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On Analysis of Variance for the Split-plot Designs.

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The purpose of the present paper is to make clear the mechanism of analysis of variance for the split-plot designs from the formal point of view. In my opinion it seems that no attempt has been made to give rigorous treatments which apply to a general case of the processes, although usually explanations are given in the literature through typical examples. The main theorems fundamental in our treatment in §2 will meet all our requirements. Throughout this paper we use frequently the results and notations in [2] and [3], which will be assumed to be known.

1. Some notations and definitions

Let I be an index set of N elements, say, 1, 2, ..., N. Let $\mathfrak{A}=(A_1, A_2, \cdots, A_l)$ be a classification of I, where A_i are non void disjoint subsets and $I=\sum Ai$. Each A_i is termed an $\mathfrak{A}$-class. Let N_{A_i} denote the number of elements contained in A_i. $\mathfrak{A}$ is called regular when N_{A_i} are independent of i.

Let $x_1, x_2, \cdots, x_N$ be random variables with normal distributions having the same mean zero and the satandard deviation σ^2. The sum of squares $S_{\mathfrak{A}}$ between $\mathfrak{A}$-classes is given by $\sum \frac{(A_i)^2}{N_{A_i}} - \frac{(I)^2}{N}$, where (A) denotes $\sum_{i \in A} x_i$. We consider an N-dimensional Euclidean space R^N. With each A we associate a vector e_A whose i-th component is 1 or 0 according as $i \in A$ or not. Let P_A denote the projective operator whose range is the subspace spanned by e_A. If we put

$$P_{\mathfrak{A}} = \sum P_{A_i} - P_I,$$

then $P_{\mathfrak{A}}$ is projective and $\|P_{\mathfrak{A}}\mathfrak{x}\|^2 = S_{\mathfrak{A}}$, where $\mathfrak{x}$ is a joint variable $(x_1, x_2, \cdots, x_N)$. We denote by P_0 the projection associated with the total variations.

2. The main theorems

This section is devoted to the establishment of the main theorems, which will make clear the mechanism of the analysis of variance for the split-plot designs.

In this section $\mathfrak{C}$ stands for a fixed regular classification where $N_{c_i}=s$. And we make for $\mathfrak{C}$ the following assumptions according to the usual process in the split-plot designs:

$$(1) \qquad E(x_i, x_j) = \begin{cases} \rho\sigma^2 & \textit{for } i, j \ (i \neq j) \textit{ in the same } \mathfrak{C}\textit{-class}, \\ 0 & \textit{for } i, j \textit{ in the different } \mathfrak{C}\textit{-classes}, \end{cases}$$

where ρ *is a positive constant* <1.

THEOREM 1. *For any projection* $P \leqq P_0$, *the following statements hold.*

(2) $P \leqq P_{\mathfrak{C}}$ *if and only if* $\frac{1}{(1+(s-1)\rho)\sigma^2}\|P\mathfrak{x}\|^2$ *is distributed in a* χ^2*-distribution.*

(3) $PP_{\mathfrak{C}}=0$ *if and only if* $\frac{1}{(1-\rho)\sigma^2}\|P\mathfrak{x}\|^2$ *is distributed in a* χ^2*-distribution.*

PROOF. Let M be the moment matrix of $\mathfrak{x}$ ([1], p. 295). Then $M=(1-\rho)\sigma^2E+\rho\sigma^2K$, where E is a unit matrix and

$$K = \begin{pmatrix} \overbrace{1\ 1\cdots 1}^{s} & & & 0 \\ \cdots\cdots\cdots & & & \\ 1\ 1\cdots 1 & & & \\ & 1\ 1\cdots 1 & & \\ & \cdots\cdots\cdots & & \\ & 1\ 1\cdots 1 & & \\ & & \ddots & \\ & & & 1\ 1\cdots 1 \\ 0 & & & \cdots\cdots\cdots \\ & & & 1\ 1\cdots 1 \end{pmatrix}$$

A calculation shows us that det. $M=\{\sigma^2+(s-1)\rho\sigma^2\}^l(\sigma^2-\rho\sigma^2)^{(S-1)l}\neq 0$, so that M is non singular.

Let C, A be the matrices corresponding to $P_{\mathfrak{C}}$ and P respectively. It is not difficult to see that we can write C in the form

$$C=\frac{1}{s}K-\frac{1}{N}J, \quad \text{where } J=\begin{pmatrix} 1\ 1\cdots 1 \\ \cdots\cdots\cdots \\ 1\ 1\cdots 1 \end{pmatrix}.$$

Then

$$(4) \qquad \begin{aligned} AMA &= (1-\rho)\sigma^2A+\rho\sigma^2AKA \\ &= (1-\rho)\sigma^2A+\rho s\sigma^2A\left(C+\frac{1}{N}J\right)A \\ &= (1-\rho)\sigma^2A+\rho s\sigma^2ACA. \end{aligned}$$

Now we give the proof of the theorem.

