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Exceptional Values of Meromorphic Functions in a Neighborhood of the set of Singularities

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1. Let E be a compact set in the z-plane and let Ω be its complement with respect to the extended z-plane. Then it is well known that E is of capacity zero[1] if and only if Ω is a domain and admits no Green's function on it. Suppose that E is of capacity zero. We shall consider a single-valued meromorphic function $w=f(z)$ on Ω which has an essential singularity at each point of E, that is, the cluster set of $f(z)$ at each point of E is the whole w-plane.

We shall say that a value w is exceptional for $f(z)$ at a point ζ of E if there exists a neighborhood of ζ where the function does not take this value w. It is well known that the set of all exceptional values of $f(z)$ at a point ζ of E is a K_σ-set, by which we mean the union of an enumerable number of compact sets, and is of capacity zero. Here arises the following question: *Can we replace "a K_σ-set of capacity zero" by "at most two" or "at most enumerable"?*

In this paper, we shall show that, for every K_σ-set K of capacity zero in the w-plane, we can find a function $f(z)$ which has K as the set of its exceptional values at each singularity. From this fact we can of course conclude that the above question is answered in the negative. Furthermore we shall show by an example that, even if E is of logarithmic measure zero, the set of exceptional values is not always enumerable.

2. Theorem 1. *For every K_σ-set K of capacity zero in the w-plane, there exist a compact set E of capacity zero in the z-plane and a single-valued meromorphic function $f(z)$ on its complementary domain Ω such that $f(z)$ has an essential singlarity at each point of E, and the set of exceptional values at each singularity coincides with K.*

Proof.[2] Let $\{K_n\}_{n=1,2,\dots}$ be a non-decreasing sequence of compact sets whose union is equal to K and let R_n be the complement of K_n with respect to the extended w-plane. We observe first that each R_n can be considered as a Riemann surface with null boundary. The Dirichlet integral, taken in a domain (or an open set) G, of a function g will be denoted by $D_G(g)$. If, in

1) In this paper, capacity is always logarithmic.
2) When K itself is a compact set, the proof is considerably simpler. See the remark of Theorem 3.

particular, g is a harmonic measure defined with respect to G, we shall write simply $D(g)$ for $D_G(g)$. Before determining an exhaustion $\{R_{n,k}\}_{k=0,1,2,\dots}$ of R_n, we assume that it will always satisfy the following conditions:

1) the relative boundary $\partial R_{n,k}$ of $R_{n,k}$ consists of a finite number of closed analytic curves,

2) each component of $R_n - R_{n,k}$ is non-compact.

We shall denote by $N_n(k)$ the number of closed curves which are the components of $\partial R_{n,k}$, and by $\omega_{n,k}(w)$ the harmonic measure of $\partial R_{n,k}$ with respect to the open set $R_{n,k}-\bar{R}_{n,k-1}$. If we denote the components of $R_{n,k}-\bar{R}_{n,k-1}$ by $\{\Delta^{\nu}_{n,k}\}_{\nu=1,2,\dots,N_n(k-1)}$, then $\omega_{n,k}(w)$ is equal, in each $\Delta^{\nu}_{n,k}$, to the harmonic measure of $\partial\Delta^{\nu}_{n,k}\cap\partial R_{n,k}$ with respect to $\Delta^{\nu}_{n,k}$.

We shall determine $\{R_{n,k}\}$ by induction as follows: First $\{R_{1,k}\}_{k=0,1,2,\dots}$ are chosen so that

$$D(\omega_{1,k}) \leqq \frac{1}{2k}.$$

We take a small slit $S^{\nu}_{1,k}$ in each $\Delta^{\nu}_{1,k}$ such that

$$S^{\nu}_{1,k}\cap K_{k+1}=\phi$$

and, for the harmonic measure $\omega'_{1,k}(w)$ of $\partial R_{1,k}+\bigcup_{\nu=1}^{N_1(k-1)} S^{\nu}_{1,k}$ with respect to $R_{1,k}-\bar{R}_{1,k-1}-\bigcup_{\nu=1}^{N_1(k-1)} S^{\nu}_{1,k}$,

$$D(\omega'_{1,k}) \leqq 2D(\omega_{1,k}).$$

We set $P_1=P_2=1$ and

$$P_{n+1}=\sum_{j=1}^{n} P_j N_j(n-1).$$

Let us assume that we have chosen $\{R_{j,k}\}_{j=1,2,\dots,n;\,k=0,1,2,\dots}$ and $\{S^{\nu}_{j,k}\}_{j=1,2,\dots,n;\,k=0,1,2,\dots;\,\nu=1,2,\dots,N_j(k-1)}$ with the following properties:

