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Application of Majorized Group of Transformations to Ordinary Differential Equations with Periodic Coefficients

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1. Introduction. In the previous paper[1], by means of majorized group of transformations, we have proved the following two theorems:

Theorem 1[2]. *Given an analytic transformation*

$$T: \ x_i' = \varphi_i(x) = \lambda_i x_i + [x]_2,$$

where $|\lambda_i| = 1$ *and* $[x]_2$ *denotes a sum of the terms of the second and higher orders with respect to* x_j. *Then there exists a set of analytic functions* $f_i(x)$ *of the form*

$$(*) \qquad f_i(x) = x_i + [x]_2$$

satisfying the relation

$$(**) \qquad f_i(\varphi) = \lambda_i f_i(x),$$

if either of the following two conditions is fulfilled:

1° *a set of* $\{T^k\}$ $(k = 0, \pm 1, \pm 2, \cdots)$ *is majorized, namely there exists a set of analytic functions* $\Phi_i(x)$ *such that* $\varphi_i(x, k) \ll \Phi_i(x)$, *where* $\varphi_i(x, k)$ *are the functions such that* $x_i' = \varphi_i(x, k)$ *represents a transformation* T^k;

2° *the arguments of* λ_i*'s are all commensurable with* 2π *and there exist formal series* $f_i(x)$ *of the form* (*) *satisfying the relation* (**) *formally.*

A set of functions $f_i(x)$ *meeting the requirement is given by*

$$f_i(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{p=0}^{n-1} \frac{1}{\lambda_i^p} \varphi_i(x, p).$$

In the case 2°, *the above functions can be written in the finite form as follows:*

$$f_i(x) = \frac{1}{q} \sum_{p=0}^{q-1} \frac{1}{\lambda_i^p} \varphi_i(x, p),$$

1) M. Urabe, *Application of majorized group of transformations to functional equations.* J. Sci. Hiroshima Univ., Ser. A, **16**, 267-283 (1952). In the sequel, we denote this paper by [P].

2) [P], p. 271 and p. 273.

where q is a positive integer such that $\lambda_i^q = 1$.

Theorem 2[3]. *Given a system of differential equations*

$$\frac{dx_i}{dt} = X_i(x) = \lambda_i x_i + [x]_2,$$

where $\Re(\lambda_i) = 0$. Then there exists an analytic transformation

$$(*) \qquad y_i = f_i(x) = x_i + [x]_2$$

by which the initial system is reduced to the system

$$(**) \qquad \frac{dy_i}{dt} = \lambda_i y_i,$$

if either of the following two conditions is fulfilled:

1° *The solution $\varphi_i(x, t)$ such that $\varphi_i(x, 0) = x_i$ is majorized with respect to x_j, namely there exists a set of analytic functions $\Phi_i(x)$ such that $\varphi_i(x, t) \ll \Phi_i(x)$;*

2° *λ_i's are all mutually commensurable and there exists a formal transformation of the form (*) by which the initial system is reduced to the system (**).*

A set of functions $f_i(x)$ meeting the requirement is given by

$$f_i(x) = \lim_{n\to\infty} \frac{1}{n} \int_0^n e^{-\lambda_i \tau} \varphi_i(x, \tau)\, d\tau.$$

In the case 2°, the above functions can be written in the finite form as follows:

$$f_i(x) = \frac{1}{\omega} \int_0^\omega e^{-\lambda_i \tau} \varphi_i(x, \tau)\, d\tau,$$

where ω is a positive number such that $\lambda_i \omega$ are all integral multiples of $2\pi i$.

Afterwards, in 1954, making use of formal transformations, Y. Sibuya[4] has given another proof to Theorem 2 just mentioned and, besides, he has shown that the analogous results are valid also for a system of differential equations with periodic coefficients. His results are as follows:

Theorem 3. *Given a system of differential equations*

$$(1.1) \qquad \frac{dx_i}{dt} = X_i(x, t) \qquad (i = 1, 2, \cdots, n),$$

where $X_i(x, t)$ fulfill the conditions:

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3) [P], p. 278 and p. 281.

4) Y. Sibuya, *Sur un système des équations différentielles ordinaires non linéaires à coefficients constants ou périodiques.* J. Fac. Sci. Univ. Tokyo, Sec. I, 7, 19–32 (1954).

