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Topologies on Rings of Operators

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In the course of the investigations of the properties of a ring $\mathbb{M}$ of operators on a Hilbert space six different topologies have been introduced into the ring $\mathbb{M}$ by various writers ([**12**], [**13**], [**4**], [**5**], [**9**]) : *uniform*, $\sigma(\mathbb{M}, \mathbb{M}^*)$, *ultrastrong*, *ultraweak*, *strong* and *weak topologies*. In comparing the different topologies we use the words "stronger than" to mean "at least as strong" and give this meaning to the symbol $>$. Among these topologies the following relations hold :

$$\begin{array}{ccccc} \text{Uniform top.} & > & \text{Ultrastrong top.} & > & \text{Strong top.} \\ \vee & & \vee & & \vee \\ \sigma(\mathbb{M}, \mathbb{M}^*) & > & \text{Ultraweak top.} & > & \text{Weak top.} \end{array}$$

It is the main purpose of this paper to study the conditions for any assigned two topologies on a ring $\mathbb{M}$ among these six ones cited above to coincide.

§ **1** includes the preliminaries to the rest of the paper. Although most of the results of §1 can be found in the literature as the references indicate, the proofs are given them for the sake of completeness of our treatment. In §**2** we give the conditions of equivalence of two topologies "ultraweak" and "weak" (or "ultrastrong" and "strong") on $\mathbb{M}$. The results obtained are closely related to those of Dye [**7**], Griffin [**9**] and Dixmier [**5**]. § **3** is devoted to the discussions of some properties of cyclic projections. In § **4** we show that any two topologies on $\mathbb{M}$ (besides the cases mentioned above) coincide if and only if $\mathbb{M}$ is finite-dimensional.

The results of the papers of Dixmier [**1**], [**2**] and [**3**] are assumed to be known and will be used without further reference.

§ I. General remarks on the spatial properties of a rings of operators

In what follows, unless otherwise stated, $\mathbb{M}$ stands for a ring of opreators (containing the identity operator I) on a Hilbet space H. Let K be a Hilbert space of dimension α. The mapping $A \rightarrow A \otimes I$ of $\mathbb{M}$ onto a ring $\mathbb{M} \otimes I$ on the Hilbert space $H \otimes K$ is called the *ampliation* of order α and $M \otimes I$ will be denoted by $\mathbb{M}^{(\alpha)}$, the *α-fold copy of* $\mathbb{M}$ (Cf. [**5**], [**6**]). It is not difficult to see that if there exists a homogeneous partition $\{\mathfrak{M}_\iota\}$ of H in $\mathbb{M}'$, then $\mathbb{M}$ is spatially isomorphic to $(\mathbb{M}_{\mathfrak{m}_1})^{(\alpha)}$, where $\mathbb{M}_{\mathfrak{m}}$ denotes as usual the ring of operators on $\mathfrak{M}$ formed by the portions on $\mathfrak{M}$ of operators in $\mathbb{M}$.

A linear form φ on $\mathbb{M}$ is called a state if $\varphi(A^*A) \geq 0$ for every $A \in \mathbb{M}$, and a trace if furthermore $\varphi(AB) = \varphi(BA)$ holds for every A, $B \in \mathbb{M}$. The following statements for a linear form φ on $\mathbb{M}$ are well-known ([**4**], [**6**]):

(*1*) φ is weakly (resp. ultraweakly) continuous if and only if φ is strongly (resp. ultrastrohgly) continuous;

(*2*) *if* $\varphi(A) = \sum_{j=1}^{n} (Ay_j, z_j)$ $\{$*resp.* $= \sum_{j=1}^{\infty} (Ay_j, z_j)$, $\sum \|y_j\|^2 < +\infty$, $\sum \|z_j\|^2 < +\infty$,$\}$ is a state, then φ is of the form $\varphi(A) = \sum_{j=1}^{n} (Ax_j, x_j)$ $\{$*resp.* $= \sum_{j=1}^{\infty} (Ax_j, x_j)$, $\sum \|x_j\|^2 < +\infty\}$.

By virtue of (*1*) and (*2*) we see [**5**] that

(*3*) the weak and ultraweak topologies on $\mathbb{M}$ coincide if and only if so do the strong and ultrastrong topologies on $\mathbb{M}$.

A state φ is called countably additive if $\varphi(\sum P_n) = \sum \varphi(P_n)$ for each sequence $\{P_n\}$ of mutually orthogonal projections in $\mathbb{M}$, and completely additive if $\varphi(\sum P_\iota) = \sum \varphi(P_\iota)$ for each family $\{P_\iota\}$ of mutually orthogonal projections in $\mathbb{M}$. A state φ is called *normal* if $\varphi(A_\delta) \uparrow \varphi(A)$ for each monotone incresing directed set $\{A_\delta\}$ of positive operators $\mathbb{M}$ with $A_\delta \uparrow A$.

Lemma I. *The following statements for a state φ on $\mathbb{M}$ are equivalent:*

(*i*) *φ is countably additive and there exists a σ-finite* (= *countably decomposable*) *projection Q in $\mathbb{M}$ with $\varphi(Q^\perp) = 0$;*

(*ii*) *φ is completely additive;*

(*iii*) *φ is normal;*

(*iv*) *φ is ultraweakly continuous.*

Proof. The implications (*iv*) $\rightarrow$ (*iii*) $\rightarrow$ (*ii*) $\rightarrow$ (*i*) are clear. To prove (*i*) $\rightarrow$ (*iv*), owing to the Schwarz inequality for a state, we may assume that $Q = I$, so that φ is completely additive and I is σ-finite. The following proof is patterned after the proof of the implication (*iii*) $\rightarrow$ (*iv*) due to Dixmier [**5**]. For any non-zero projection Q in $\mathbb{M}$, and for a vector z with $\varphi(Q) \leq (Qz, z)$, there exists a non-zero projection Q' in $\mathbb{M}$ such that for each projection $P \in \mathbb{M}$ with $P \leq Q'$ we have $\varphi(P) \leq (Pz, z)$. Indeed, if we suppose the contrary, the complete additivity of φ would yield $\varphi(Q) > (Qz, z)$, a contradiction. Therefore there exists a sequence $\{Q_n\}$ of mutually orthogonal projections in $\mathbb{M}$ such that $\sum Q_n = I$ and for each n there exists a vector $z_n \in Q_n H$ with the property that $\varphi(P) \leq (Pz_n, z_n)$ for every projection P in $\mathbb{M}$ with $P \leq Q_n$. By making use of the spectral resolution of positive operators in $\mathbb{M}$, we have $\varphi(Q_n A^* A Q_n) \leq \|Az_n\|^2$. Then by the Schwarz inequality we have $|\varphi(Q_n A)|^2 \leq \varphi(Q_n)\varphi(Q_n A^* A Q_n) \leq \varphi(I)\|Az_n\|^2$. A lemma of Riesz shows us that $\varphi(Q_n A)$ is of the form (Ax_n, y_n). Therefore $\psi_n(A) = \varphi((\sum_{j=1}^n Q_j)A)$ is weakly continuous. And $|\varphi(A) - \psi_n(A)|^2 = |\varphi((1 - \sum_{j=1}^n Q_j)A)|^2 \leq \varphi(1 - \sum_{j=1}^n Q_j)\varphi(A^* A)\ (\leq \varphi(I)^2 \|A\|^2$. Since the set of ultraweakly continuous linear forms on $\mathbb{M}$ is a closed subspace of the dual $\mathbb{M}^*$ [**5**], it follows from $\|\varphi - \psi_n\| \to 0$ that φ is ultraweakly continuous. The proof is complete.

Now we define the length of a normal state φ on $\mathbb{M}$. φ is of the form $\sum (Ax_n, x_n)$, $\sum \|x_n\|^2 < +\infty$. The least number (finite or $+\infty$) of the canonical states (Ax, x) by whose sum φ may be represented will be called the *length* of φ. That every normal state is of finite length is equivalent to saying that in the ring considered the two topologies "weak" and "ultraweak" coincide.

A projection Q (resp. the range of Q) in $\mathbb{M}$ is termed a *carrier projection* (resp. *subspace*) of a normal state φ [7] if Q is the minimal projection in $\mathbb{M}$ such $\varphi(Q^\perp) = 0$. From the form of φ given above we see that $Q = \cup P_{[\mathbb{M}' x_n]}$. Evidently Q is σ-finite. Conversely any σ-finite projction Q can be written in the form $\cup P_{[\mathbb{M}' x_n]}$, $\sum \|x_n\|^2 < +\infty$, so that Q is the carrier projection of a normal state $\sum (Ax_n, x_n)$. The length of a σ-finite projection Q is defined as the least number (finite or $+\infty$) of cyclic projections $P_{[\mathbb{M}' x]}$ by whose union Q may be represented. We shall prove later on that a normal state and its carrier projection have the same length. The length of a ultraweakly continuous linear form on $\mathbb{M}$ may be defined as in the case of a nomal state on $\mathbb{M}$.