1) $$D(\omega_{j,k}) \leqq \frac{1}{2kP_j},\ S^{\nu}_{j,k}\cap K_{k+1}=\phi,\ \bigcup_{p=1}^{j-1}\bigcup_{\nu=1}^{N_p(j-2)} S^{\nu}_{p,j-1}\subset R_{j,j-1}$$

and

2) $$D(\omega'_{j,k}) \leqq 2D(\omega_{j,k}),$$

where $\omega'_{j,k}(w)$ is the harmonic measure of $\partial R_{j,k}+\bigcup_{\nu=1}^{N_j(k-1)} S^{\nu}_{j,k}$ with respect to $R_{j,k}-\bar{R}_{j,k-1}-\bigcup_{\nu=1}^{N_j(k-1)} S^{\nu}_{j,k}$. We choose $\{R_{n+1,k}\}_{k=0,1,2,\dots}$ so that 1) and 2) are satisfied for $j=n+1$.

We shall construct a covering surface $\hat{R}$ of the extended w-plane such that it is of planar character, belongs to the class O_G and covers infinitely often no point of K but does all points lying outside K.

First we connect R_2 with R_1 crosswise across the slit $S^1_{1,1}$ and denote by $\hat{R}^2$ the resulting surface. Next we take $P_3 = N_1(1) + N_2(1)$ replicas of R_3 and connect each one of them with $\hat{R}^2$ crosswise across each $S^{\nu}_{j,2}$ ($j=1, 2$; $\nu=1, 2, \cdots, N_j(1)$). Thus we obtain a surface $\hat{R}^3$ which contains $\hat{R}^2$ as its subsurface. Supposing that $\hat{R}^n$ is obtained, we connect each one of P_{n+1} replicas of R_{n+1} with $\hat{R}^n$ crosswise across each $S^{\nu}_{j,n}$ ($j=1, 2, \cdots, n$; $\nu=1, 2, \cdots, N_j(n-1)$) so that a surface $\hat{R}^{n+1}$ of planar character is obtained. Here we note that, for each j and ν, the slit $S^{\nu}_{j,n}$ appears on $\hat{R}^n$ just P_j times and hence we need $P_{n+1} = \sum_{j=1}^{n} P_j N_j(n-1)$ replicas of R_{n+1}. The limiting surface $\hat{R}$ is of planar character too. We shall show that $\hat{R}$ belongs to O_G.

We take the part of $\hat{R}$ corresponding to $R_{1,0}$ as the first domain of exhaustion and denote it by $\hat{R}_0$. We consider the part of $\hat{R}$ corresponding to $R_{1,1}$ and $R_{2,1}$. Then it is a domain containing $\hat{R}_0$, and is taken as the second domain of exhaustion. It will be denoted by $\hat{R}_1$. Next we consider the part of $\hat{R}$ corresponding to $R_{1,2}$, $R_{2,2}$ and $R_{3,2}$. This domain contains $\hat{R}_1$ and will be denoted by $\hat{R}_2$. In this manner we obtain an exhaustion $\{\hat{R}_n\}$ of $\hat{R}$. We shall use the following criterion due to Sario [6]:

A Riemann surface belongs to O_G if there is an exhaustion $\{F_n\}$ with the property that

$$\sum_{n=1}^{\infty} \frac{1}{D(\omega_n)} = \infty,$$

where ω_n is the harmonic measure of ∂F_n with respect to the open set $F_n - \bar{F}_{n-1}$.