Necessity. Ad (2). As $P \leqq P_{\mathfrak{C}}$, we have $AC = A$, and hence (4) yields $AMA = (1-\rho)\sigma^2 A + \rho s \sigma^2 A = (1 + (s-1)\rho)\sigma^2 A$. Therefore we have

$$\frac{1}{(1+(s-1)\rho)\sigma^2} AM \frac{1}{(1+(s-1)\rho)\sigma^2} A = \frac{1}{(1+(s-1)\rho)\sigma^2} A.$$

It follows from Theorem 1 in [2] that $\frac{1}{(1+(s-1)\rho)\sigma^2} \|P\mathfrak{x}\|^2$ is distributed in a χ^2-distribution.

Ad (3). As $PP_{\mathfrak{C}} = 0$, we have $AC = 0$, and hence (4) yields

$$AMA = (1-\rho)\sigma^2 A.$$

Thus $\frac{1}{(1-\rho)\sigma^2} \|P\mathfrak{x}\|^2$ is distributed in a χ^2-distribution.

Sufficiency. Ad (2). As $\frac{1}{(1+(s-1)\rho)\sigma^2} \|P\mathfrak{x}\|^2$ is distributed in a χ^2-distribution, it follows from the theorem cited above that $AMA = (1 + (s-1)\rho)\sigma^2 A = (1-\rho)\sigma^2 A + \rho s \sigma^2 A$. Then (4) gives $ACA = A$. Since $(AC - A)(AC - A)^* = (AC-A)(CA-A) = ACA - ACA - ACA + A = 0$, we have $AC = A$. This implies $PP_{\mathfrak{C}} = P$, that is $P \leqq P_{\mathfrak{C}}$, as desired.

Ad (3). As $\frac{1}{(1-\rho)\sigma^2} \|P\mathfrak{x}\|^2$ is distributed in a χ^2-distribution, we have $AMA = (1-\rho)\sigma^2 A$ by the same reason as above. Then (4) gives $ACA = 0$. Since $AC(AC)^* = ACA = 0$, we have $AC = 0$. This implies $PP_{\mathfrak{C}} = 0$. Thus the theorem is completely proved.

We note that d. f. (degrees of freedom) of those χ^2-distributions in Theorem 1 coincide with the formal d. f. of $\|P\mathfrak{x}\|^2$, that is, $\mathrm{tr}\, A$ (cf. §5 of [2]).

REMARK: The projections P in (2), (3) are respectively related to the analysis of variance with respect to Units and Sub-units in the split-plot designs.

THEOREM 2. *Let P satisfy (2) or (3) in Theorem 1, and Q be any projection, then $PQ = 0$ holds if and only if $\|P\mathfrak{x}\|^2$ and $\|Q\mathfrak{x}\|^2$ are independent.*

PROOF. Let A, B be matrices corresponding to P, Q respectively. Then after a calculation we have

$$\begin{aligned} AMB &= (1-\rho)\sigma^2 AB + \rho s \sigma^2 ACB \\ &= \begin{cases} (1 + (s-1)\rho)\sigma^2 AB & \text{for } P \leqq P_{\mathfrak{C}}, \\ (1-\rho)\sigma^2 AB & \text{for } PP_{\mathfrak{C}} = 0. \end{cases} \end{aligned}$$

Consequently $AMB = 0$ if and only if $AB = 0$. It follows from Theorem 3 of [2] that $PQ=0$ holds if and only if $\|P\mathfrak{x}\|^2$ and $\|Q\mathfrak{x}\|^2$ are independent, completing the proof.

From this theorem we see that ρ has no concern with the mutual independency of quadratic quantities appearing in the analysis of variance for the split-plot designs.

As a consequence of the preceding theorems we have

THEOREM 3. *Let* $\mathfrak{A}_1, \mathfrak{A}_2, \cdots, \mathfrak{A}_m$ *be regularly orthogonal classifications* ([3], p. 460, Definition), *and let* $\mathfrak{A}_1, \mathfrak{A}_2, \cdots, \mathfrak{A}_m, \mathfrak{B}_1, \mathfrak{B}_2, \cdots, \mathfrak{B}_n$ be *orthogonal. Put* $\mathfrak{C}=\mathfrak{A}_1\wedge\mathfrak{A}_2\wedge\cdots\wedge\mathfrak{A}_m$ *and suppose that* $\mathfrak{C}$ *fulfils the assumptions* (1), *then*

$$\frac{1}{(1+(s-1)\rho)\sigma^2}\|P_{\mathfrak{A}_1}\mathfrak{x}\|^2,\ \frac{1}{(1+(s-1)\rho)\sigma^2}\|P_{\mathfrak{A}_2}\mathfrak{x}\|^2,\ \cdots,$$