In the following discussions we shall often make use of a theorem established by the present author and K. Yoshinaga ([**15**], Theorem 3) to the effect that in the left ring $\mathbb{L}$ of an H-system every normal state on $\mathbb{L}$ is at most of length *1*. The proof was carried out by an elementary way. The following lemma is well-known and plays a fundamental role in the study of the spatial properties of a ring of operators.

Lemma 2. ([**9**]. [**16**], [**6**]). *For a given $x \in H$ we have the following statements:*

(*i*) $[\mathbb{M}x]$ *is finite in* $\mathbb{M}'$ *if so is for* $[\mathbb{M}'x]$ *in* $\mathbb{M}$;

(*ii*) $[\mathbb{M}x]$ *is irreducible in* $\mathbb{M}'$ *if so is for* $[\mathbb{M}'x]$ *in* $\mathbb{M}$;

(*iii*) $[\mathbb{M}x]$ *is properly infinite and semi-finite in* $\mathbb{M}'$ *if so is for* $[\mathbb{M}'x]$ *in* $\mathbb{M}$;

(*iv*) $[\mathbb{M}x]$ *is purey infinite in* $\mathbb{M}'$ *if so is for* $[\mathbb{M}'x]$ *in* $\mathbb{M}$.

Proof. By virtue of the $*$-isomorphisms of $\mathbb{M}_{[\mathbb{M}'x]}\to\mathbb{M}_{[\mathbb{M}x]\cap[\mathbb{M}'x]}$ and $\mathbb{M}'_{[\mathbb{M}x]}\to\mathbb{M}'_{[\mathbb{M}x]\cap[\mathbb{M}'x]}$ under the natural mappings we may assume that $H=[\mathbb{M}x]=[\mathbb{M}'x]$. Suppose $\mathbb{M}$ is semi-finite and let $\mathbb{L}$ be the left ring of an associated H-system $\hat{H}$ of $\mathbb{M}$. $\mathbb{M}$ and $\mathbb{L}$ are $*$-isomorphic and we denote by $\hat{A}$ the operator in $\mathbb{L}$ corresponding to A in $\mathbb{M}$ under this $*$-isomorphism. Put $\varphi(A)=(Ax, x)$ and define $\hat{\varphi}(\hat{A})$ by the equation $\hat{\varphi}(\hat{A})=\varphi(A)$. Then $\hat{\varphi}$ is a normal state on $\mathbb{L}$, so that we can write $\hat{\varphi}(\hat{A})=(\hat{A}\hat{x},\hat{x})$ for some positive element $\hat{x}$ in $\hat{H}$. The carrier projection of $\hat{\varphi}$ is the identity since so is for φ. This means that $[\mathbb{L}'\hat{x}]=\hat{H}$. Let S be the conjugation of $\hat{H}$, that is, $S\hat{y}=\hat{y}^*$. As $\mathbb{L}'=S\mathbb{L}S$ and $\hat{x}$ is positive, we see that $[\mathbb{L}\hat{x}]=S[\mathbb{L}'\hat{x}]=\hat{H}$. Therefore the mapping $Ax\to\hat{A}\hat{x}$ can be uniquely extended to the unitary one W from $H=[\mathbb{M}x]$ onto $\hat{H}$. And it is easy to verify that $\hat{A}=WAW^{-1}$, so that $A\to\hat{A}$ is spatial. Thus we may identify $\mathbb{M}$ with $\mathbb{L}$ and x with $\hat{x}$. In an H-system, $\mathbb{L}$ is finite (resp. commutative, properly infinite) if and only if $\mathbb{L}'$ is finite (resp. commutative, properly infinite. So we have the statements (*i*), (*ii*) and (*iii*).

Next we shall prove (*iv*). If not, we may assume that $[\mathbb{M}'x]$ is purely infinite and $[\mathbb{M}x]$ is semi-finite. In the above discussion if we replace $\mathbb{M}$ by $\mathbb{M}'$, we shall obtain that $[\mathbb{M}'x]$ is semi-finite by (*iii*). This is a contradiction. The proof is complete.

As an immediate consequence of this lemma, it it not difficult to see that $\mathbb{M}$ is of type I (resp. II, III) if and only if $\mathbb{M}'$ is of type I (resp. II, III).

Lemma 3. ([**5**]). *Let* $\mathbb{M}'$ *be properly infinite. Every ultraweakly continuous linear form* φ *on* $\mathbb{M}$ *is at most of length 1. Especially every normal state on* $\mathbb{M}$ *is at most of length 1.*

Proof. As $\mathbb{M}'$ is assumed to be properly infinite, there exists a homogeneous partition $\{\mathfrak{M}_n\}_{0<n<+\infty}$ of H such that $\mathfrak{M}_i\sim\mathfrak{M}_j$ (mod $\mathbb{M}'$). Then $\mathbb{M}$ is identified with $\{\mathbb{M}_{\mathfrak{M}_1}\}^{(\aleph_0)}$. Owing to this identification the lemma will be clear from the expression of φ as series of terms (Ax_n, x_n), where $x_n, y_n\in\mathfrak{M}_1$ and $\sum\|x_n\|^2$, $\sum\|y_n\|^2<+\infty$.

A normal state ρ on $\mathbb{M}$ is called *absolutely continuous* with respect to a normal state φ on $\mathbb{M}$ [**7**] if $\varphi(P)=0$ implies $\rho(P)=0$ for every projection P in $\mathbb{M}$. The condition is clearly equivalent to saying that the carrier projection of ρ is contained in that of φ.

Lemma 4. *Let* φ *be a normal state on* $\mathbb{M}$ *of length n. Every normal state* ρ *on* $\mathbb{M}$ *absolutely continuous with respect* φ *is at most of length n.*

Proof. By considering the n-fold copy of $\mathbb{M}$, the proof is reduced to the case $n=1$. that is, φ is of the form (Ax, x). As ρ is absolutely continuous with respect to φ, the carrier projection of ρ is contained in the carrier projection Q of φ. Owing to the Schwarz inequality we have $\rho(A)=\rho(QAQ)$, so that we may assume $Q=I$, that is, $H=[\mathbb{M}'x]$. Let $\mathfrak{M}'=[\mathbb{M}x]$ and consider the ring $\mathbb{M}_{\mathfrak{M}'}$. If we put $\rho_1(A_{\mathfrak{M}'})=\rho(A)$, $\varphi_1(A_{\mathfrak{M}'})=\varphi(A)$, then ρ_1 and φ_1 are normal states on $\mathbb{M}_{\mathfrak{M}'}$ and ρ_1 is absolutely continuous with respect to φ_1. Thus we may assume that $H=[\mathbb{M}x]=[\mathbb{M}'x]$. By Lemma 3 we may also assume that $\mathbb{M}'$ is finite and a fortiori $\mathbb{M}$ is finite. Let $\hat{H}$ be an H-system associated with $\mathbb{M}$. Conserve the notations in the proof of Lemma 2. The proof of that lemma shows us that $\mathbb{M}$ is spatially isomorphic to the left ring $\mathbb{L}$ of $\hat{H}$. Define $\hat{\rho}$ by the equation $\hat{\rho}(\hat{A})=\rho(A)$. Then $\hat{\rho}$ is a normal state on $\mathbb{L}$ so that we can write $\hat{\rho}(\hat{A})=(\hat{A}\hat{y}, \hat{y})$, $\hat{y}$ being a vector in $\hat{H}$. Hence there exists a vector y in $H=[\mathbb{M}x]\frown[\mathbb{M}'x]$ such that $\rho(A)=(Ay, y)$, as was to be proved.

Remark. From the proof of the above lemma we see that if a normal state ρ on $\mathbb{M}$ is absolutely continuous with respect to a canonical state (Ax, x), then there exists a vector y in $[\mathbb{M}x]\frown[\mathbb{M}'x]$ such that we can write $\rho(A)=(Ay, y)$ for every A in $\mathbb{M}$.

As a consequence of this lemma we have

Corollary. *Let Q be a σ-finite projection in $\mathbb{M}$ and let ρ be a normal state on $\mathbb{M}$ with Q as its carrier projection. Then ρ and Q are of the same length.*

Proof. In order to show that the length of Q is not less than that of ρ, we may assume that Q is of finite length, say n. We can write Q in the form $\cup_{j=1}^{n} P_{[\mathbb{M}'x_j]}$. Put $\varphi(A)=\sum_{j=1}^{n}(Ax_j, x_j)$. Then the length of φ is at most n and ρ is absolutely continuous with respect to φ. The preceding lemma shows us that the length of ρ is at most n, as desired. The converse is trivial from the representation of ρ as a sum of canonical states induced by the representation of Q as a union of cyclic projections. The proof is complete.

