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On a Method to Compute Periodic Solutions of the General Autonomous System

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1. Introduction

Previously [4, 8][1] the writer devised a method to compute a periodic solution for the general autonomous differential system and, as an example, applying that method to van der Pol's equation, he computed the periodic solutions for the values 0 (0.2) 1.0 of the damping coefficient [6]. But, for that method, he had to prove directly the convergence of the iterative process using somewhat troublesome estimations of various quantities, because that method was completely different from Newton's method for the solution of equations.

Recently, however, the writer found that, without causing any radical change in the actual computation, the method can be altered so that it may be reduced to solution of certain equations by means of Newton's method, consequently the derivation of the method may be greatly simplified.

This note is devoted to the explanation of this modified method.

2. Moving orthonormal system along an orbit

The modified method is also based on the variation of orbits of the autonomous system and, for the study of this, the moving orthonormal system [5, 7, 8] along an orbit (not necessarily closed) is used. So the results about the moving orthonormal system which are essential for this note are stated in this section.

Given the autonomous system

$$\frac{dx}{dt}=X(x) \tag{1}$$

where $X(x)$ is an N-times $(N\geqq 1)$[2] continuously differentiable function with respect to x in a domain G of n-dimensional Euclidean space R^n, and let

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1) The numbers in the brackets refer to the references listed at the end of the note.

2) In the actual computation, N must be not-small, because, otherwise, we could not apply any integration formula to the given system.

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(2) $$C_0 : x=\varphi_0(t) \qquad (t_1 \leqq t \leqq t_2)$$

be an orbit (not necessarily closed) of (1) lying in G. In the sequel, by the moving orthonormal system along C_0 is meant an orthonormal system

$$\left\{ \hat{X}[\varphi_0(t)]=\frac{X[\varphi_0(t)]}{\|X[\varphi_0(t)]\|},\ \xi_\nu(t) \right\}^{1)} \qquad (\nu=2, 3, \cdots, n)$$

such that

1° each $\xi_\nu(t) \in C_t^N [t_1, t_2]$;

2° each $\xi_\nu(t)$ $(\nu=2, 3, \cdots, n)$ is continuous when this is considered as a function of the space point. This implies $\xi_\nu(t)$ is periodic with the same period as $\varphi_0(t)$ when C_0 is closed.

Such an orthonormal system always exists and moreover there are infinitely many such systems [5, 7, 8]. But, in the sequel, we use a special one of them which is constructed in the following way:

Take a unit vector e_1 so that it never coincides with $-\hat{X}[\varphi_0(t)]$ when $n \geqq 3$[2]. The possibility of such a choice of e_1 follows readily from the lemma of Diliberto and Hufford [2, 7, 8]. Next, construct an orthonormal system $\{e_i\}$ $(i=1, 2, \cdots, n)$ which includes e_1. If

(3) $$\xi_\nu=e_\nu-\frac{\cos\theta_\nu}{1+\cos\theta_1}(e_1+\hat{X}) \quad (\nu=2, 3, \cdots, n)$$

where the angles θ_i $(i=1, 2, \cdots, n)$ are defined by

(4) $$e_i^* \hat{X}[\varphi_0(t)]=\cos\theta_i{}^{2)} \quad (i=1, 2, \cdots, n),$$

then the desired orthonormal system is $\{\hat{X}, \xi_\nu\}$ $(\nu=2, 3, \cdots, n)$.

The vectors given by (3) are those obtained as the last positions of e_ν's rotated about the $(n-2)$-subspace perpendicular to both e_1 and $\hat{X}$ by the angle θ_1; of course, assuming that e_1 never coincides with both $-\hat{X}$ and $\hat{X}$ when $n \geqq 3$. For the details, refer to [5, 7, 8].

