformations defined by (2.2) (regarding $u^h\,(h=1,2,3)$ as parameters) form a *3*-parameter sub-group, G_3, of the proper Lorentz group. Now we shall obtain spin transformations corresponding to the transformations of the group G_3. For this purpose, we introduce the Dirac matrices γ^{μ} satisfying the commutation relations:

$$\gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2g^{\mu\nu}I \qquad (\mu,\ \nu = 1,\ \cdots,\ 4) \tag{2.7}$$

of which each γ^{μ} being realized by 4×4 (*4*-columns, *4*-rows) matrix. As well known, for every Lorentz transformation: $x'^{\lambda} = L^{\lambda}{}_{\mu}x^{\mu}$, there exists certain non-singular matrix S satisfying the relations:

$$L^{\lambda}{}_{\mu}\gamma^{\mu} = S\gamma^{\lambda}S^{-1} \qquad (\lambda,\ \mu = 1,\ \cdots,\ 4) \tag{2.8}$$

Our problem is to determine the matrix S for every transformation of the group G_3, i. e. to determine S such that

$$A^{\lambda}{}_{\mu}\gamma^{\mu}S = S\gamma^{\lambda} \qquad (\lambda,\ \mu = 1,\ \cdots,\ 4) \tag{2.9}$$

$A^{\lambda}{}_{\mu}$ being given by (2.3). To solve this problem, we first introduce the matrices $\gamma^{[\lambda}\gamma^{\mu]}$ and γ_5 defined by

$$\gamma^{[\lambda}\gamma^{\mu]} = \frac{1}{2}(\gamma^{\lambda}\gamma^{\mu} - \gamma^{\mu}\gamma^{\lambda}), \quad \gamma_5 = -\frac{1}{4\,!}\sqrt{g}\;\varepsilon_{\alpha\beta\gamma\delta}\,\gamma^{\alpha}\gamma^{\beta}\gamma^{\gamma}\gamma^{\delta} \tag{2.10}$$

Here we define $\varepsilon_{\alpha\beta\gamma\delta}$ to be $+1$ if α, β, γ, δ is an even prermutation of say *1*, *2*, *3*, *4*; -1 if odd, and zero otherwise. And g denotes the determinant of $g_{\mu\nu}$, accordingly $\sqrt{g} = i$. It is known that γ^{λ}, $\gamma^{[\lambda}\gamma^{\mu]}$, γ_5, $\gamma^{\lambda}\gamma_5$ $(\lambda,\ \mu = 1,\ \cdots,\ 4)$ and unit matrix I together constitute basis of 4×4 matrix, namely any 4×4 matrix can be expressed by linear combination of these *16* matrices multiplied by certain coefficients. Hence we express the matrix S in the form:

$$S = sI + s^5\gamma_5 + s_{\alpha\beta}\gamma^\alpha\gamma^\beta + s_\alpha\gamma^\alpha + s_\alpha{}^5\gamma^\alpha\gamma_5 \tag{2.11}$$

where s, s^5, s_α, $s_\alpha{}^5$ and $s_{\alpha\beta} = -s_{\beta\alpha}$ are quantities (scalars, vectors and anti-symmetric tensor) to be determined by (2.9). Substituting (2.11) into (2.9), the right side of (2.9) is expanded in terms of the basis as follows:

$$S\gamma^\lambda = s\gamma^\lambda - s^5\gamma^\lambda\gamma_5 + 2s_\alpha{}^\lambda\gamma^\alpha + s^\lambda I + s_\alpha\gamma^{[\alpha}\gamma^{\lambda]} - s^{\lambda 5}\gamma_5$$
$$+ s_{\alpha\beta}\frac{1}{\sqrt{g}}\varepsilon^{\alpha\beta\lambda\nu}\gamma_\nu\gamma_5 - s_\alpha{}^5\frac{1}{2\sqrt{g}}\varepsilon^{\alpha\lambda\kappa\nu}\gamma_\kappa\gamma_\nu \tag{2.12}$$