$$\frac{1}{(1+(s-1)\rho)\sigma^2}\|P_{\mathfrak{A}_1\mathfrak{A}_2\cdots\mathfrak{A}_m}\mathfrak{x}\|^2,$$

$$\frac{1}{(1-\rho)\sigma^2}\|P_{\mathfrak{B}_1}\mathfrak{x}\|^2,\ \frac{1}{(1-\rho)\sigma^2}\|P_{\mathfrak{B}_2}\mathfrak{x}\|^2,\ \cdots,\ \frac{1}{(1-\rho)\sigma^2}\|P_{\mathfrak{B}_1\mathfrak{B}_2}\mathfrak{x}\|^2,\ \cdots,$$

$$\frac{1}{(1-\rho)\sigma^2}\|P_{\mathfrak{B}_1\mathfrak{B}_2\cdots\mathfrak{B}_n\mathfrak{A}_1\mathfrak{A}_2\cdots\mathfrak{A}_m}\mathfrak{x}\|^2$$

are distributed in χ^2*-distributions, and are all mutually independent. The degrees of freedom of these* χ^2*-distributions coincide with the formal ones.*

3. Example

As an illustration of our theorem 3, we consider an example which is chosen from Cochran and Cox ([4] p. 238) and slightly modified. In their terminology: "*The factor A applied to the Units has* α *levels, while the sub-units treatments are the* 8 *combinations of factor B, C and D at* 2 *levels each, and ABCD is completely confounded between sub-blocks for the aim of obtaining more precise estimate of A and BCD.*"

Notation : $\mathfrak{A}_1$ = classification due to replications.

$\mathfrak{A}_2$ = classification due to the factor *A*.

$\mathfrak{A}_3$ = classification due to the factor *BCD*.

$\mathfrak{B}_1$ = classification due to the factor *B*.

$\mathfrak{B}_2$ = classification due to the factor *C*.

Then the interpretations by our notations and d. f. in the analysis of variance for this split-plot design are shown in the following table.

TABLE. $\alpha \times 2^3$ *design with split-plot confounding* (*r replicates*).

Units	Interpretations	d. f.
Between sub-blocks	$\mathfrak{A}_1 \wedge \mathfrak{A}_2$	$\alpha r-1$
A	$\mathfrak{A}_2$	$\alpha-1$
BCD	$\mathfrak{A}_3$	1
Error (a)	$(\mathfrak{A}_1 \wedge \mathfrak{A}_2)\mathfrak{A}_3 - \mathfrak{A}_2$	$\alpha(r-1)$
Total between units	$\mathfrak{A}_1 \wedge \mathfrak{A}_2 \wedge \mathfrak{A}_3$	$2\alpha r-1$
Sub-units		
B	$\mathfrak{B}_1$	1
C	$\mathfrak{B}_2$	1
BC	$\mathfrak{B}_1\mathfrak{B}_2$	1
AB	$\mathfrak{B}_1\mathfrak{A}_2$	$\alpha-1$
CD	$\mathfrak{B}_1\mathfrak{A}_3$	1
AC	$\mathfrak{B}_2\mathfrak{A}_2$	$\alpha-1$
BD	$\mathfrak{B}_2\mathfrak{A}_3$	1
ACD	$\mathfrak{B}_1\mathfrak{A}_2\mathfrak{A}_3$	$\alpha-1$
ABD	$\mathfrak{B}_2\mathfrak{A}_2\mathfrak{A}_3$	$\alpha-1$
ABC	$\mathfrak{B}_1\mathfrak{B}_2\mathfrak{A}_2$	$\alpha-1$
D	$\mathfrak{B}_1\mathfrak{B}_2\mathfrak{A}_3$	1
AD	$\mathfrak{B}_1\mathfrak{B}_2\mathfrak{A}_2\mathfrak{A}_3$	$\alpha-1$
Error (b)	$\{\mathfrak{A}_1 + \mathfrak{A}_1(\mathfrak{A}_2 \wedge \mathfrak{A}_3)\}(\mathfrak{B}_1 \wedge \mathfrak{B}_2)$	$6\alpha(r-1)$
Grand total	$\mathfrak{A}_1 \wedge \mathfrak{A}_2 \wedge \mathfrak{A}_3 \wedge \mathfrak{B}_1 \wedge \mathfrak{B}_2$	$8\alpha r-1$

References

[1] H. Cramér: Mathematical Methods of Statistics, Princeton, 1946

[2] T. Ogasawara and M. Takahashi: Independence of quadratic quantities in a normal system, this journal 15 (1951) 1-9.

[3] T. Ogasawara and M. Takahashi: Orthgonality relation in the analysis of variance I, this journal 16 (1953) 457-470.

[4] W. G. Cochran and G. M. Cox: Experimental designs, London, 1950.