Let

(5) $$C : x=\varphi(\tau)$$

be any orbit of the given system (1) lying near C_0, where τ is the time variable along C. Then any point of C is expressed as

(6) $$x=\varphi(\tau)=\varphi_0(t)+\sum_{\nu=2}^{n}\rho^\nu\xi_\nu(t),$$

and, as is readily seen [5, 7, 8], τ and ρ^ν $(\nu=2, 3, \cdots, n)$ become continuously differentiable functions of t. Consequently, substituting (6) into the equation

1) The symbol $\|\cdots\|$ denotes an Euclidean norm of a vector.

2) When n=2, e_1 can be taken arbitrarily.

3) The symbol * denotes the transposed.

of C:

$$\frac{d\varphi(\tau)}{d\tau}=X[\varphi(\tau)],$$

i. e.

$$\frac{d\varphi(\tau)}{dt}=X(\varphi_0+\sum_{\nu}\rho^{\nu}\xi_{\nu})\cdot\frac{d\tau}{dt},$$

we have:

(7) $$X[\varphi_0(t)]+\sum_{\nu}\frac{d\rho^{\nu}}{dt}\cdot\xi_{\nu}+\sum_{\nu}\rho^{\nu}\cdot\frac{d\xi_{\nu}}{dt}=X'\cdot\frac{d\tau}{dt},$$

where

(8) $$X'=X(\varphi_0+\sum_{\nu}\rho^{\nu}\xi_{\nu}).$$

The equation (7) can be divided into two equations, namely into that in the tangential direction of C_0 and into that in the normal hyperplane of C_0 as follows:

(9) $$\frac{d\tau}{dt}=\frac{\|X\|^2+\sum_{\nu=2}^{n}\rho^{\nu}X^*\dot{\xi}_{\nu}}{X^*X'}\qquad (X=X[\varphi_0(t)]),$$

(10) $$\frac{d\rho}{dt}=R(\rho,\ t),$$

where ρ and $R(\rho,\ t)$ are the $(n-1)$-dimensional vectors whose components are respectively ρ^{ν} and

(11) $$R^{\nu}(\rho,\ t)=\frac{\|X\|^2+\sum_{\mu=2}^{n}\rho^{\mu}X^*\dot{\xi}_{\mu}}{X^*X'}\cdot\xi_{\nu}^*X'-\sum_{\mu=2}^{n}\rho^{\mu}\xi_{\nu}^*\dot{\xi}_{\mu}\quad (\nu=2,\ 3,\ \cdots,\ n).$$

It is evident that $R(\rho,\ t)\in C_{\rho}^{N}$ and also that, when C_0 is closed, $R(\rho,\ t)$ is periodic in t with the same period as $\varphi_0(t)$.

The linear variational equation of (10) for the solution $\rho=0$ is readily obtained as follows:

(12) $$\frac{d\rho}{dt}=\Xi(t)\rho,$$

where $\Xi(t)$ is a matrix whose elements $\Xi_{\mu}^{\nu}(t)$ are

$$\Xi_{\mu}^{\nu}(t)=\left.\frac{\partial R^{\nu}(\rho,\ t)}{\partial\rho^{\mu}}\right|_{\rho=0}=\xi_{\nu}^*A\xi_{\mu}-\xi_{\nu}^*\dot{\xi}_{\mu}\in C_t^{N-1}$$

where A is a matrix whose elements are $\left.\partial X^i(x)/\partial x^j\right|_{x=\varphi_0(t)}$. In the sequel, by $\Psi(t)=(\Psi_{\mu}^{\nu}(t))$, we shall denote the fundamental matrix of (12) such that

$\Psi(0)=E$ where E is a unit matrix.

3. The method to compute a priodic solution

In order to get a periodic solution of the given system (1), it is needless to say that it is enough to get a closed orbit near C_0.