which is obtained by using (2.7) and the following relations:

$$\gamma^\lambda\gamma_5 + \gamma_5\gamma^\lambda = 0 \tag{2.13}$$

$$-\frac{1}{\sqrt{g}}\varepsilon^{\alpha\beta\gamma\delta}\gamma_5\gamma_\delta = \gamma^{[\alpha}\gamma^\beta\gamma^{\gamma]}, \quad 2\gamma_5\gamma^{[\gamma}\gamma^{\delta]} = \sqrt{g}\,\varepsilon^{\alpha\beta\gamma\delta}\gamma_\alpha\gamma_\beta \tag{2.14}$$

$\varepsilon^{\alpha\beta\gamma\delta}$ being defined similarly as $\varepsilon_{\alpha\beta\gamma\delta}$. In the same way, $\gamma^\mu S$ on the left side of (2.9) is expanded in terms of the basis. Then comparing the coefficients of the basis of both sides of (2.9), we have:

coefficient of I: $(A^\lambda{}_\mu - \delta^\lambda{}_\mu)s^\mu = 0$ (2.15)

〃 γ_5: $(A^\lambda{}_\mu + \delta^\lambda{}_\mu)s^{\mu 5} = 0$ (2.16)

〃 γ^α: $2(A^\lambda{}_\mu + \delta^\lambda{}_\mu)s^\mu{}_\alpha + (A^\lambda{}_\alpha - \delta^\lambda{}_\alpha)s = 0$ (2.17)

〃 $\gamma_\nu\gamma_5$: $(A^\lambda{}_\mu - \delta^\lambda{}_\mu)\frac{1}{\sqrt{g}}\varepsilon^{\alpha\beta\mu\nu}s_{\alpha\beta} + (A^{\lambda\nu} + g^{\lambda\nu})s^5 = 0$ (2.18)

〃 $\gamma_{[\kappa}\gamma_{\nu]}$: $\frac{1}{2}(A^\lambda{}_\mu - \delta^\lambda{}_\mu)\frac{1}{\sqrt{g}}\varepsilon^{\mu\alpha\kappa\nu}s_\alpha{}^5 + (A^\lambda{}_\mu + \delta^\lambda{}_\mu)s^{[\nu}g^{\kappa]\mu} = 0$ (2.19)

Calculating the determinant of the matrix $\|A^\lambda{}_\mu + \delta^\lambda{}_\mu\|$, we have

$$|A^\lambda{}_\mu + \delta^\lambda{}_\mu| = 4\{(kU) + (kU')\}^2/(kU)(kU') \tag{2.20}$$

which does not vanish. Hence there exists the inverse matrix; the following relations are obtained

$$\frac{1}{2}\left[\delta^\beta{}_\lambda + \frac{(U^\beta - U'^\beta)k_\lambda - k^\beta(U_\lambda - U'_\lambda)}{(kU) + (kU')}\right]\left[A^\lambda{}_\mu + \delta^\lambda{}_\mu\right] = \delta^\beta{}_\mu \tag{2.21}$$

Using the above relations, we can solve the equations (2.15)—(2.19). Namely, from (2.17), multiplying (2.17) by the inverse matrix of $\|A^\lambda{}_\mu + \delta^\lambda{}_\mu\|$ and using

(2.21), we have

$$2\,s^{\beta}{}_{\alpha} - \frac{(U^{\beta} - U'^{\beta})k_{\alpha} - k^{\beta}(U_{\alpha} - U'_{\alpha})}{(kU) + (kU')}\,s = 0$$

or

$$s_{\alpha\beta} = -\,s\,\{k_{\alpha}(U_{\beta} - U'_{\beta}) - k_{\beta}(U_{\alpha} - U'_{\alpha})\}/2\{(kU) + (kU')\} \tag{2.22}$$

From the remaining equations of (2.15)—(2.19), we have

$$s^{\lambda 5} = 0, \quad s^{\lambda} = 0, \quad s^{5} = 0 \tag{2.23}$$

Hence the matrix S satisfying the relations (2.9) is given by

$$S = s\left\{I - \frac{k_{\alpha}(U_{\beta} - U'_{\beta})}{(kU) + (kU')}\,\gamma^{[\alpha}\gamma^{\beta]}\right\} \tag{2.24}$$

s being arbitrary scalar.