As an immediate consequence of this lemma, it is clear that if P and Q are σ-finite projections in $\mathbb{M}$ such that $P\leq Q$, then the length of P is at most equal to that of Q. As already remarked, the two topologies "weak" and "ultraweak" coincide if and only if every normal state on $\mathbb{M}$ is of finite length. Therefore the preceding corollary yields the following.

Theorem I. *The weak and ultraweak (resp. strong and ultrastrong) topologies on $\mathbb{M}$ coincide if and only if every σ-finite projection in $\mathbb{M}$ is of finite length. Especially if $\mathbb{M}$ is σ-finite, the condition is reduced to that I is of finite length.*

This is a slight generalization of a result due to Dye ([7], Cor. 5. 2).

Let $\tilde{\mathbf{M}}$ be another ring of operators on a Hilbert space $\tilde{H}$. Suppose that $\mathbf{M}$ is $*$-isomorphic to $\tilde{\mathbf{M}}$ under the mapping $A \to \tilde{A}$. Let K be the direct sum of H and $\tilde{H}$ whose elements are denoted by $\{x, y\}$ where $x \in H$, $y \in \tilde{H}$. Then the set of operators $[A, \tilde{A}]$ defined by $[A, \tilde{A}]\ \{x, y\} = \{Ax, \tilde{A}y\}$ forms a ring $\mathbf{N}$ of operators on K. $\mathbf{M}$ (resp. $\tilde{\mathbf{M}}$) is identified with $\mathbf{N}_H$ (resp. $\mathbf{N}_{\tilde{H}}$). The mapping $A \to \tilde{A}$ is spatial if and only if $H \sim \tilde{H}$ (mod N'). This follows from a lemma of R. Pallu de La Barrière ([**16**], p. 34), or from a direct caluculation using the usual matrix representations of operators on K. It is noted that H, $\tilde{H} (\eta \mathbf{N}')$ have the same central envelope K in $\mathbf{N}'$. To study the equivalence $H \sim \tilde{H} (mod \mathbf{N}')$ we begin with the following

Lemma 5. *Let $\mathfrak{M}$, $\mathfrak{N}\ \eta\ \mathbf{M}$ have the same central envelope H. If any of the following conditions is satisfied, than $\mathfrak{M} \sim \mathfrak{N}$ (mod $\mathbf{M}$).*

(*i*) *$\mathfrak{M}$ and $\mathfrak{N}$ are irreducible;*

(*ii*) *$\mathfrak{M}$ and $\mathfrak{N}$ are purely infinite and σ-finite;*

(*iii*) *$\mathfrak{M}$ and $\mathfrak{N}$ are semi-finite, properly infinite and have the same algebraic invariant α* (Cf. [**16**]).

Proof. By virtue of the comparability theorem of Dixmier we may assume that $\mathfrak{M} \lesssim \mathfrak{N}$ (*mod* $\mathbf{M}$).

Ad (*i*). Let $\mathfrak{N}_1$ be such that $\mathfrak{M} \sim \mathfrak{N}_1 \subseteq \mathfrak{N}$. As H is the central envelope of $\mathfrak{M}$, we have $\mathfrak{N}_1^\natural = \mathfrak{M}^\natural = H$. If $\mathfrak{N} \ominus \mathfrak{N}_1 \neq (0)$, we can find non-zero subspaces $\mathfrak{N}'$, $\mathfrak{N}_1'\ \eta\ \mathbf{M}$ such that $\mathfrak{N}' \sim \mathfrak{N}_1'$, $\mathfrak{N}' \subseteq \mathfrak{N} \ominus \mathfrak{N}_1$, $\mathfrak{N}_1' \subseteq \mathfrak{N}_1$ since $(\mathfrak{N} \ominus \mathfrak{N}_1)^\natural \cap \mathfrak{N}_1^\natural \neq (0)$. This contradicts the irreducibility of $\mathfrak{N}$.

Ad (*ii*). Let $\{H_\alpha\}$ be a maximal central partition such that each $H_\alpha \cap \mathfrak{N}$ admits of a homogeneous partition $\{\mathfrak{N}_{\alpha, j}\}_{0<j<\infty}$ with $H_\alpha \cap \mathfrak{M} \sim \mathfrak{N}_{\alpha, j}$. By the comparatibility theorem cited above it is not difficult to see that $H = \sum \oplus H_\alpha$, so that $\mathfrak{N}$ admits of a homogeneous partition $\{\mathfrak{N}_k\}_{0<k<\infty}$ such that $\mathfrak{M} \sim \mathfrak{N}_k$ (*mod* $\mathbf{M}$). On the other hand as $\mathfrak{M}$ is purely infinite and σ-finite, $\mathfrak{M}$ admits of a homogeneous partition $\{\mathfrak{M}_k\}_{0<k<\infty}$ such that $\mathfrak{M} \sim \mathfrak{M}_k$ (mod $\mathbf{M}$). Hence $\mathfrak{M} \sim \mathfrak{N}$ (*mod* $\mathbf{M}$).

Ad (iii). Let $\{\mathfrak{M}_\iota\}_{\iota \in \mathfrak{J}}$ be a homogeneous partiton of $\mathfrak{M}$ such that $\mathfrak{M}_1$ is finite, where the power of the index set $\mathfrak{J}$ is α. It follows from $\mathfrak{M} \lesssim \mathfrak{N}$ that there exists $\mathfrak{N}_1 \subseteq \mathfrak{N}$ such that $\mathfrak{M}_1 \sim \mathfrak{N}_1$ (mod $\mathbf{M}$). Since $\mathfrak{N}_1^\natural = \mathfrak{M}_1^\natural = H$ and $\mathfrak{N}_1$ is finite, there exists a homogeneous partition $\{\mathfrak{N}_\iota\}_{\iota \in \mathfrak{J}}$ of $\mathfrak{N}$ of which $\mathfrak{N}_1$ is an element. Hence $\mathfrak{M} \sim \mathfrak{N}$ (mod $\mathbf{M}$). The proof is complete.

From this lemma we have immediately the following

Corollary. (*Cf.* [*8*], [*17*], [*16*]). *Let* $\mathbf{M}$, $\tilde{\mathbf{M}}$ *be rings of operators on Hilbert spaces*

H. $\tilde{H}$ respectively. If any of the following conditions is satisfied, then any $*$-isomorphism of $\mathbb{M}$ onto $\tilde{\mathbb{M}}$ is spatial.

(*i*) $\mathbb{M}'$ *and* $\tilde{\mathbb{M}}'$ *are commutative;*

(*ii*) $\mathbb{M}'$ *and* $\tilde{\mathbb{M}}'$ *are properly infinite and σ-finite;*

(*iii*) $\mathbb{M}'$ *and* $\tilde{\mathbb{M}}'$ *are properly infinite, semi-finite and have the same algebraic invariant.*

Remark. If we are only concerned separable Hilbert spaces, then any ring of operators is σ-finite, and the above corollary shows us that any $*$-isomorphism of rings of type III is spatial. Concerning the spatial isomorphism of rings of type III a more general result has been obtained by Griffin [**8**].

To go further into the discussin of spatial isomorphism of rings of operators it seems convenient to consider the unitary invariant C of a ring (Cf. [**10**], [**9**], [**16**]). To this end we shall first consider a dimension function of a ring of operators in a certain sense of Segal [**18**]. Let $\mathbb{M}$ be a semi-finite ring. We denote by Ω the spectre of the center $\mathbb{M}^{\natural}$ of $\mathbb{M}$. $\mathbb{M}^{\natural}$ is identified with the set of bounded continuous functions on Ω. Let $\boldsymbol{Z}$ be the set of non-negative valued (inclusive $+\infty$) continuous functions on Ω. The sum and product of any two elements of $\boldsymbol{Z}$ are defined to be continuous as observed by Dixmier [**3**]. A function d defined on $\mathbb{M}_P$ with values in $\boldsymbol{Z}$ will be called a dimension function of $\mathbb{M}$ if the following axioms are satisfied (For a detailed discussion of a dimension function see the paper of S. Maeda, this journal, 211–237):

(*a*) $d(P)=0$ *if and only if* $P=0$;

(*b*) $d(P+Q)=d(P)+d(Q)$ *for* $PQ=0$;

(*c*) $d(UPU^*)=d(P)$ *for every* $U\in\mathbb{M}_U$;

(*d*) $d(PQ)=d(P)Q$ *if* Q *is central;*

(*e*) $d(P)$ *is finite-valued except on a non-dense subset of* Ω *if* P *is finite.*