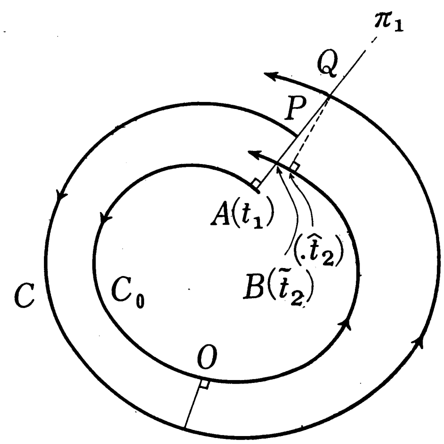
Let us assume that C_0 is approximately closed and its approximate period is $\omega_0>0$. By this is meant that, when C_0 is followed starting from the point

$$A:\ x=\varphi_0(t_1) \qquad (0>t_1\fallingdotseq-\omega_0/2)$$

to the point

$$B:\ x=\varphi_0(\tilde{t}_2) \qquad (0<\tilde{t}_2\fallingdotseq\omega_0/2<t_2)$$

lying on the normal hyperplane π_1 of C_0 at A, the distance AB is short. The point B is really uniquely determined, because $\tilde{t}_2$ is a root of the equation



$$X^*[\varphi_0(t_1)]\cdot\{\varphi_0(t)-\varphi_0(t_1)\}=0 \tag{13}$$

and such a root $t=\tilde{t}_2$ is uniquely determined since the derivative of the left-hand side of (13) with respect to t is

$$X^*[\varphi_0(t_1)]X[\varphi_0(t)]\fallingdotseq\|X[\varphi_0(t_1)]\|^2\neq 0$$

for $t=\tilde{t}_2$. The actual value of $\tilde{t}_2$ can be found by solving (13) by means of Newton's method. The reason why t_1 and t_2 are chosen in the above manner is merely to reduce the errors produced in step-by-step numerical integration of the differential equations.

Now let

$$\rho=\rho(t,c)$$

be the solution of (10) such that $\rho(0,c)=c$ and put

$$\tau=\tau(t,c)\overset{\text{def}}{=\!=}\int_0^t\frac{\|X\|^2+\sum_\nu\rho^\nu(t,c)X^*\xi_\nu}{X^*X[\varphi_0+\sum_\nu\rho^\nu(t,c)\xi_\nu]}\,dt. \tag{14}$$

Then, from the differentiability of $R(\rho,t)$ and $X(x)$, it is evident that

$$\rho(t,c),\ \tau(t,c)\in C^N_{t,c}. \tag{15}$$

Further, as is seen from (11), for small $\|c\|$, $\|\rho(t,c)\|$ is small, consequently $d\tau(t,c)/dt$ is nearly equal to 1 for small $\|c\|$. Then it is evident that the relation (14) can be solved as

(16) $$t=t(\tau, c)\in C^N_{\tau,c}.$$

Let

(17) $$C: x=\varphi(\tau, c)=\varphi_0(t)+\sum_{\nu=2}^{n}\rho^\nu(t, c)\xi_\nu(t)\ (\underline{\underline{\mathrm{def}}}\ \hat{\varphi}(t, c))$$

be the orbit of the initial system (1) corresponding to $\rho=\rho(t, c)$. Such an orbit can be computed by step-by-step numerical integration of the initial system starting from the initial value

$$\varphi(0, c)=\varphi_0(0)+\sum_{\nu=2}^{n}c^\nu\xi_\nu(0).$$

Let P be the point of C such that

$$P: x=\hat{\varphi}(t_1, c)=\varphi(\tau_1, c),$$

namely the point where C meets π_1 at the time near t_1, and

$$Q: x=\hat{\varphi}(\hat{t}_2, c)=\varphi(\tau_2, c)$$

be the point where C meets π_1 at the time near $\tilde{t}_2$. In other words, P and Q are assumed to be respectively the first points where C meets π_1 when C is followed in the decreasing and increasing sense of the time starting from the point $x=\varphi(0, c)$. Since P and Q lie in π_1, the τ_i $(i=1, 2)$ must satisfy the equation

(18) $$X^*[\varphi_0(t_1)]\cdot[\varphi(\tau, c)-\varphi_0(t_1)]=0.$$

Then, since the derivative of the left-hand side of this equation with respect to τ is

$$X^*[\varphi_0(t_1)]X[\varphi(\tau, c)]\doteqdot\|X[\varphi_0(t_1)]\|^2\neq 0$$

for $\tau=\tau_i$ $(i=1, 2)$, the τ_i $(i=1, 2)$ are uniquely determined so that $\tau_i=\tau_i(c)\in C^N_c$ and $\tau_1(0)=t_1$, $\tau_2(0)\doteqdot\tilde{t}_2$ respectively, and moreover the actual values of the τ_i $(i=1, 2)$ can be computed from (18) by Newton's method when the actual value of c is known.