If the γ^{μ}'s are realized by the following matrices:

$$\gamma^{1a}{}_{b} = \begin{Vmatrix} 0 & \begin{matrix} 0 & i \\ i & 0 \end{matrix} \\ \begin{matrix} 0 & i \\ i & 0 \end{matrix} & 0 \end{Vmatrix} \quad \gamma^{2a}{}_{b} = \begin{Vmatrix} 0 & \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} \\ \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & 0 \end{Vmatrix} \quad \gamma^{3a}{}_{b} = \begin{Vmatrix} 0 & \begin{matrix} i & 0 \\ 0 & -i \end{matrix} \\ \begin{matrix} i & 0 \\ 0 & -i \end{matrix} & 0 \end{Vmatrix} \quad \gamma^{4a}{}_{b} = \begin{Vmatrix} 0 & \begin{matrix} -i & 0 \\ 0 & -i \end{matrix} \\ \begin{matrix} i & 0 \\ 0 & i \end{matrix} & 0 \end{Vmatrix} \tag{2.25}$$

(0 being zero matrix), from (2.24), S has the form

$$S = \begin{Vmatrix} S_1 & 0 \\ 0 & S_2 \end{Vmatrix} \quad \text{with} \quad S_1 = \begin{Vmatrix} \alpha & \beta \\ \gamma & \delta \end{Vmatrix} \quad S_2 = \begin{Vmatrix} \bar{\delta} & -\bar{\gamma} \\ -\bar{\beta} & \bar{\alpha} \end{Vmatrix} \tag{2.26}$$

where, putting $\{k_{\mu}(U_{\nu} - U'_{\nu}) - k_{\nu}(U_{\mu} - U'_{\mu})\}/\{(kU) + (kU')\} = a_{\mu\nu}$,

$$\begin{aligned} \alpha &= s(1 + ia_{12} + a_{34}) & \beta &= s(-a_{13} + a_{14} - ia_{24} + ia_{23}) \\ \gamma &= s(a_{13} + a_{14} + ia_{24} + ia_{23}) & \delta &= s(1 - ia_{12} - a_{34}) \end{aligned} \tag{2.27}$$

and $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$, $\bar{\delta}$ denote the expressions obtained from α, β, γ, δ after replacing i by $-i$. We restrict ourselves to the case where the determinant $|S_1| = |S_2|$ is equal to *1*. Then it must be that

$$s = \pm\,\{(kU) + (kU')\}/2\sqrt{(kU)(kU')} \tag{2.28}$$

Hence, from (2.24), we have

$$S = \pm\,\frac{1}{2\sqrt{(kU)(kU')}}\left\{\{(kU) + (kU')\}I - k_{\alpha}(U_{\beta} - U'_{\beta})\gamma^{[\alpha}\gamma^{\beta]}\right\} \tag{2.29}$$

which gives the spin transformation corresponding to the transformation (2.2). From the above, we can see that

$$Spur\, S_1 = Spur\, S_2 = \pm\,\{(kU) + (kU')\}/\sqrt{(kU)(kU')} = real \tag{2.30}$$

which does not take place in the proper Lorentz group.

§ 3. 2-spinor representation of the new fundamental group G_3.

In §2, starting from the transformation (2.2), we have obtained the spin transformation corresponding to (2.2). In this section, by considering certain special *2*-spinor transformations, we shall investigate spin representation of the group G_3.