(i) $X_i(x, t)$ *are analytic with respect to* x_j *at* $x_j = 0$;

(ii) $X_i(x, t)$ *are continuous with respect to* t *for* $-\infty < t < \infty$;

(iii) $X_i(x, t)$ *are periodic with respect to* t *with period* $\omega > 0$;

(iv) $X_i(0, t) = 0$.

Then there exists an analytic transformation with periodic coefficients of the form

$$y_i = f_i(x, t) = x_i + [x\,;t]^{5)}, \tag{1.2}$$

by which the system (1.1) *is reduced to the system*

$$\frac{dy_i}{dt} = \lambda_i y_i, \tag{1.3}$$

if either of the following two conditions is fulfilled:

1° *the solution* $\varphi_i(x, t)$ *of* (1.1) *such that* $\varphi_i(x, 0) = x_i$ *is majorized with respect to* x_j [6];

2° *the products* $\lambda_i \omega$ *of the characteristic exponents* λ_i *and the period* ω *are all integral multiples of* $2\pi i$ *and there exists a formal transformation of the form* (1.2) *by which the system* (1.1) *is reduced to the system* (1.3).

In this note, we show that Theorem 3 is simply proved from Theorem 1 if we make use of the majorized group of transformations. Lastly, as supplement to Theorem 3, we add a theorem corresponding to Theorem 7 of [P] [7].

2. Case 1°. Let the expansions of $X_i(x, t)$ be

$$X_i = \sum_j c_{ij} x_j + \sum_{\mathfrak{p}}{}'' c_{i\mathfrak{p}}(t)\, x_1^{p_1} x_2^{p_2} \cdots x_n^{p_n}, \tag{2.1}$$

where $\sum_{\mathfrak{p}}''$ denotes summation over $\mathfrak{p} = (p_1, p_2, \cdots, p_n)$ such that $s(\mathfrak{p}) = p_1 + p_2 + \cdots + p_n \geqq 2$. By our hypotheses, the coefficients c_{ij} and $c_{i\mathfrak{p}}(t)$ are periodic with period $\omega > 0$. Applying a suitable linear transformation with periodic coefficients, without loss of generality, we may suppose that the coefficients c_{ij} are constants and moreover the matrix $C = \|c_{ij}\|$ is of the Jordan's canonical form. Since X_i are analytic with respect to x_j, for $|t| < T$ where T is an arbitrary fixed positive

5) $[x\,;t]_2$ denotes a sum of the terms of the second and higher orders with respect to x_j, the coefficients of which are periodic functions of t.

6) In this case, the set of transformations $x_i' = \varphi_i(x, t)$ does not make a one-parameter group unlike in the case of Theorem 2. Consequently, here, we do not mean the majorizedness of the group of transformations, but merely mean the majorizedness of the set of functions $\varphi_i(x, t)$, namely the existence of a set of analytic functions $\Phi_i(x)$ such that $\varphi_i(x, t) \ll \Phi(x)$ for $-\infty < t < \infty$.

7) [P], p. 281.

number, the solution $\varphi_i(x, t)$ of (1.1) is analytic with respect to x_j at $x_j = 0$. Let the expansions of $\varphi_i(x, t)$ be

$$\varphi_i(x, t) = \sum_j a_{ij}(t) x_j + \sum_{\mathfrak{p}}{}'' a_{i\mathfrak{p}}(t) x_1^{p_1} x_2^{p_2} \cdots x_n^{p_n}, \tag{2.2}$$

then, substituting (2.2) into (1.1) and comparing the coefficients of x_j, we have:

$$\frac{da_{ij}}{dt} = \sum_k c_{ik} a_{kj}.$$

Consequently

$$\| a_{ij} \| = e^{tC} K, \tag{2.3}$$

where K is a constant matrix. From the initial condition for the solution $\varphi_i(x, t)$, it must be that

$$a_{ij}(0) = \delta_{ij} \quad a_{i\mathfrak{p}}(0) = 0 \quad (s(\mathfrak{p}) \geqq 2), \tag{2.4}$$

where δ_{ij} is a Kronecker's delta. Then, from (2.3), we see that

$$\| a_{ij}(t) \| = e^{tC}. \tag{2.5}$$

Now, by the hypotheses, the solution $\varphi_i(x, t)$ is majorized, consequently $a_{ij}(t)$ must be bounded for $-\infty < t < \infty$, consequently C must be of the diagonal form and the real parts of all the characteristic roots λ_i's of C must be zero. Thus, from (2.5) follows

$$a_{ij}(t) = \delta_{ij} e^{\lambda_i t}$$

and, from (2.2) follows

$$\varphi_i(x, t) = e^{\lambda_i t} x_i + \sum{}'' a_{i\mathfrak{p}}(t) x_1^{p_1} x_2^{p_2} \cdots x_n^{p_n}. \tag{2.6}$$