We shall denote the harmonic measure of $\partial \hat{R}_n$ with respect to $\hat{R}_n - \bar{\hat{R}}_{n-1}$ by $\hat{\omega}_n$. We observe that $\hat{R}_1 - \bar{\hat{R}}_0$ consists of $R_{1,1} - \bar{R}_{1,0}$ and $R_{2,1}$ connected crosswise across $S^1_{1,1}$. We define a continuous function u_1 on $\hat{R}_1 - \bar{\hat{R}}_0$ by

$$u_1 = \begin{cases} \omega'_{1,1} & \text{on } R_{1,1} - \bar{R}_{1,0} - S^1_{1,1}, \\ 1 & \text{elsewhere.} \end{cases}$$

This function has the same boundary value as $\hat{\omega}_1$ and is piecewise continuously differentiable. Therefore, by the Dirichlet principle,

$$D(\hat{\omega}_1) \leqq D_{\hat{R}_1 - \bar{\hat{R}}_0}(u_1) = D(\omega'_{1,1}) \leqq 2D(\omega_{1,1}) \leqq \frac{2}{2} = 1$$

To evaluate $D(\hat{\omega}_2)$ we note that $\hat{R}_2 - \bar{\hat{R}}_1$ has two connected components; one consists of $R_{1,2} - \bar{R}_{1,1}$ and $N_1(1)$ replicas of $R_{3,2}$ connected crosswise across $\{S^{\nu}_{1,2}\}_{\nu=1,2,\ldots,N_1(1)}$, and the other consists of $R_{2,2} - \bar{R}_{2,1}$ and $N_2(1)$ replicas of $R_{3,2}$, connected crosswise across $\{S^{\nu}_{2,2}\}_{\nu=1,2,\ldots,N_2(1)}$. We define a continuous function u_2 on $\hat{R}_2 - \bar{\hat{R}}_1$ by

$$u_2=\begin{cases}\omega'_{1,2} & \text{on } R_{1,2}-\bar{R}_{1,1}-\bigcup_{\nu=1}^{N_1(1)} S^{\nu}_{1,2},\\ \omega'_{2,2} & \text{on } R_{2,2}-\bar{R}_{2,1}-\bigcup_{\nu=1}^{N_2(1)} S^{\nu}_{2,2},\\ 1 & \text{elsewhere.}\end{cases}$$

This function has the same boundary value as $\hat{\omega}_2$ and it follows by the Dirichlet principle that

$$D(\hat{\omega}_2)\leqq D_{\hat{R}_2-\hat{R}_1}(u_2)=D(\omega'_{1,2})+D(\omega'_{2,2})$$
$$\leqq 2D(\omega_{1,2})+2D(\omega_{2,2})\leqq\frac{2}{2\cdot 2}+\frac{2}{2\cdot 2}=1.$$

We continue this calculation. By the construction of $\hat{R}^n$ we attach one each replica of R_n to $\hat{R}^{n-1}$ at each slit $S^{\nu}_{j,n-1}$ $(1\leqq j\leqq n-1,\ 1\leqq\nu\leqq N_j(n-2))$. The number of these replicas is equal to P_n which was defined before. We define a continuous function u_n on $\hat{R}_n-\bar{\hat{R}}_{n-1}$ by

$$u_n=\begin{cases}\omega'_{j,k} \text{ on the replicas of } R_{j,n}-\bar{R}_{j,n-1}-\bigcup_{\nu=1}^{N_j(n-1)} S'_{j,n} \quad (1\leqq j\leqq n),\\ 1 \quad \text{elsewhere.}\end{cases}$$

It follows that

$$D(\hat{\omega}_n)\leqq D_{\hat{R}_n-\hat{R}_{n-1}}(u_n)=D(\omega'_{1,n})+D(\omega'_{2,n})+P_3D(\omega'_{3,n})+\cdots\cdots+P_nD(\omega'_{n,n})$$
$$\leqq 2P_1D(\omega_{1,n})+2P_2D(\omega_{2,n})+\cdots+2P_nD(\omega_{n,n})\leqq\sum_{k=1}^{n}\frac{P_k}{nP_k}=1.$$

Consequently

$$\sum_{n=1}^{\infty}\frac{1}{D(\hat{\omega}_n)}\geqq\sum_{n=1}^{\infty}1=\infty.$$

Therefore $\hat{R}$ belongs to O_G on account of Sario's criterion.