Such a dimension function d is unique in a certain sense, that is, if we let d' be another dimension function of $\mathbb{M}$, it is obtained from d by multiplying an element in $\boldsymbol{Z}$ which is positive and finite except on a non-dense subset of Ω. In the sequel if $\mathbb{M}$ is finite, we shall normalize a dimemsion function d in such a way that $d(I)=1$. Any dimension function is the restriction on $\mathbb{M}_P$ of a $\natural$-application which is normal, faithful and essential. If $\mathfrak{M}\,\eta\,\mathbb{M}$, $d(\mathfrak{M})$ is defined to be $d(P_{\mathfrak{M}})$. It is noted that for finite $\mathfrak{M}$, $\mathfrak{N}\,\eta\,\mathbb{M}$, the condition $\mathfrak{M}\sim\mathfrak{N}$ is equivalent to $d(\mathfrak{M})=d(\mathfrak{N})$. Let $\mathfrak{M}\,\eta\,\mathbb{M}$ be finite and suppose that $\mathfrak{M}^{\natural}=H$ The centers of $\mathbb{M}$ and $\mathbb{M}_{\mathfrak{M}}$ are $*$-isomorphic under the natural mapping, so that we can identify the spectre of $(\mathbb{M}_{\mathfrak{M}})^{\natural}$ with that of $\mathbb{M}^{\natural}$. Then the normalized dimension function of $\mathbf{M}_{\mathfrak{M}}$, denoted by $d_{\mathfrak{M}}$, is given by the equation $d_{\mathfrak{M}}(\mathfrak{N})=\dfrac{d(\mathfrak{N})}{d(\mathfrak{M})}$ $\mathfrak{N}\,\eta\,\mathbf{M}_{\mathfrak{M}}$. If $\mathfrak{M}'\,\eta\,\mathbf{M}'$ and $\mathfrak{M}'^{\natural}=H$, then $\mathbf{M}$ and $\mathbf{M}_{\mathfrak{M}'}$ are $*$-isomorphic under the natural mapping and therefore so for their centers, and so we identify the spectres of these centers. Then as any $\mathfrak{N}\,\eta\,\mathbf{M}_{\mathfrak{M}'}$ is uniquely written as $\mathfrak{N}_1\cap\mathfrak{M}'$, where $\mathfrak{N}_1\,\eta\,\mathbf{M}$, any dimension function $d_{\mathfrak{M}'}$ of $\mathbf{M}_{\mathfrak{M}'}$ is defined by means of a dimension function d of $\mathbf{M}$, that is, $d_{\mathfrak{M}'}(\mathfrak{N})=d(\mathfrak{N}_1)$. Moreover if $\mathbf{M}$ is finite and d is normalized, so is for $d_{\mathfrak{M}'}$. A similar result holds also for $\mathbf{M}_{\mathfrak{M}\cap\mathfrak{M}'}$, where $\mathfrak{M}\,\eta\,\mathbf{M}$, $\mathfrak{M}'\,\eta\,\mathbf{M}$, $\mathfrak{M}^{\natural}=\mathfrak{M}'^{\natural}=H$ and $\mathfrak{M}$ is finite. $\mathfrak{N}\,\eta\,\mathbf{M}_{\mathfrak{M}\cap\mathfrak{M}'}$ is uniquely written as $\mathfrak{N}_1\cap\mathfrak{M}'$, $\mathfrak{N}_1\,\eta\,\mathbf{M}_{\mathfrak{M}}$. Then the normalized dimension function $d_{\mathfrak{M}\cap\mathfrak{M}'}$ of $\mathbf{M}_{\mathfrak{M}\cap\mathfrak{M}'}$ is given by the formula $d_{\mathfrak{M}\cap\mathfrak{M}'}(\mathfrak{N})=d(\mathfrak{N}_1)/d(\mathfrak{M})$.

Now we suppose that $\mathbf{M}$ and $\mathbf{M}'$ are finite. We denote by the same symbol d the normalized dimension functions of $\mathbf{M}$ and $\mathbf{M}'$. A theorem of Kaplansky [**10**] tells us that there exists a $C\in\boldsymbol{Z}$, positive and finite-valued except on a non-dense subset, such that

$$(4)\qquad Cd([\mathbf{M}x])=d([\mathbf{M}'x])\ \textit{for every}\ x\in H.$$

The existence of such a C is clear if we can show that

$$(5)\qquad d([\mathbf{M}x])\,d([\mathbf{M}'y])=d([\mathbf{M}'x])\,d(\mathbf{M}y])\ \textit{for every}\ x,\,y\in H.$$

To prove (5) we may assume that the central envelopes of these four cyclic subspaces coincide with H, and by virtue of the comparability theorem of Dixmier that $[\mathbf{M}y]\lesssim[\mathbf{M}x]$ and therefore $[\mathbf{M}'y]\lesssim[\mathbf{M}'x]$ by a theorem of Murray and von Neumann [**11**]. We show that we may also assume that $y\in[\mathbf{M}x]\cap[\mathbf{M}'x]$. Indeed, as $\mathbf{M}$ is finite, there exists a $U\in\mathbf{M}_U$ with $U[\mathbf{M}'y]=[\mathbf{M}'Uy]\subset[\mathbf{M}'x]$. Then $[\mathbf{M}Uy]=[\mathbf{M}y]$. There exists also a $V\in\mathbf{M}'_U$ with $V[\mathbf{M}Uy]=[\mathbf{M}VUy]\subset[\mathbf{M}x]$. We have $[\mathbf{M}'Uy]=[\mathbf{M}'VUy]$. If we put $y_1=VUy$, then $y_1\in[\mathbf{M}x]\cap[\mathbf{M}'x]$ and $d([\mathbf{M}y_1])=d([\mathbf{M}y])$, $d([\mathbf{M}'y_1])=d([\mathbf{M}'y])$, as desired. Let $\mathfrak{M}=[\mathbf{M}'x]$ and $\mathfrak{M}'=[\mathbf{M}x]$. It is easy to see that $[\mathbf{M}'_{\mathfrak{M}\cap\mathfrak{M}'}y]=[\mathbf{M}'y]\cap[\mathbf{M}x]$ and $[\mathbf{M}_{\mathfrak{M}\cap\mathfrak{M}'}y]=[\mathbf{M}y]\cap[\mathbf{M}'x]$. Then by the preceding discussions on dimension functions we have

$$(6)\qquad \begin{aligned} d_{\mathfrak{M}\cap\mathfrak{M}'}([\mathbf{M}_{\mathfrak{M}\cap\mathfrak{M}'}y]) &= d([\mathbf{M}y])/d(\mathbf{M}x]),\\ d_{\mathfrak{M}\cap\mathfrak{M}'}([\mathbf{M}'_{\mathfrak{M}\cap\mathfrak{M}'}y]) &= d([\mathbf{M}'y])/d([\mathbf{M}'x]). \end{aligned}$$

Therefore if we can show that $d_{\mathfrak{M}\cap\mathfrak{M}'}([\mathbf{M}_{\mathfrak{M}\cap\mathfrak{M}'}y])=d_{\mathfrak{M}\cap\mathfrak{M}'}([\mathbf{M}'_{\mathfrak{M}\cap\mathfrak{M}'}y])$, then (5) follows from (6). As shown in the proof of Lemma 2, $\mathbf{M}_{\mathfrak{M}\cap\mathfrak{M}'}$ is spatially isomorphic with the left ring $\mathbf{L}$ of an H-system $\hat{H}$. Thus the proof of the theorem of Kaplansky is reduced to the case where $\mathbb{M}$ is the left ring of an H-system $\hat{H}$. Let S be the conjugation of $\hat{H}$. It is almost clear that $d(\mathfrak{M})=d(S\mathfrak{M})$ for every $\mathfrak{M}\eta\mathbb{L}$. For any $\hat{x}\in\hat{H}$, there exists a $U\in\mathbb{L}_U$ such that $\hat{x}_1=U\hat{x}$ is a self-adjoint element of $\hat{H}$. Then we see that $[\mathbb{L}\hat{x}]=[\mathbb{L}\hat{x}_1]$ and $[\mathbb{L}'\hat{x}]\sim[\mathbb{L}'\hat{x}_1]=S[\mathbb{L}\hat{x}]$. These equations yield $d([\mathbb{L}\hat{x}])=d([\mathbb{L}'\hat{x}])$ for every $\hat{x}\in\hat{H}$. Thus the theorem of Kaplansky is completely proved.