Substituting (2.6) into (1.1) and comparing the coefficients of the products of x_j's, we have:

$$\frac{da_{i\mathfrak{p}}}{dt} = \lambda_i a_{i\mathfrak{p}} + L_{i\mathfrak{p}}\left(\prod_k a_{k\mathfrak{r}_{k1}} a_{k\mathfrak{r}_{k2}} \cdots a_{k\mathfrak{r}_{kq(k)}}\right), \tag{2.7}$$

where $L_{i\mathfrak{p}}$ denote the linear combinations of the arguments with periodic coefficients for $\sum q(k) \geqq 2$ and $\mathfrak{r}_{kj} = (r_{kj1}, r_{kj2}, \cdots, r_{kjn})$ such that $\sum_{k,j} r_{kjl} = p_l$. From (2.7), owing to (2.4), the coefficients $a_{i\mathfrak{p}}(t)$ are uniquely determined by successive integration.

Now let us consider the transformation

$$T:\ x_i' = \varphi_i(x, \omega) = e^{\lambda_i \omega} x_i + [x]_2. \tag{2.8}$$

Since X_i are periodic with respect to t, it is evident that

$$\varphi_i[\varphi(x, \omega), t] = \varphi_i(x, t+\omega).$$

From this, it is seen that the transformation

$$x_i' = \varphi_i(x, k\omega)$$

represents a k-iterated transformation T^k of T. Since the solution $\varphi_i(x, t)$ is majorized, the group of transformations $\{T^k\}$ $(k=0, \pm 1, \pm 2, \cdots)$ is majorized. Then, by Theorem 1, there exists a set of analytic functions

$$f_i(x) = x_i + [x]_2 \tag{2.9}$$

such that

$$f_i[\varphi(x, \omega)] = e^{\lambda_i \omega} f_i(x). \tag{2.10}$$

Making use of these functions, we consider the coordinate transformation

$$y_i = f_i(x), \tag{2.11}$$

and put

$$f_i[\varphi\{f^{-1}(y), t\}] = \psi_i(y, t).$$

Then $\psi_i(y, t)$ becomes the solution of the transformed differential equations such that $\psi_i(y, 0) = y_i$. Since $f_i(x)$ are of the form (2.9), the differential equations do not alter in the linear parts by the transformation (2.11), consequently the functions $\psi_i(y, t)$ are also expanded like (2.6) and do not alter in the linear parts. In the sequel, for brevity, instead of y_i and $\psi_i(y, t)$, we use the letter x_i and the expression $\varphi_i(x, t)$. Then the results obtained till now are valid also for newly defined x_i and $\varphi_i(x, t)$, and moreover the functions $f_i(x)$ defined by (2.9) become x_i. Then, from (2.10) and (2.6) follows

$$a_{i\mathfrak{p}}(\omega) = 0 \qquad (s(\mathfrak{p}) \geqq 2). \tag{2.12}$$

We shall prove by induction that $a_{i\mathfrak{p}}(t)$ are written in the form as follows:

$$a_{i\mathfrak{p}}(t) = e^{\lambda(\mathfrak{p})t} b_{i\mathfrak{p}}(t), \tag{2.13}$$

where $\lambda(\mathfrak{p}) = \lambda_1 p_1 + \lambda_2 p_2 + \cdots + \lambda_n p_n$ and $b_{i\mathfrak{p}}(t)$ are the periodic functions with period ω. For $s(\mathfrak{p}) = 1$, (2.13) is evident from (2.6). We assume that (2.13) hold for $s(\mathfrak{p}) < s$. Then, for $s(\mathfrak{p}) = s$ $(s \geqq 2)$, by (2.4), from (2.7), $a_{i\mathfrak{p}}(t)$ are expressed as follows:

$$a_{i\mathfrak{p}}(t) = e^{\lambda_i t} \int_0^t e^{-\lambda_i \tau} e^{\lambda(\mathfrak{p})\tau} M_{i\mathfrak{p}}(\tau)\, d\tau,$$