Since $\hat{R}$ is of planar character and belongs to the class O_G, we can map $\hat{R}$ one-to-one conformally onto a domain Ω on the z-plane which is the complement of a compact set E of capacity zero. If we denote by $\hat{f}$ and φ this mapping function of $\hat{R}$ onto Ω and the projection of $\hat{R}$ into the extended w-plane, respectively, then $f(z)=\varphi\circ\hat{f}^{-1}(z)$ is a function satisfying the conditions of the theorem. In fact, let ζ be an arbitrary point of E and let r be a positive number such that the circle c: $|z-\zeta|=r$ does not pass any point of E. Since the image $\hat{f}^{-1}(c)$ of c on $\hat{R}$ is a compact set, there exists an n such that $\hat{R}_n$ contains $\hat{f}^{-1}(c)$. Hence the circular disc (c): $|z-\zeta|<r$ contains the image of at least one component of $\hat{R}-\bar{\hat{R}}_n$, but every component of $\hat{R}-\bar{\hat{R}}_n$ contains at least one replica of R_m for every m greater than n as its subdomain. It follows from this that, for each point ζ of E and its arbitrary neighborhood $v(\zeta)$, $f(z)$ takes in $v(\zeta)$ infinitely often each value contained in the complement

of K with respect to the extended w-plane. Hence we can conclude that $f(z)$ has an essential singularity at each point ζ of E and has K as the set of exceptional values at ζ. Thus our theorem is established.

3. REMARK 1. From our theorem, we see that there exists a set of exceptional values which is everywhere dense in the w-plane. For one point is a compact set of capacity zero and hence the set of all rational points is a K_σ-set, which is everywhere dense in the w-plane.

REMARK 2.[3] Let D be a domain in the z-plane, let Γ be its boundary and let E be a closed set of capacity zero contained in Γ. We suppose that $w=f(z)$ is non-constant, single-valued and meromorphic in D, and associate with every point z_0 of Γ the following sets of values.

i) *The cluster set* $C_D(f, z_0)$. $\alpha \in C_D(f, z_0)$ if there exists a sequence of points $\{z_n\}$ with the following properties:

$$z_n \in D, \lim_{n\to\infty} z_n = z_0 \text{ and } \lim_{n\to\infty} f(z_n) = \alpha.$$

ii) *The boundary cluster set* $C_{\Gamma-E}(f, z_0)$. $\alpha \in C_{\Gamma-E}(f, z_0)$ if there is a sequence of points $\{\zeta_n\}$ of $\Gamma-(z_0 \cup E)$ such that

$$w_n \in C_D(f, \zeta_n) \quad \text{for each } n,$$

$$z_0 = \lim_{n\to\infty} \zeta_n \text{ and } \alpha = \lim_{n\to\infty} w_n.$$

When z_0 is a point of E such that $U(z_0) \cap (\Gamma - E) \neq \phi$ for any neighborhood $U(z_0)$ of z_0 and the open set $\Omega = C_D(f, z_0) - C_{\Gamma-E}(f, z_0)$ is not empty, it is known that $f(z)$ takes every value of Ω infinitely often in any neighborhood of z_0 except for a possible set of capacity zero. We raise the question as to whether the number of exceptional values in each component of Ω can be reduced at most to two. This is actually true if we add some condition on E, for instance, "E is contained in a single boundary component of Γ"[4] or "each point of E belongs to a boundary component of Γ consisting of a non-degenerate continuum"[5]. Here we remark that our quention is not true in general. In fact, we use the same notations as in the proof of Theorem 1 and suppose that K itself is compact and non-enumerable. If we denote by s a closed segment in the w-plane not touching K, then there exist an infinite number of segments s_n on $\hat{R}$ whose projections are just s, and their images $\gamma_n = \hat{f}(s_n)$ cluster to E. Hence we can find a point z_0 of E such that any neighborhood $U(z_0)$ of z_0 contains an infinite number of γ_n's. Set $D = \Omega - \bigcup_{n=1}^{\infty} \gamma_n$. Then D is a