Let $\mathbb{M}$ and $\mathbb{M}'$ be finite as above. If $\mathfrak{M}\eta\mathbb{M}$, $\mathfrak{M}'\eta\mathbb{M}'$ have the central envelope H, then the unitary invariants $C_{\mathfrak{M}}$ and $C_{\mathfrak{M}'}$ of $\mathbb{M}_{\mathfrak{M}}$ and $\mathbb{M}_{\mathfrak{M}'}$ respectively are give by the following formulas [**16**]:

$$C_{\mathfrak{M}}=C/d(\mathfrak{M}) \tag{7}$$

$$C_{\mathfrak{M}'}=Cd(\mathfrak{M}') \tag{8}$$

Indeed, for any $x\in\mathfrak{M}$, we have $[\mathbb{M}_{\mathfrak{M}}x]=[\mathbb{M}x]\cap\mathfrak{M}$ and $[\mathbb{M}'_{\mathfrak{M}}x]=[\mathbb{M}'x]$, so that $d_{\mathfrak{M}}([\mathbb{M}_{\mathfrak{M}}x])=d([\mathbb{M}x])$, $d_{\mathfrak{M}}([\mathbb{M}'_{\mathfrak{M}}x])=d([\mathbb{M}x])/d(\mathfrak{M})$. Then the definition of the unitary invariant $C_{\mathfrak{M}}$ yields (*7*). Similarly we obtain (*8*). If $C\geq 1$, it follows from the equation (*8*) that there exists an $\mathfrak{M}'\eta\mathbb{M}'$ with $\mathfrak{M}'^{\natural}=H$ such that $C_{\mathfrak{M}'}=1$. To show this we may separate the proof in two cases: (*a*) $\mathbb{M}$ *and* $\mathbb{M}'$ are of type II; (*b*) $\mathbb{M}$ and $\mathbb{M}'$ are of type I and homogeneous of orders m and n respectively. In case (*a*), the existence of the the required $\mathfrak{M}'$ follows from the fact that for any non negative valued function $g\leq 1$ on Ω there exists an $\mathfrak{M}'\eta\mathbb{M}'$ such that $d(\mathfrak{M}')=g$ (Cf [**16**] and the paper of S. Maeda cited above). This is purely of dimension theoretic character and easily proved by making use of homogeneous partitions of H. In case (*b*), $C=n/m$ and $n\geq m$ by our assumption. H admits of a homogeneous partition $\{\mathfrak{M}'_j\}_{1\leq j\leq n}\mathbb{M}'$. If we put $\mathfrak{M}'=\sum_{j=1}^{m}\oplus\mathfrak{M}'_j$, then we have $d(\mathfrak{M}')=m/n$, as desired.

Let H_1 be a central subspace of H. The spectre of $\mathbb{M}^{\natural}_{H_1}$ is considered as a subspace of Ω and is denoted by Ω_{H_1}. Suppose that $\mathbb{M}$ is finite. Let $\mathfrak{M}'\eta\mathbb{M}'$ be finite such that $\mathfrak{M}'^{\natural}=H$. The unitary invariant $C_{\mathfrak{M}'}$ of $\mathbb{M}_{\mathfrak{M}'}$ is a function defined on Ω_{H_1}. If $\mathfrak{N}'\eta\mathbb{M}'$ with $\mathfrak{N}'^{\natural}=H_2$ is also finite, then

$$C_{\mathfrak{M}'}/d(\mathfrak{M}')=C_{\mathfrak{N}'}/d(\mathfrak{N}') \text{ on } \Omega_{H_1\cap H_2} \tag{9}$$

holds for any dimension function d of $\mathbb{M}'$. To see this we may assume that $\mathfrak{M}'^{\natural}=\mathfrak{N}'^{\natural}=H$ and $\mathfrak{M}'\subset\mathfrak{N}'$. The normalized dimension function of $\mathbb{M}'_{\mathfrak{N}'}$ is given by $d/d(\mathfrak{N}')$, and therefore (*8*) yields (*9*). Similary if $\mathbb{M}'$ is finite and $\mathfrak{M}$, $\mathfrak{N}\eta\mathbb{M}$ are also finite, then for any dimention function d of $\mathbb{M}$ we have

$$C_{\mathfrak{M}}d(\mathfrak{M})=C_{\mathfrak{N}}d(\mathfrak{N}) \text{ on } \Omega_{\mathfrak{M}^{\natural}\cap\mathfrak{N}^{\natural}}. \tag{10}$$

For any semi-finite ring $\mathbb{M}$ R. Pallu de La Barrière introduced the concept of the unitary invariant C of $\mathbb{M}$ in such a way that if H' is the central subspace of H such that both $\mathbb{M}_{H'}$ and $\mathbb{M}'_{H'}$ are finite, then C coincides on $\Omega_{H'}$ with $C_{H'}$ already defined (For details see [**16**]). Griffin [**9**] also defined a unitary invariant of $\mathbb{M}$, which is the inverse of C.

Theorem 2. ([*9*], [*16*]). *Let* $\mathbb{M}$, $\tilde{\mathbb{M}}$ *be semi-finite rings of operators on Hilbert spaces* H, $\tilde{H}$ *with unitary invariants* C, $\tilde{C}$ *respectively. Suppose that* C *vanishes identically on no non-void open subset of the spectre of* $\mathbb{M}^{\natural}$. *If* φ *is a* $*$*-isomorphism of* $\mathbb{M}$ *onto* $\tilde{\mathbb{M}}$ *taking* C *into* $\tilde{C}$, *then* φ *is spatial.*

Proof. We may carry out the proof by separating the cases: (a) $\mathbb{M}$, $\mathbb{M}'$, $\tilde{\mathbb{M}}$ and $\tilde{\mathbb{M}}'$ are finite; (b) $\mathbb{M}'$ and $\tilde{\mathbb{M}}'$ are properly infinite and have the same algebraic invariant. The case (b) is a part of Corollary of Lemma 5. We turn to the case (a). We may identify $\mathbb{M}$ with $\mathbb{N}_H$ and $\tilde{\mathbb{M}}$ with $\mathbb{N}_{\tilde{H}}$ (For the notations see p. 50). Then the mapping φ becomes $A_H \rightarrow A_{\tilde{H}}$, $A \in \mathbb{N}$ and $C = \bar{C}_H$, $\tilde{C} = \bar{C}_{\tilde{H}}$, $\bar{C}$ being the invariant of $\mathbb{N}$. The hypotheses of the theorem yield $\bar{C}_H = \bar{C}_{\tilde{H}}$. Then using the equation (8) we see that $\bar{C}d(H) = \bar{C}d(\tilde{H})$, and threfore $d(H) = d(\tilde{H})$. Since H is finite in $\mathbb{N}'$, it follows that $H \sim \tilde{H}$ (mod $\mathbb{N}'$). This shows us that the mapping $A_H \rightarrow A_{\tilde{H}}$ is spatial, completing the proof.

As a consequence of this theorem the unitary invariant of a semi-finite ring $\mathbb{M}$ is identically 1 if and only if $\mathbb{M}$ is unitary equivalent to the left ring of an H-system associated with $\mathbb{M}$. For the unitary invariant of the left ring of an H-system is identically 1 as easily seen.

If the unitary invariant C of a semi-finite ring $\mathbb{M}$ is such that $C \geq 1$, we can find an $\mathfrak{M}' \eta \mathbb{M}'$ with $(\mathfrak{M}')^{\natural} = H$ such that $C_{\mathfrak{M}'} = 1$ [**16**]. Such an $\mathfrak{M}'$ is termed a *separating normal subspace for* $\mathbb{M}$ [**16**]. The proof presents no difficulties and is carried out by a similar way to the case where $\mathbb{M}$ and $\mathbb{M}'$ are finite. We omit the details.

§ 2. Conditions of equivalence of the two topologies "weak" and "ultraweak"

Lemma 6. *If* $\mathbb{M}$ *is finite, the following statements are equivalent:*

(i) $C \geq 1$;

(ii) *Every nomal trace on* $\mathbb{M}$ *is at most of length 1;*

(iii) *Every normal state on* $\mathbb{M}$ *is at most of length 1;*

(iv) *Every ultraweakly continuous linear form on* $\mathbb{M}$ *is at most of length 1.*

Proof. $(i)\rightarrow(iv)$. Let φ be any ultraweakly continuous linear form on $\mathbf{M}$. As observed in §1, $C\geq 1$ implies the existence of a separating normal subspace $\mathfrak{M}'$ for $\mathbf{M}$. Then the mappng $A\rightarrow A_{\mathfrak{M}'}$ of $\mathbf{M}$ onto $\mathbf{M}_{\mathfrak{M}'}$ is $*$-isomorphic and therefore bicontinuous in the ultraweak topology [**5**]. By setting $\rho(A_{\mathfrak{M}'})=\varphi(A)$ and by using the fact that $\mathbf{M}_{\mathfrak{M}'}$ is spatially isomorphic to the left ring of an H-system, it follows from Theorem 3 of [**15**] that there exist two vectors $x, y\in\mathfrak{M}'$ such that $\rho(A_{\mathfrak{M}'})=(A_{\mathfrak{M}'}x, y)=(Ax, y)$, and a fortiori $\varphi(A)=(Ax, y)$.

The implications $(iv)\rightarrow(iii)\rightarrow(ii)$ are clear from §1.