where $M_{i\mathfrak{p}}(\tau)$ are periodic. Then, from (2.12), it follows that

$$\begin{aligned} a_{i\mathfrak{p}}(t+\omega) &= e^{\lambda_i(t+\omega)} \int_\omega^{t+\omega} e^{[\lambda(\mathfrak{p})-\lambda_i]\tau} M_{i\mathfrak{p}}(\tau)\, d\tau \\ &= e^{\lambda_i(t+\omega)} \int_0^t e^{[\lambda(\mathfrak{p})-\lambda_i](\tau+\omega)} M_{i\mathfrak{p}}(\tau)\, d\tau \\ &= e^{\lambda(\mathfrak{p})\omega} a_{i\mathfrak{p}}(t), \end{aligned}$$

from which we see that (2.13) hold also for $s(\mathfrak{p}) = s$. Thus the validity of (2.13) for any $\mathfrak{p}$ follows.

Then, from (2.6), the solution $\varphi_i(x, t)$ can be written as follows:

(2.14) $$\begin{aligned} \varphi_i(x, t) &= e^{\lambda_i t} x_i + \sum_{\mathfrak{p}}{}'' b_{i\mathfrak{p}}(t) (x_1 e^{\lambda_1 t})^{p_1} \cdots (x_n e^{\lambda_n t})^{p_n} \\ &= \Phi_i(x_1 e^{\lambda_1 t}, x_2 e^{\lambda_2 t}, \cdots, x_n e^{\lambda_n t}, t), \end{aligned}$$

where $\Phi_i(u_1, u_2, \cdots, u_n, t)$ denote the functions analytic with respect to the arguments u_j having the coefficients periodic with respect to t. Thus, returning to the initial variables, we see that *the solution of the initial system is experessed as*

(2.15) $$x_i = f_i^{-1}[\Phi(C_1 e^{\lambda_1 t}, C_2 e^{\lambda_2 t}, \cdots, C_n e^{\lambda_n t}, t)],$$

where C_j are arbitrary constants. Here, if we put $y_i = C_i e^{\lambda_i t}$, then it holds that

(2.16) $$\frac{dy_i}{dt} = \lambda_i y_i.$$

This says that *the initial system* (1.1) *is transformed to the system* (2.16) *by the transformation*

(2.17) $$y_i = \Phi_i^{-1}[f(x), t].$$

Now, from (2.9) and (2.14), it is evident that the functions $\Phi_i^{-1}[f(x), t]$ are of the form (1.2). Thus we see that Theorem 3 is valid for the case 1°.

3. Case 2°. Let the formal tranformation given by the hypotheses be

(3.1) $$y_i = x_i + \sum_{\mathfrak{p}}{}'' k_{i\mathfrak{p}}(t)\, x_1^{p_1} \cdots x_n^{p_n},$$

where $k_{i\mathfrak{p}}(t)$ are periodic with period ω. Then the set of formal series

(3.2) $$x_i = y_i + \sum_{\mathfrak{p}}{}'' l_{i\mathfrak{p}}(t)\, y_1^{p_1} \cdots y_n^{p_n}$$

obtained by solving (3.1) inversely with respect to x_i becomes the formal solution of (1.1) if we put

(3.3) $$\frac{dy_i}{dt} = \lambda_i y_i.$$

Since the solution of (3.3) is expressed as $y_i = C_i e^{\lambda_i t}$ for arbitrary constants C_i, the set of formal series

(3.4) $$\psi_i(C, t) = C_i e^{\lambda_i t} + \sum_{\mathfrak{p}}{}'' l_{i\mathfrak{p}}(t)\,(C_1 e^{\lambda_1 t})^{p_1} \cdots (C_n e^{\lambda_n t})^{p_n}$$

becomes the formal solution of (1.1) with respect to C_j. Now, (3.2) is a formal solution of (3.1), consequently $l_{i\mathfrak{p}}(t)$ are priodic with period ω. Then, from (3.4) and the hypothesis that $\lambda_i \omega$ are all integral multiples of $2\pi i$, we see that