We now consider geometrical objects, ψ_A, which are defined over a two dimensional complex space (the spin space) and obey the transformation law

$$\psi'_A = s^B{}_A \psi_B \qquad (A, B = 1, 2) \tag{3.1}$$

with the unimodular matrix $\|s^B{}_A\|$ in general complex. Further, we introduce a geometrical object ψ^A transforming contragrediently to ψ_A according to

$$\psi'^A = (s^{-1})^A{}_B \psi^B \quad \text{where} \quad (s^{-1})^A{}_B s^B{}_C = \delta^A{}_C \tag{3.2}$$

We associate covariant and contravariant components according to the rule

$$\psi_A = \varepsilon_{AB}\psi^B \quad \text{or} \quad \psi^A = \varepsilon^{BA}\psi_B \tag{3.3}$$

where

$$\left\| \varepsilon_{AB} \right\| = \left\| \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} \right\| = \left\| \varepsilon^{AB} \right\| \tag{3.4}$$

Owing to the transformation (3.1), we consider the transformation:

$$X'_{\dot{A}B} = \bar{s}^C{}_A s^D{}_B X_{\dot{C}D} \qquad (A, B, C, D = 1, 2) \tag{3.5}$$

where the placing of the dot over the indices A, C serves as a reminder that a bar (complex conjugate) is to be placed over the corresponding $s^C{}_A$ in the law of transformation. In order to establish a correspondence between second-rank Hermitian matrices, say $\|X_{\dot{A}B}\|$, and vectors, say v^λ, in R_4, we introduce the spin-tensors $\|g_{\mu\dot{A}B}\|$ defined by †)

$$\left\| g_{1\dot{A}B} \right\| = \left\| \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \right\|, \quad \left\| g_{2\dot{A}B} \right\| = \left\| \begin{matrix} 0 & i \\ -i & 0 \end{matrix} \right\|, \quad \left\| g_{3\dot{A}B} \right\| = \left\| \begin{matrix} 1 & 0 \\ 0 & -1 \end{matrix} \right\|, \quad \left\| g_{4\dot{A}B} \right\| = \left\| \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \right\| \tag{3.6}$$

and put

$$X_{\dot{A}B} = g_{\mu\dot{A}B} v^\mu \quad \text{or} \quad v^\mu = \frac{1}{2} g^{\mu\dot{A}B} X_{\dot{A}B} \tag{3.7}$$

†) Using (3.6), the matrices $\gamma^{\mu a}{}_b$ defined by (2.25) are expressed as

$$\gamma^{\mu a}{}_b = \left\| \begin{matrix} 0 & -i\|g^\mu{}_{\dot{A}B}\| \\ i\|g^{\mu\dot{A}B}\| & 0 \end{matrix} \right\|$$

Then the spin transformation (3.5), of the type $\psi_A \to \psi'_A = s^B{}_A \psi_B$, induces vector transformation $v^\mu \to v'^\mu$ of the form:

$$v'^\mu = L^\mu{}_\nu v^\nu \qquad (\mu, \nu = 1, \cdots, 4) \tag{3.8}$$

with

$$L^\mu{}_\nu = \frac{1}{2} g^{\mu \dot{A} B} \bar{s}^C{}_A s^D{}_B g_{\nu \dot{C} D} \tag{3.9}$$

It is well known that the transformation: $x'^\mu = L^\mu{}_\nu x^\nu$ in R_4, represents transformation of *proper* Lorentz group.