3) This remark is due to Kuroda.
4) Cf. Noshiro [5].
5) Cf. Hervé [4].

domain and has $\Gamma=E+\bigcup_{n=1}^{\infty}\gamma_n$ as its boundary. The point z_0 satisfies that $U(z_0)\cap(\Gamma-E)\neq\phi$ for any neighborhood $U(z_0)$ of z_0. For f restricted to D, $C_D(f, z_0)$ is the whole w-plane, $C_{\Gamma-E}(f, z_0)=s$ and hence Ω consists of a single component which is the complement of s with respect to the extended w-plane. But $f(z)$ does not take any value belonging to K; this is contained in Ω and non-enumerable.

4. In this and the following sections, we shall be concerned with functions which have as the set of singularities a compact set E of finite logarithmic measure.

We shall state the definition of logarithmic measure. Let e be a bounded set in the z-plane. We cover e by an at most enumerable number of discs $\{S_i\}$ with diameters $d_i<\rho$ and put

$$m_0(e, \rho)=\inf\sum_i \frac{1}{\log\dfrac{1}{d_i}},$$

where the lower bound is taken over all the $\{S_i\}$. $m_0(e, \rho)$ increases as $\rho\to 0$, so that

$$\lim_{\rho\to 0} m_0(e, \rho)=m_0(e) \quad (0\leqq m_0(e)\leqq\infty)$$

exists. We call this quantity $m_0(e)$ the logarithmic measure of e.

By a theorem of Erdös and Gills [1][6], a set of finite logarithmic measure is of capacity zero. But we shall see in the following that there exists a single-valued meromorphic function such that the set of singularities is of logarithmic measure zero and yet the set of exceptional values at each singularity is non-enumerable. First we prove the following lemma.

LEMMA. *Let γ be a continuum in the unit disc $|z|<1$ whose diameter is equal to d. If $D(\omega_\gamma)$ is sufficiently small, then*

$$D(\omega_\gamma)>\frac{\pi}{\log 1/d},$$

where ω_γ is the harmonic measure of γ with respect to the unit disc.

Proof. We map the unit disc by a linear transformation $\zeta=T(z)$ onto itself so that a certain point $z_0=re^{i\theta}$ of γ is transformed to the origin. Set $\rho=\max_{T(\gamma)\ni\zeta}|\zeta|$. From Teichmüller's result concerning extremal region[7], we see that

$$D(\omega_\gamma)=D(\omega_{T(\gamma)})\geqq D(\omega_{\gamma_\rho}),$$

6) Cf. Kametani [2].
7) See Teichmüller [7].

where γ_ρ is a radius of the circle c_ρ: $|\zeta|=\rho$. As $D(\omega_\gamma)\to 0$, $\rho\to 0$ and

$$\frac{1}{D(\omega_{\gamma_\rho})}=\frac{1}{2\pi}\log\frac{4}{\rho}+O(\rho^2)$$ [8].

By the inverse transformation $z=T^{-1}(\zeta)$, the disc $|\zeta|\leqq\rho$ is mapped onto a disc containing γ, whose diameter is equal to $2\rho\frac{1-r^2}{1-\rho^2r^2}$.

Therefore

$$d\leqq 2\rho\frac{1-r^2}{1-\rho^2r^2}\leqq 2\rho,$$

and we see that

$$\frac{1}{D(\omega_\gamma)}\leqq\frac{1}{2\pi}\log\frac{1}{d}+\frac{1}{2\pi}\log 8+O(\rho^2).$$

Hence there exists a positive number δ such that, if $D(\omega_\gamma)\leqq\delta$, then

$$\frac{1}{2\pi}\log 8+O(\rho^2)\leqq\frac{1}{2\pi}\log\frac{1}{d},$$

and we have

$$D(\omega_\gamma)\geqq\frac{\pi}{\log\frac{1}{d}}.$$

Next we give a sufficient condition for a compact set E to be of finite logarithmic measure.