$(ii)\rightarrow(i)$. As $\mathbf{M}$ is finite, it is a central direct sum of σ-finite finite rings. So we may assume for the proof of the implication concerned that $\mathbf{M}$ is σ-finite and $\mathbf{M}'$ is finite. Then $\mathbf{M}$ has a faithful trace φ, which is of the form $\varphi(A)=(Ax, x)$ by our assumption. Since φ is faithful, the carrier projection of φ is I, and therefore $[\mathbf{M}'x]=H$. It follows from the definition of C that $C=d([\mathbf{M}'x])/d([\mathbf{M}x])=1/d(\mathbf{M}x])\geq 1$, as desired. The proof is complete.

By making use of the ampliation of order n of $\mathbf{M}$ and by noting that the unitary invariant of $\mathbf{M}^{(n)}$ is nC, we have the the following lemma as an immediate consequence of the above lemma 6.

Lemma 7. *If* $\mathbf{M}$ *is finite the following statements are equivalent:*

(*i*) $C\geq 1/n$;

(*ii*) *Every normal trace on* $\mathbf{M}$ *is at most of length* n;

(*iii*) *Every normal state on* $\mathbf{M}$ *is at most of length* n;

(*iv*) *Every ultraweakly continuous linear form on* $\mathbf{M}$ *is at most of length* n.

We note that this lemma tells us that if n is the least positive integer with $C\geq 1/n$, the maximal length of normal states (traces) on $\mathbf{M}$ is just n.

Lemma 8. *If* $\mathbf{M}$ *is finite and* $\underset{x\in\Omega}{g.l.b.}\,C(x)=0$, *then there exists a normal trace on* $\mathbf{M}$ *whose length is infiite.*

Proof. It follows from the hypotheses of the lemma that there exists a central partition $\{\mathfrak{M}_n\}$ of H with the properties that $C_{\mathfrak{M}_n}(x)>0$ and $\lim_{n\to\infty}\underset{x\in\Omega_{\mathfrak{M}_n}}{g.l.b.}\,C_{\mathfrak{M}_n}(x)=0$. Choose the least positive integer p_n such that $C_{\mathfrak{M}_n}\geq 1/p_n$. Clearly $\lim_{n\to\infty}p_n=\infty$. As remarked above, there exists for each n a normal trace φ_n on $\mathbf{M}_{\mathfrak{M}_n}$ whose length is p_n. Normalize φ_n so that we may have $\sum_{n=1}^{\infty}\varphi_n(I_{\mathfrak{M}_n})<+\infty$. Define a normal state φ by the equation $\varphi(A)=\sum_{n=1}^{\infty}\varphi_n(A_{\mathfrak{M}_n})$. We show that φ is of infinite length. Suppose the contrary. Then φ is of the form $\varphi(A)=\sum_{j=1}^{m}(Ax_j, x_j)$. $\varphi_n(A_{\mathfrak{M}_n})=\varphi(P_{\mathfrak{M}_n}AP_{\mathfrak{M}_n})=\sum_{j=1}^{m}(AP_{\mathfrak{M}_n}x_j, P_{\mathfrak{M}_n}x_j)$, so that the length of φ_n is at most m. This is a contradiction since $\lim_{n\to\infty}p_n=+\infty$, completing the proof.

Lemma 9. *If* $\mathbf{M}$ *is properly infinite and* $\mathbf{M}'$ *is finite, then there exists a normal state on* $\mathbf{M}$ *whose length is infinite.*

Proof. By Lemma 2 any cyclic subspace $\eta\mathbf{M}$ is finite since $\mathbf{M}'$ is finite. Let x_1 be a non-zero vector of H. As $\mathbf{M}$ is properly infinite, there exists a homogeneous partition $\{\mathfrak{M}_n\}_{1\leq n<\infty}$ with $\mathfrak{M}_n=[\mathbf{M}'x_n]$ for some $x_n\in H$, where we may assume that $\sum\|x_n\|^2<+\infty$. It follows from $\mathfrak{M}_i\sim\mathfrak{M}_j$ that $\mathfrak{N}=\sum\oplus\mathfrak{M}_n$ is infinite, and therefore $\varphi(A)=\sum(Ax_n, x_n)$ is of infinite length. For otherwise the carrier subspace of φ, that is, $\mathfrak{N}$ would be finite since any cyclic subspace $\eta\mathbf{M}$ is finite. The proof is complete.

From Lemmas 7–9 together with Theorem 1 and Lemma 3 we have the following

Theorem 3. *The following statements for a ring* $\mathbf{M}$ *are equivalent:*

(*i*) *The weak and ultraweak topologies coincide;*

(*ii*) *The strong and ultrastrong topologies coincide;*

(*iii*) *The strong topology is stronger than the ultraweak topology;*

(*iv*) *If* H^f *is the central subspace such that* $\mathbf{M}'_{H^f}$ *is finite, then* $\mathbf{M}_{H^f}$ *is also finite and* $C_{H^f}(x)>0$*;*

(*v*) *Let* H^f *be the same as in* (*iv*). $\mathbf{M}_{H^f}$ *is a central direct sum of* σ*-finte rings and every* σ*-finite central projection in* $\mathbf{M}_{H^f}$ (*or* $\mathbf{M}$) *is of finite length;*

(*vi*) *Every* σ*-finite projection in* $\mathbf{M}$ *is of finite length.*

Moreoverif $\mathbf{M}$ *is finite, these conditions are equivalent to*

(*vii*) *The mapping* $A\to A^\natural$ (canonical $\natural$*-application*) *is continuous in the weak* (*strong*) *topologies.*

Let any of these equivalent conditions be satisfied, and let n *be the least positive integer such that* $C_{H^f}\geq 1/n$ *if* $H^f\neq(0)$. *The maximal length of normal states* (*normal traces, ultraweakly continuous linear forms,* σ*-finite central projections,* σ*-finite projactions*) *is* n *if* $H^f\neq(0)$. *The maximal length of normal states* (*ultraweakly continuous linear forms,* σ*-finite projections*) *is* 1 *if* $H^f=(0)$.

Proof. (*i*), (*ii*), (*vi*) are equivalent by Theorem I, and (*i*), (*iv*), (*v*) by Lemmas 3, 7, 8, 9. Cleary (*i*) implies (*iii*). Conversely (*i*) implies that any normal state on $\mathbf{M}$ is strongly continuous, and therefore weakly continuous (§**1**), so that (*iii*) imples (*i*). Thus (*i*)–(*vi*) are equivalent.

Let $\mathbf{M}$ be finite. The mapping $A\to A^\natural$ is continuous in the ultraweak (ultrastrong) topologies since the mapping is normal [**5**]. Note that for any commutative ring of operators the condition (*iv*) holds, and a fortiori (*i*) and (*ii*). Hence (*i*) and (*ii*) imply (*vii*) for $\mathbf{M}$. Now we show the implication (*vii*)$\to$(*v*).

The first part of (*v*) is clear since $\mathbb{M}$ is finite. To complete the proof we may assume that $\mathbb{M}$ is σ-finite and $\mathbb{M}'$ is finite. Then there exists a faithful normal trace φ on $\mathbb{M}$. $\varphi(A) = \varphi(A^{\natural})$. As remarked above (*i*) and (*ii*) hold for $\mathbb{M}^{\natural}$, and therefore $A^{\natural} \to \varphi(A^{\natural})$ is weakly (strongly) continuous. Hence $A \to \varphi(A^{\natural})$ is weakly (strongly) continuous, so that φ is of finite length. Since φ is taken to be faithful, the carrier projection of φ is I. Thus the length of I is finite.

The rest of the statements of the theorem follows from Lemmes 3, 7, 8, 9,

The results obtained in this theorem are closely related to the works of Dye [**7**], Griffin [**9**] and Dixmier [**5**]. Dye ([**5**], Cor. 5.2) showed that in a σ-finite and finite ring the condition (*ii*) is equivalent to the condition that the identity is of finite length. This is a special case of our theorem. Basing on the result of Dye, Dixmier ([**5**], Prop. 4) obtained the equivalence of (*ii*) and (*v*). As for a finite ring, Griffin ([**9**], Theorem 8) obtained the equivalence of (*iv*) and (*vii*).

If a ring $\mathbb{M}$ is commutative, we know that its unitary invariant $C \geq 1$, so that any ultraweakly continuous linear form on $\mathbb{M}$ is of the form (Ax, y). This is a result of R. Pallu de La Barrière [**16**].