Specially when $X_{\dot{A}B}$ takes the form $X_{\dot{A}B} = \bar{\xi}_A \xi_B$, the corresponding vector $w^\mu = \frac{1}{2} g^{\mu \dot{A} B} \bar{\xi}_A \xi_B$ is a null-vector viz. $w^\mu w_\mu = 0$ which is shown by using the relations $g^{\mu \dot{A} B} g_{\mu \dot{C} D} = 2 \delta^A{}_C \delta^B{}_D$ and $\xi_A \xi^A = 0$. Hence, by choosing ξ_A suitably, we can identify w^λ with the null-vector k^λ defined by (2.4), so we can put the correspondence between ξ_A and k^λ as follows:

$$\bar{\xi}_A \xi_B = g_{\lambda \dot{A} B} k^\lambda \quad \text{or} \quad k^\lambda = \frac{1}{2} g^{\lambda \dot{A} B} \bar{\xi}_A \xi_B \tag{3.10}$$

By the transformation (2.2), k^λ is transformed into k'^λ as follows:

$$k'^\lambda = k^\lambda (kU)/(kU') \tag{3.11}$$

Corresponding to the above, if we consider spin transformation: $\psi'_A = s^B{}_A \psi_B$ under the condition that $s^B{}_A$ satisfy the relations

$$s^B{}_A \xi_B = a \xi_A \qquad (A, B = 1, 2) \tag{3.12}$$

for fixed ξ_A (a being unfixed scalar), by the induced vector transformation, (3.10) is transformed as follows:

$$k'^\lambda = \frac{1}{2} g^{\lambda \dot{A} B} \bar{s}^C{}_A s^D{}_B \bar{\xi}_C \xi_D = \frac{1}{2} g^{\lambda \dot{A} B} \bar{a} \bar{\xi}_A a \xi_B = \bar{a} a k^\lambda \tag{3.13}$$

which is the same form as (3.11) putting $\bar{a}a = (kU)/(kU')$.

The actual form of $s^B{}_A$ satisfying the relations (3.12), is given by

$$s^B{}_A = a \delta^B{}_A + \xi^B \eta_A. \tag{3.14}$$

Under the condition that $|s^B{}_A| = 1$, η_A is the solution of $a^2 + a\xi^A \eta_A = 1$, namely, η_A is expressed in the form

$$\eta_A = \rho \xi_A + \zeta_A \quad \text{with} \quad \zeta_A = (1/a - a) \bar{\xi}^A / (\xi^1 \bar{\xi}^1 + \xi^2 \bar{\xi}^2) \tag{3.15}$$

ρ being arbitrary. Hence the unimodular matrix $\|s^B{}_A\|$ satisfying the condition

(3.12) for fixed ξ_A, is given by

$$s^B{}_A = a\delta^B{}_A + \rho\xi^B\xi_A + \xi^B\zeta_A \tag{3.16}$$

Therefore, all the spin transformations $\psi'_A = s^B{}_A\psi_B$ in which $s^B{}_A$ are given by (3.16) (a and ρ being arbitrary), have the property that the induced vector transformations of the form (3.8) transform the null-vector k^λ defined by (3.10) into null-vector $a\bar{a}k^\lambda$. However, besides the transformations of the group G_3, space rotations around the direction (d^1, d^2, d^3) make the null-vector $k^\lambda = (d^1, d^2, d^3, 1)$ invariant. Such space rotations and the transformations of the group G_3, together form a 4-parameter sub-group, say G_4, of the proper Lorentz group. In fact, as indicated in the classification at the end of the section, we see that *the unimodular spin transformations: $\psi'_A = s^B{}_A\psi_B$ under the condition that $s^B{}_A$ satisfy the relations $s^B{}_A\xi_B = a\xi_A$ for fixed ξ_A (a being unfixed), induce the transformations of the group G_4, which leaves the plane wave front defined by $k_\mu x^\mu = 0$ invariant.*

We shall obtain the condition that the above-mentioned spin transformations induce the transformations of the group G_3. From (3.16), taking spur of $s^B{}_A$, we have

$$\text{spur } s = s^A{}_A = 2a + \xi^A\zeta_A = a + 1/a \tag{3.17}$$

On the other hand, from (2.30), we see that

$$\text{spur } S_1 = \pm\{\sqrt{(kU)/(kU')} + \sqrt{(kU')/(kU)}\} \tag{3.18}$$

But, in usual spinor analysis, it is known that †)

$$S_1 = \|\tilde{s}^A{}_B\| \quad (\text{transposed matrix of } \|s^A{}_B\|) \tag{3.19}$$