Then, from (16), we know that the function

$$\hat{t}_2=\hat{t}_2(c)=t[\tau_2(c), c]$$

belongs to C^N_c. Therefore, the problem of finding a closed orbit near C_0 is reduced to the one of determining a solution c of the continuously differentiable equations

(19) $$F^\nu(c)\ \underline{\underline{\mathrm{def}}}\ \xi^*_\nu(t_1)\{\hat{\varphi}[\hat{t}_2(c), c]-\hat{\varphi}(t_1, c)\}=0\quad(\nu=2, 3, \cdots, n).$$

In order to solve these equations by Newton's method, it is enough to know the value of the derivative of the function $F(c)=\{F^\nu(c)\}$ $(\nu=2, 3, \cdots, n)$, since, by assumption, $c=0$ is already an approximate solution.

Now, from (17), for small $\|c\|$, we have

$$\frac{\partial\hat{\varphi}(t, c)}{\partial t}=\frac{\partial\varphi(\tau, c)}{\partial\tau}\cdot\frac{\partial\tau(t, c)}{\partial t}\doteqdot X[\varphi(\tau, c)]=X[\hat{\varphi}(t, c)]$$

and

$$\frac{\partial\hat{\varphi}(t, c)}{\partial c^{\mu}}=\sum_{\lambda=2}^{n}\frac{\partial\rho^{\lambda}(t, c)}{\partial c^{\mu}}\xi_{\lambda}(t)\doteqdot\sum_{\lambda=2}^{n}\Psi_{\mu}^{\lambda}(t)\xi_{\lambda}(t)\quad(\mu=2, 3, \cdots, n),$$

because, for small $\|c\|$,

$$\frac{\partial\tau(t, c)}{\partial t}\doteqdot 1 \text{ and } \frac{\partial\rho^{\nu}(t, c)}{\partial c^{\mu}}\doteqdot\left.\frac{\partial\rho^{\nu}(t, c)}{\partial c^{\mu}}\right|_{c=0}=\Psi_{\mu}^{\nu}(t)$$

$$(\nu, \mu=2, 3, \cdots, n).$$

Then, using these relations, from (19), we have:

(20) $$\frac{\partial F^{\nu}(c)}{\partial c^{\mu}}\doteqdot\xi_{\nu}^{*}(t_1)\cdot\{X[\hat{\varphi}(\tilde{t}_2, c)]\frac{\partial\tilde{t}_2(c)}{\partial c^{\mu}}+\sum_{\lambda=2}^{n}\Psi_{\mu}^{\lambda}(\tilde{t}_2)\xi_{\lambda}(\tilde{t}_2)-\sum_{\lambda=2}^{n}\Psi_{\mu}^{\lambda}(t_1)\xi_{\lambda}(t_1)\}$$

$$\doteqdot\xi_{\nu}^{*}(t_1)\cdot\left\{X[\varphi_0(t_1)]\frac{\partial\tilde{t}_2(c)}{\partial c^{\mu}}+\sum_{\lambda=2}^{n}\Psi_{\mu}^{\lambda}(\tilde{t}_2)\xi_{\lambda}(\tilde{t}_2)-\sum_{\lambda=2}^{n}\Psi_{\mu}^{\lambda}(t_1)\xi_{\lambda}(t_1)\right\}$$

$$\doteqdot\sum_{\lambda=2}^{n}\Psi_{\mu}^{\lambda}(\tilde{t}_2)\xi_{\nu}^{*}(t_1)\xi_{\lambda}(t_1)-\sum_{\lambda=2}^{n}\Psi_{\mu}^{\lambda}(t_1)\xi_{\nu}^{*}(t_1)\xi_{\lambda}(t_1)=\Psi_{\mu}^{\nu}(\tilde{t}_2)-\Psi_{\mu}^{\nu}(t_1).$$