Let $A \to \tilde{A}$ be a $*$-isomorphic mapping of $\mathbb{M}$ onto $\tilde{\mathbb{M}}$. Suppose that the condition (*i*) holds for $\tilde{\mathbb{M}}$. Then the mapping $A \to \tilde{A}$ is continuous in the weak (strong) topologies if and only if (*i*) holds also for $\mathbb{M}$ (Cf. [**9**], p. 503, Lemma 2). Indeed, assume that the mapping $A \to \tilde{A}$ is continuous in the weak (strong) topologies. Let φ be any normal state on $\mathbb{M}$, and define $\tilde{\varphi}$ by the equation $\tilde{\varphi}(\tilde{A}) = \varphi(A)$. Since the mapping $\tilde{A} \to A$ is normal and (*i*) holds for $\tilde{\mathbb{M}}$, $\tilde{\varphi}$ is of finite length. Hence, as the mapping $A \to \tilde{A}$ is continuous in the weak (strong) topolgies, φ is weakly continuous. Thus (*i*) holds for $\mathbb{M}$. The converse is trivial. Using this remark we show the following

Corollary ([*16*], *Theorem A*). *Let* $\mathbb{M}$, $\tilde{\mathbb{M}}$ *be rings of operators on Hilbert spaces* H, $\tilde{H}$ *respectively. Assume that the unitary invariant* C *(resp.* $\tilde{C}$*) of the semi-finite part of* $\mathbb{M}$ *(resp.* $\tilde{\mathbb{M}}$*) satisfies the condition that* C *(resp.* $\tilde{C}$*) vanishes on no non-void open subsets. Let* H_0 *(resp.* $\tilde{H}_0$*) be the central subspace* $\eta\mathbb{M}$ *(resp.* $\eta\tilde{\mathbb{M}}$*) such that* $\mathbb{M}_{H_0}$ *(resp.* $\tilde{\mathbb{M}}_{\tilde{H}_0}$*) is finnite. Put* $\tilde{C}_0 = min\ (1, \tilde{C}_{\tilde{H}_0})$. *Now, if* Φ *is a* $*$*-isomorphic mapping of* $\mathbb{M}$ *onto* $\tilde{\mathbb{M}}$, Φ *is continuous in the weak (strong) topologies if and only if* $C_{H_0}/\Phi^{-1}(\tilde{C}_0) > 0$ *at every point of* Ω_{H_0}.

Proof. By the preceding remark the proof of the corollary is redeuced to the case where $\mathbb{M}$, $\tilde{\mathbb{M}}$ are finite and $C, \tilde{C} \leq 1$.

Necessity. Let $\tilde{\mathfrak{M}}\eta\tilde{\mathbb{M}}$ be chosen so that $\tilde{C}_{\tilde{\mathfrak{M}}} = 1$. And let $\mathfrak{M}\eta\mathbb{M}$ be such that $\Phi(P_{\mathfrak{M}}) = P_{\tilde{\mathfrak{M}}}$. Then $\mathbb{M}_{\mathfrak{M}}$ and $\tilde{\mathbb{M}}_{\tilde{\mathfrak{M}}}$ are $*$-isomorphic under the mapping $\Phi_{\mathfrak{M}}$: $A_{\mathfrak{M}} \to \tilde{A}_{\tilde{\mathfrak{M}}}$. It is not difficult to see that $\Phi_{\mathfrak{M}}$ must be continuous in the weak (strong) topologies, so that by the preceding remark (*i*) holds for $\mathbf{M}_{\mathfrak{M}}$ and therefore by Theorem 3 we have $C_{\mathfrak{M}}=C/d(\mathfrak{M})>0$ at every point *of* Ω. On the other hand, $1=\tilde{C}_{\tilde{\mathfrak{M}}}=\tilde{C}/d(\tilde{\mathfrak{M}})$ and $d(\tilde{\mathfrak{M}})=\Phi(d(\mathfrak{M}))$, so that $C/\Phi^{-1}(\tilde{C})>0$ at every point of Ω.

Sufficiency. let n be the least positive integer such that $C/\Phi^{-1}(C)\geq 1/n$ Then $nC\geq\Phi^{-1}(C)$. As the ampliation of order n of $\mathbf{M}$ is bicontinuous in the weak (strong) topologies, we may assume for the proof that $n\doteq 1$. As done in §**I**, we can choose $\mathfrak{M}'\eta\mathbf{M}'$ such that $Cd(\mathfrak{M}')=\Phi^{-1}(\tilde{C})$, so that we may identify $\tilde{\mathbf{M}}$ with $\mathbf{M}_{\mathfrak{M}'}$ and Φ becomes then the natural mapping $A\rightarrow A_{\mathfrak{M}}$ of $\mathbf{M}$ onto $\mathbf{M}_{\mathfrak{M}'}$. This mapping is evidently continuous in the weak (strong) topologies. The proof is complete.

§ 3. Cyclic projections

This section is devoted to the discussions concerning some properties of cyclic projections in a ring of operators. A ring $\mathbf{M}$ is called *essentially finite* [7] if every cyclic projection in $\mathbf{M}$ is finite.

Theorem 4. *A ring* $\mathbf{M}$ *is essentially finite if and only if there exists a central partition* $H=H_1\oplus H_2$ *such that* $\mathbf{M}_{H_1}$ *and* $\mathbf{M}'_{H_2}$ *are finite.*

Proof. *Sufficiency.* It is obvious from the fact that $[\mathbf{M}x]$ is finite if and only if $[\mathbf{M}'x]$ is finite.

Necesstiy. It is clear that $\mathbf{M}$ is semi-finite. Consider a central partition $H=H_1\oplus H_2$ of H such that $C_{H_1}\geq 1$ and $C_{H_2}\leq 1$. Then the proof will be complete if we can show the following two assertions:

(*a*) *If* $\mathbf{M}$ *is essentially finite and* $C\geq 1$, *then* $\mathbf{M}'$ *is finite;*

(*b*) *If* $\mathbf{M}$ *is essentially finite and* $C\leq 1$, *then* $\mathbf{M}'$ *is finite.*

(*b*) is converted into (*a*) if we replace $\mathbf{M}$ by $\mathbf{M}'$, so we shall prove (*a*). To this end it is sufficent to draw a contradiction under the hypothesis that $\mathbf{M}$ is properly infinite. Choose an arbitrary vector $x\neq 0$. Put $\mathfrak{M}=[\mathbf{M}'x]$. As $\mathfrak{M}$ is finite, there exists a homogeneous partition $\{\mathfrak{M}_j\}_{1\leq j<+\infty}$ such that $\mathfrak{M}=\mathfrak{M}_1$. Clearly $\mathfrak{N}=\sum\oplus\mathfrak{M}_j$ is infinite. On the other hand $\mathfrak{N}$ is σ-finite, so that by Theorem 3, $\mathfrak{N}$ is a cyclic subspace and therefore finite. This is a contradiction, as desired.

Lemma 10. *Let* $\{P_\delta\}$ *be a monotone increasing directed set of cyclic projections* $\in\mathbf{M}$ *contained in a* σ*-finite projection* $\in\mathbf{M}$. *Then* $\cup P_\delta$ *is also a cyclic projection.*

Proof. We may assume that $I=\cup P_\delta$ and by Lemma 3 that $\mathbf{M}'$ is finite. Consider the case where $\mathbf{M}$ is finite. Let $\mathfrak{M}_\delta$ be the range of P_δ. For $\delta>\delta_0$, by the relation (*7*) we have $C=C_{\mathfrak{M}_\delta}d(\mathfrak{M}_\delta)$ on $\Omega_{\mathfrak{M}_\delta}\natural$. As $C_{\mathfrak{M}_\delta}\geq 1$ by Lemma 6 and $d(\mathfrak{M}_\delta)\uparrow 1$, we have $C\geq 1$. Therefore, by the same lemma, I is a cyclic projection in $\mathbf{M}$. Hence if we can show that the other cases can not occur, the proof will be complete. To this end it is sufficient to draw a contradiction assuming that $\mathbf{M}$ is properly infinite. By the relation (*10*), for $\delta>\delta_0$, $C_{\mathfrak{M}_0}d(\mathfrak{M}_{\delta_0})$ $=C_{\mathfrak{M}_\delta}d(\mathfrak{M}_\delta)$ on $\Omega_{\mathfrak{M}_{\delta_0}}\natural$ for any dimension function d of $\mathbf{M}$. As $C_{\mathfrak{M}_\delta}\geq 1$ by Lemma 6 and $d(\mathfrak{M}_\delta)\uparrow +\infty$, $C_{\mathfrak{M}_{\delta_0}}d(\mathfrak{M}_{\delta_0})=\infty$. This is a contradiction, as desired.

Corollary *If* $\{P_n\}$ *is a monotone increasing sequence of cyclic projections in* $\mathbf{M}$, *then so is also for* $\cup P_n$.

Proof. As $\cup_n P_n$ is σ-finite, the statement is an immediate consequence of the preceding lemma.

Corollary. *Assume that* $\mathbf{M}$ *is smi-finite. Then* $\mathbf{M}$ *is essentially finite if and only if for any monotone increasing sequence* $\{P_n\}$ *of finite cyclic projections in* $\mathbf{M}$, $\cup_n P_n$ *is also finite.*

Proof. We may carry out the proof by separating the cases: (*a*) $C\geq 1$; (*b*) $C\leq 1$. We need only to show the "if" part.