(3.5) $$\psi_i(C, \omega) = \psi_i(C, 0).$$

Let the formal series obtained by substituting

(3.6) $$C_i = x_i + \sum_{\mathfrak{p}}{}'' k_{i\mathfrak{p}}(0)\, x_1^{p_1} \cdots x_n^{p_n}$$

into $\psi_i(C, t)$ be $\varphi_i(x, t)$, then $\varphi_i(x, t)$ is also a formal solution of (1.1) and moreover

$$\varphi_i(x, 0) = \psi_i(C, 0) = C_i + \sum_{\mathfrak{p}}{}'' l_{i\mathfrak{p}}(0)\, C_1^{p_1} \cdots C_n^{p_n} = x_i,$$

since (3.2) is a formal solution of (3.1). As is seen in §2, such a formal solution is unique, consequently it coincides with an actual solution of (1.1), in other words, the set of formal series $\varphi_i(x, t)$ converges and expresses the actual solution of (1.1) such that $\varphi_i(x, 0) = x_i$. From (3.4), and (3.6), $\varphi_i(x, t)$ are of the form (2.6). Moreover, from (3.5), it holds that $\varphi_i(x, \omega) = x_i$. Then the transformation T defined by (2.8) becomes an identical transformation, consequently, of course, the group of transformations $T^k (k = 0, \pm 1, \pm 2, \cdots)$ becomes majorized. Then, by §2, the conclusion of Theorem 3 follows. Thus we see that Theorem 3 is valid also for the case 2°.

In addition, since $\varphi_i(x, \omega) = x_i = e^{\lambda_i \omega} x_i$, in the present case, the functions $f_i(x)$ required to satisfy (2.10) may be supposed to be x_i. Then, from (2.17), we see that, in the present case, *an analytical transformation meeting the requirement is given by*

(3.7) $$y_i = \Phi_i^{-1}(x, t)$$

for Φ_i *defined by* (2.14).

4. Supplement to the case 2°. In the case where $\lambda_i\omega$ are all integral multiples of $2\pi i$, let us consider the following three conditions:

1° *the solution* $\varphi_i(x, t)$ *is majorized*;

2° *the relation* $\varphi_i(x, \omega)=x_i$ *holds*;

3° *there exists a formal transformation of the form* (1.2) *by which the system* (1.1) *is reduced to the system* (1.3).

When 1° is fulfilled, by (2.10) it is valid that

$$f_i\left[\varphi(x, \omega)\right]=f_i(x),$$

from which follows 2° since $f_i(x)$ are of the form (2.9).

When 2° is fulfilled, if we take sufficiently small $\delta>0$, then there exists a positive number M such that

$$|\varphi_i(x, t)|\leqq M \tag{4.1}$$

for $|x|\leqq\delta$ and $-\infty<t<\infty$, because, from the present conditions, follows the periodicity with respect to t of the functions $\varphi_i(x, t)$. From (4.1) readily follows 1°.

When 3° is fulfilled, by the reasonings of §3, 2° is valid.

Now, when 1° is valid, by the reasonings of §2, the conclusion of Theorem 3 is valid, consequently, of couse, 3° is valid.

Thus, corresponding to Theorem 7 in [P] [8], we have

Theorem 4. *When* $\lambda_i\omega$ *are all integral multiples of* $2\pi i$, *the above three conditions* 1°, 2° *and* 3° *are equivalent to one another and, provided that one of these conditons is fulfilled, the system* (1.1) *is reduced to the system* (1.3) *by an analytic transformation of the form* (1.2) *given by* (3.7).

5. Remarks. Of the functions $f_i(x)$ for which (2.10) holds, the functions given by Theorem 1 are

$$f_i(x)=\lim_{n\to\infty}\frac{1}{n}\sum_{p=0}^{n-1}e^{-p\lambda_i\omega}\varphi_i(x, p\omega). \tag{5.1}$$

But if we adopt the method of proof of Theorem 2 [9], we can obtain another functions satisfying (2.10). Namely we consider the functions

$$F_i^{n,t}(x)=\frac{1}{n\omega}\int_t^{t+n\omega}e^{-\lambda_i\tau}\varphi_i(x, \tau)\,d\tau, \tag{5.2}$$

8) [P], p. 281.