Hence, it must be that

$$\pm\{\sqrt{(kU)/(kU')} + \sqrt{(kU')/(kU)}\} = a + 1/a \tag{3.20}$$

from which we see that a is real. Actually, we can see that *the unimodular spin transformations $\psi'_A = s^B{}_A\psi_B$ under the condition that $s^B{}_A$ satisfy the relations $s^B{}_A\xi_B = a\xi_A$ for fixed ξ_A (a being unfixed but real), induce the transformations of the group G_3.* The above

†) Considerations on the Lorentz invariance of the Dirac equation $(\partial_\mu\gamma^\mu - i\kappa)\Psi = 0$, lead to the transformation: $\Psi' = S\Psi$ $(\Psi'^a = S^a{}_b\Psi^b)$ with the matrix S determined by (2.8). Here Ψ is defined as a combination of two 2-spinors ψ_A, $\phi^{\dot{A}}$:

$$\Psi^a = \begin{Vmatrix} \Psi^1 \\ \Psi^2 \\ \Psi^3 \\ \Psi^4 \end{Vmatrix} = \begin{Vmatrix} \psi_1 \\ \psi_2 \\ \phi^{\dot{1}} \\ \phi^{\dot{2}} \end{Vmatrix} \quad \text{hence} \quad S = \begin{Vmatrix} S^a{}_b \end{Vmatrix} = \begin{Vmatrix} \|\tilde{s}^A{}_B\| & O \\ O & \|\overline{(s^{-1})^A{}_B}\| \end{Vmatrix}$$

For, under proper Lorentz transformations we have $\psi_A \to \psi'_A = s^B{}_A\psi_B$ and $\phi^{\dot{A}} \to \phi'^{\dot{A}} = \overline{(s^{-1})}^A{}_B\phi^{\dot{B}}$. We should note the inversion of indices (transposed matrix $\|\tilde{s}^A{}_B\|$) due to the fact that the two constituent 2-spinors transform (conjugate) contragrediently.

result may be clarified by the following classification :

A) The case where $a = \pm 1$ and ρ is arbitrary in general complex. Spin transformations $\psi'_A = s^B{}_A \psi_B$ with the matrices $\|s^B{}_A\| = \pm \|\delta^B{}_A + \rho \xi^B \xi_A\|$ regarding ρ as parameter (complex)), induce the transformations of G_3 for which $(kU)/(kU') = 1$ i.e. $\{1-(du)/c\} = \sqrt{1-u^2/c^2}$. These transformations themselves constitute 2-parameter sub-group, say G_2, of G_3.

B) The case where $\rho = 0$ and a is arbitrary but *real*. Spin Transformations with the matrices $\|s^B{}_A\| = \pm \|a\delta^B{}_A + \xi^B \zeta_A\|$, ζ_A being given by (3.15), (regarding a as parameter (real)) induce one-parameter group of transformations in R_4 which together with G_2 constitute G_3.

C) The case where $\rho = 0$ and $\bar{a}a = 1$ i.e. $a = e^{i\theta}$ (θ is real). Spin transformations with the matrices $\|s^B{}_A\| = \pm \|a\delta^B{}_A + \xi^B \zeta_A\|$ in which $a = e^{i\theta}$, (regarding θ as parameter (real)) induce the space rotations around the direction (d^1, d^2, d^3) given by the spatial direction of the null-vector k^λ defined by (3.10). Such space rotations together with G_3 constitute the group G_4, which leaves the plane wave front defined by $k_\mu x^\mu = 0$ invariant.

If we combine the cases A) and C), we obtain another 3-parameter group of transformations which transform k^λ defined by (3.10) into itself. Also the combination of the cases B) and C) leads to a 2-parameter group of transformations in R_4.

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