Ad (*a*). Let $\mathfrak{M}'$ be a separating normal subspace for $\mathbf{M}$. $\mathbf{M}_{\mathfrak{M}'}$ is unitary equivalent to the left ring of an *H*-sysem. If follows from the proprerties of an *H*-system that every cyclic projection is the l. u. b. of a monotne increasing sequence of cyclic finite projections in $\mathbf{M}_{\mathfrak{M}'}$. Then from the proof of Theorem 4 we see that $\mathbf{M}_{\mathfrak{M}'}$ is finite. As $\mathbf{M}$ is $*$-isomorphic to $\mathbf{M}_{\mathfrak{M}'}$ under the natural mapping, $\mathbf{M}$ is also finite.

Ad (*b*). There exists a separating normal subspace $\mathfrak{M}$ for $\mathbf{M}'$ as $C\leq 1$. Then by the same reasonning as in (*a*) we conclude that $\mathbf{M}_{\mathfrak{M}}$ is finite. As $C_{\mathfrak{M}}=1$, $\mathbf{M}'_{\mathfrak{M}}$ is also finite and therefore $\mathbf{M}'$ is finite. The proof is complete.

Using Lemma *10* we shall show

Theorem 5. *Let* $\{\varphi_\delta\}$ *be a monotone increasing sequence of normal states on* $\mathbf{M}$. *If* $lim_\delta\,\varphi_\delta(A)=\varphi(A)$ *exists and is finite for every* A *in* $\mathbf{M}$, *then* φ *is a normal state and holds the equation: The length of* $\varphi=lim_\delta$ (*the length of* φ_δ).

Proof. We show that φ is normal. Let $A_\alpha\uparrow A$ where $\{A_\alpha\}$ is a monotone increasing directed set of positive operators in $\mathbf{M}$. For any δ, $\varphi(A_\alpha)\geq\varphi_\delta(A_\alpha)$ so that $lim_\alpha\,\varphi(A_\alpha)\geq\varphi_\delta(A)$ and therefore $lim_\alpha\,\varphi(A_\alpha)\geq\varphi(A)$. The inverse inequality is evident. Hence φ is normal. Let Q and Q_δ be the carrier projections of φ and φ_δ respectively. Since $Q_\delta\leq Q$, the length of $Q_\delta\leq$ the length of Q, so that lim_δ (the length of Q_δ) $\leq$ the length of Q. It is clear that $Q_\delta\uparrow Q$. Suppose that lim_δ (the length of Q_δ) is finite, say n. By considering the ampliation of order n of $\mathbf{M}$, and by using Lemma 10 we see that Q is at most of length n.

Hence lim_δ (the length of Q_δ) $\geq$ the length of Q. The proof is complete.

In an arbitrary ring the union of any two cyclic projections in the ring is not generally cyclic.

Theorem 6. *In a ring* $\mathbf{M}$ *the following statements are equivalent:*

(*i*) *Any σ-finite projection in* $\mathbf{M}$ *is cyclic;*

(*ii*) *let H^f be the central subspace $\eta\mathbf{M}$ such that $\mathbf{M}'_{H^f}$ is finite. Then $\mathbf{M}_{H^f}$ is also finite and $C_{H^f}\geq 1$;*

(*iii*) *union of any two cyclic projections in* $\mathbf{M}$ *is also cyclic.*

Proof. The equivalence of (*i*) and (*ii*) is a special case of Theorem 3. (*iii*) is clearly a consequence of (*i*). For the proof of the implication (*iii*)$\rightarrow$(*i*), let P be any σ-finite projection in $\mathbf{M}$. P is of the form $P=\cup P_n$, where P_n is cyclic. (*iii*) implies that $\cup_{j=1}^n P_j$ is cyclic, therefore it follows from the preceding corollary that P is cyclic, as desired.

Corollary. *Let* $\mathbf{M}$ *be a commutative ring.* $\mathbf{M}$ *is a masa algebra if and only if* $[\mathbf{M}x]\cup[\mathbf{M}y]$ *is cyclic for every* $x, y\in H$.

Proof. *Necessity.* If $\mathbf{M}$ is a masa algebra, then $C=1$, so that the condition (*ii*) of the preceding theorem is satisfied for $\mathbf{M}'$, and therefore $[\mathbf{M}x]\cap[\mathbf{M}y]$ is cyclic.

Sufficiency. Apply the preceding theorem to $\mathbf{M}'$. Then we see that $\mathbf{M}'$ is finite and its unitrary invariant is at least equal to *1*, so that $C=1$. Hence $\mathbf{M}$ is a masa algebra.

This is a generalization of a theorem of Segal [**17**] to the effect that a commutative ring is a masa algebra if it has a generating vector, since if a ring $\mathbf{M}$ has a generating vector, every projection in $\mathbf{M}'$ is cyclic.

§ 4. Conditions of equivalence of two topologies besides the cases considered before

There are considered six topologies on a ring $\mathbf{M}$ of operators on a Hilbert space. Among them hold the following relations as mentioned in the introduction:

$$\begin{array}{ccccc} \text{Uniform top.} & > & \text{Ultrastrong top.} & > & \text{Strong top.} \\ \vee & & \vee & & \vee \\ \sigma(\mathbf{M}, \mathbf{M}^*) & > & \text{Ultraweak top.} & > & \text{Weak top.} \end{array}$$

The ultraweak topology coincides with the weak one if and only if the ultrastrong topology coincide with the strong one. And the conditions of equivalence of these two topologies are given in Theorem 3. In this section we show that the condition for any two topologies besides the cases just mentioned to coincide is trivial, that is, $\mathbb{M}$ is finite-dimensional. This will be clear from the following theorem.

Theorem 7. *The following statements for a ring* $\mathbb{M}$ *are equivalent:*

(*i*) $\mathbb{M}$ *is finite-dimensional;*

(*ii*) *the weak and strong topologies coincide;*

(*iii*) *the ultraweak and ultrastrong topologies coincide;*

(*iv*) *the ultraweak topology is stronger than the strong one;*

(*v*) $\sigma(\mathbb{M}, \mathbb{M}^*)$ *is stronger than the strong topology;*

(*vi*) *the ultrastrong topology is stronger than* $\sigma(\mathbb{M}, \mathbb{M}^*)$*;*

(*vii*) *the ultraweak topology is stronger than* $\sigma(\mathbb{M}, \mathbb{M}^*)$.

Proof. It is clear that if $\mathbb{M}$ is finite-dimensional, then any two topologies among six ones considered above coincide, so that (*i*) implies the other statements (*ii*)–(*vii*). The implications (*iii*) $\to$ (*iv*) $\to$ (*v*) and (*ii*) $\to$ (*v*) are trivial.

(*v*) $\to$ (*i*). For any given $x \in H$ there exists a finite number of linear forms φ_j $(j=1, 2, \dots, p) \in \mathbb{M}^*$ such that $\|Ax\| \leq \sum_{j=1}^{p} |\varphi_j(A)|$. To complete the proof it is sufficient to show that there are no infinite mutually orthogonal non-zero projections in $\mathbb{M}$. If not, let $\{P_n\}$ be such an infinite sequence of projections. Let $x_n = P_n x_n$ be chosen in such a way that $\sum \|x_n\| = +\infty$ and $\sum \|x_n\|^2 < +\infty$. Put $x = \sum x_n$. Then $P_n x = x_n$. Since $\sum_{k=1}^{m} |\phi_j(P_k)| \leq \|\varphi_j\|$ for every m, $\sum_{j=1}^{p} \sum_{k=1}^{\infty} |\varphi_j(P_k)| \leq \sum_{j=1}^{p} \|\varphi_j\|$, $\sum_{n=1}^{\infty} \|x_n\| = \sum_{n=1}^{\infty} \|P_n x\| \leq \sum_{j=1}^{p} \|\varphi_j\|$. This is a contradiction.

(*vii*) $\to$ (*vi*) is trivial.

(*vi*) $\to$ (*i*). Let $\mathbb{M}_*$ be the set of all linear forms continuous in the ultraweak topology. $\mathbb{M}_*$ is considered as a closed subspace of $\mathbb{M}^*$ and holds the relation $\mathbb{M}^* = (\mathbb{M}_*)^*$ [**5**]. Let φ be any element of $\mathbb{M}^*$. The assumption shows us that φ is continuous in the ultrastrong topology, so that φ is continuous in the ultraweak topology, that is, $\varphi \in \mathbb{M}_*$. Hence $\mathbb{M} = \mathbb{M}^{**}$. Then it follows from a theorem of the present author [**13**] that $\mathbb{M}$ is finite-dimensional. The proof is complete.

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