9) [P], pp. 278-280.

where n is an arbitrary positive integer. Since the set of functions $\varphi_i(x, t)$ is majorized, by the reasonings of [P] p. 278, for a suitable sequence $\{n_l\}$, there exist limit functions $f_i(x)$ independent of t such that

$$f_i(x) = \lim_{l\to\infty} F_i^{n_l,t}(x) = x_i + [x]_2. \tag{5.3}$$

Now, from (5.2), $F_i^{n,t}[\varphi(x, \omega)]$ are calculated in the following way:

$$\begin{aligned} F_i^{n,t}[\varphi(x, \omega)] &= \frac{1}{n\omega}\int_t^{t+n\omega} e^{-\lambda_i\tau}\varphi_i[\varphi(x, \omega), \tau]\, d\tau \\ &= \frac{1}{n\omega}\int_t^{t+n\omega} e^{-\lambda_i\tau}\varphi_i(x, \tau+\omega)\, d\tau \\ &= \frac{1}{n\omega}\int_{t+\omega}^{t+\omega+n\omega} e^{-\lambda_i(\tau-\omega)}\varphi_i(x, \tau)\, d\tau \\ &= e^{\lambda_i\omega} F_i^{n,t+\omega}(x). \end{aligned}$$

Consequently, putting $n=n_l$ and making $l\to\infty$, from (5.3), we have the relation (2.10). Therefore the functions $f_i(x)$ given by (5.3) also meet the requirement. Now, from (2.10), as in [P] pp. 279–280, it is easily shown that the limit of $F_i^{n,t}(x)$ as $n\to\infty$ is unique, consequently the functions $f_i(x)$ given by (5.3) can be expressed as

$$f_i(x) = \lim_{n\to\infty}\frac{1}{n\omega}\int_0^{n\omega} e^{-\lambda_i\tau}\varphi_i(x, \tau)\, d\tau, \tag{5.4}$$

since $f_i(x)$ are independent of t. Since the set of $\varphi_i(x, t)$ is majorized, the right hand side of (5.4) can also be expressed as follows:

$$\lim_{t\to\infty}\frac{1}{t}\int_0^{t} e^{-\lambda_i\tau}\varphi_i(x, \tau)\, d\tau. \tag{5.5}$$

Thus we see that the functions given by (5.4) or (5.5) are also the functions for which (2.10) holds.

The set of functions given by (5.4) or (5.5) does not necessarily coincide with that of functions given by (5.1). The following example illustrates this fact.

Example.

$$(E) \qquad \begin{cases} \dfrac{dx_1}{dt} = \lambda_1 x_1, \\ \dfrac{dx_2}{dt} = \lambda_2 x_2 + x_1^2 \sin t, \end{cases}$$

where $\Re(\lambda_i)=0$, $\omega=2\pi$ and $\lambda_2=2\lambda_1$.

The solution $\varphi_i(x, t)$ of (E) is easily obtained as follows:

$$(\varphi) \qquad \begin{cases} \varphi_1(x, t) = x_1 e^{\lambda_1 t}, \\ \varphi_2(x, t) = x_2 e^{\lambda_2 t} + x_1^2 e^{2\lambda_1 t}(1 - \cos t). \end{cases}$$

The sets of functions given by (5.1) and (5.5) are respectively as follows:

$$(f) \qquad \begin{cases} f_1 = x_1, \\ f_2 = x_2, \end{cases} \qquad \begin{cases} f_1 = x_1, \\ f_2 = x_2 + x_1^2. \end{cases}$$

It is easily verified that both sets of functions satisfy (2.10) for same $\varphi_i(x, 2\pi)=x_i e^{2\pi\lambda_i}$. From (f), the sets of functions Φ_i are calculated and it is seen that two sets of Φ_i coincide with each other. The common set of Φ_i is as follows:

$$\begin{cases} \Phi_1 = x_1 e^{\lambda_1 t}, \\ \Phi_2 = x_2 e^{\lambda_2 t} + x_1^2 e^{2\lambda_1 t}(1 - \cos t). \end{cases}$$

Lastly, corresponding to two sets of f_i, the transformations given by (2.17) are sought as follows:

$$\begin{cases} y_1 = x_1, \\ y_2 = x_2 + x_1^2(-1 + \cos t); \end{cases} \qquad \begin{cases} y_1 = x_1, \\ y_2 = x_2 + x_1^2 \cos t. \end{cases}$$

It is easily verified that both transformations reduce the system (E) to the system

$$\begin{cases} \dfrac{dy_1}{dt} = \lambda_1 y_1, \\ \dfrac{dy_2}{dt} = \lambda_2 y_2. \end{cases}$$

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