Thus, by Newton's method, the correction δc of the approximate solution c of (19) can be computed by solving the linear equation as follows:

(21) $$[\Psi(\tilde{t}_2)-\Psi(t_1)]\delta c+\kappa=0$$

where κ is a vector whose components κ^{ν} $(\nu=2, 3, \cdots, n)$ are

(22) $$\kappa^{\nu}=\xi_{\nu}^{*}(t_1)\{\varphi(\tau_2, c)-\varphi(\tau_1, c)\}\quad(\nu=2, 3, \cdots, n).$$

The matrices $\Psi(\tilde{t}_2)$ and $\Psi(t_1)$ can be computed by numerical integration of the linear equation (12).

Thus, by the known facts about the iteration method [1, 3], it is concluded: If

(23) $$\det[\Psi(\tilde{t}_2)-\Psi(t_1)]\neq 0$$

and the first approximate solution $c=0$ is sufficiently accurate, namely

(24) $$\|\kappa_0\|=\|\varphi_0(\tilde{t}_2)-\varphi_0(t_1)\|$$

is sufficiently small, then the iteration of the above correction process converges and, in an actual computation, after a finite times of repetitions of this process, a closed orbit near C_0; namely, a periodic solution close to $\varphi_0(t)$ can be computed as accurately as we desire[1].

The convergence of the above iterative process implies the existence of a periodic solution. But, further, in the present case, as is proved in [1], a closed orbit is unique in a small neighborhood of C_0. So, from whatever C_0 the computation may be started, there is obtained the same desired periodic solution except for the errors unavoidable in the present computation.

4. Two dimensional case

As in the previous method, when the given system (1) is of two dimensions, the equation (21) can be readily solved.

In fact, when the given system is of two dimensions, the normal unit vector can be chosen as

$$\xi^1 = -\frac{X^2}{\|X\|}, \quad \xi^2 = \frac{X^1}{\|X\|}\ ^{2)}.$$

Then, after simple calculations, we see that

$$\Psi(t) = \frac{\|X_0\|}{\|X\|} e^{h(t)},$$

where X_0 is the value of X at the point $x = \varphi_0(0)$ and

$$h(t) = \int_0^t \operatorname{div} X \Big|_{x=\varphi_0(t)} dt.$$

Thus the equation (21) can be solved readily. But, making use of the fact that the values of X at the points $x = \varphi_0(t_1)$ and $x = \varphi_0(\tilde{t}_2)$ are nearly equal to each other, we may replace the solution of (21) by the simpler formula as follows:

$$\delta c = \frac{[\varphi^1(\tau_2, c) - \varphi^1(\tau_1, c)]X_1^2 - [\varphi^2(\tau_2, c) - \varphi^2(\tau_1, c)]X_1^1}{\|X_0\| \, (e^{h(\tilde{t}_2)} - e^{h(t_1)})},$$

where X_1 is the value of X at the point $x = \varphi_0(t_1)$.

Of course, τ_1 and τ_2 are the roots of the equation

$$[\varphi^1(\tau, c) - \varphi_0^1(t_1)]X_1^1 + [\varphi^2(\tau, c) - \varphi_0^2(t_1)]X_1^2 = 0$$

such that $\tau_1 \doteqdot t_1$ and $\tau_2 \doteqdot \tilde{t}_2$.

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1) For proof of this fact, refer to [3].

2) X^i $(i = 1, 2)$ are the components of the vector $X = X[\varphi_0(t)]$.

5. Remarks

In the previous method, as compared with the present method, the initial point is always corrected in the normal hyperplane of each corrected orbit and the point P: $x=\varphi(\tau_1, c)$ is taken always for a fixed value of the time, say τ_1. But the formula by which the correction is computed is of the same form in both methods, so the rates of convergence of both iterations will differ little from each other.

Thus there will arise no radical disparity between the two methods when these are applied to the actual computation.

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