THEOREM 2. *Let E be a compact set in a complex plane and let Ω be the component of the complement of E which contains the point at infinity. Assume that there is an exhaustion $\{R_n\}_{n=0,1,2,\dots}$ which satisfies conditions* 1) *and* 2) *stated in the proof of Theorem* 1 *and satisfies the following condition*:

$$+\infty>\varliminf_{n\to\infty}N(n)D(\omega_n)=d,$$

where we denote by $N(n)$ the number of closed curves which are components of ∂R_n and by ω_n the harmonic measure of ∂R_n with respect to the domain $R_n-\bar{R}_0$. Unless E consists of a finite number of continuums and contains at least one non-degenerate component, the logarithmic measure of E is not greater than $\frac{d}{\pi}$.

Proof. The case where E consists of a finite number of points is trivial. Therefore we assume that $N(n)\to\infty$ as $n\to\infty$ and hence that $D(\omega_n)\to 0$ as $n\to\infty$. Without loss of generality, we may assume that R_0 is the outside of the unit disc. For the positive number δ stated in the proof of Lemma, there is an n_0

8) Cf. Hersch [3].

such that $D(\omega_{n_0})<\delta$. Let n be larger than n_0. The complement of the domain R_n covers E completely and has $N(n)$ components $\Delta_{n,i}$ ($i=1, 2,\cdots, N(n)$). If we denote by $\{d_i\}$ the diameters of these components, then we have by Lemma

$$N(n)D(\omega_n)\geqq \pi\sum_{i=1}^{N(n)}\frac{1}{\log\dfrac{1}{d_i}},$$

because the harmonic measure of one component of ∂R_n has of course a Dirichlet integral less than $D(\omega_n)$. As $n\to\infty$, $\max_{1\leqq i\leqq N(n)} d_i$ tends to zero and hence we see that

$$d=\varliminf_{n\to\infty} N(n)D(\omega_n)\geqq \pi\, m_0(E).$$

Thus our theorem is established.

5. We shall give a sufficient condition for a compact set E_w in the w-plane to be exactly equal to the set of exceptional values of a single-valued meromorphic function with singularities of logarithmic measure zero. In the next section, we shall give an example of a compact set E_w which is non-enumerable and satisfies our condition. Hence we shall be able to conclude that even if the set of singularities is of logarithmic measure zero, the question raised in §1 is answered in the negative.

THEOREM 3. *Let E_w be a compact set in the w-plane and let R be the component of the complement of E_w which contains the point at infinity. If there exists an exhaustion $\{R_n\}_{n=0,1,2,\dots}$ of R such that*

$$(*)\qquad +\infty>\varliminf_{n\to\infty} N(n)\Big(\prod_{\nu=1}^{n}(N(\nu-1)+1)\Big)^2 D(\omega_n)=d,$$

then we can find a compact set E in the z-plane whose logarithmic measure is not greater than $\dfrac{d}{\pi}$, and a single-valued meromorphic function $f(z)$ on its complementary domain Ω such that it has an essential singularity at each point of E and the set of exceptional values at each singlularity coincides with E_w.

Proof. Let $\{R_n\}_{n=0,1,2,\dots}$ be an exhaustion satisfying the condition (*). Then, there exist $N(n-1)$ components of the open set $R_n-\bar{R}_{n-1}$. Denote these components by $\Delta_{n,m}$ ($m=1, 2,\cdots, N(n-1)$). We take a slit $S_{n,m}$ on each $\Delta_{n,m}$ in arbitrary way.

In a similar manner as in the proof of Theorem 1, we construct a covering surface of the w-plane which is of planar character. First we connect one replica of R with another crosswise across the slit $S_{1,1}$. If we denote the resulting surface by $\hat{R}^1$, then $\hat{R}^1$ has $N(0)+1=2$ sheets. Next, we take $(N(0)+1)N(1)$ replicas of R and connect each one of them with $\hat{R}^1$ crosswise across each slit $S_{2,k}$ lying on $\hat{R}^1$. The resulting surface $\hat{R}^2$ has $(N(0)+1)$ $(N(1)+1)$

sheets. Supposing that $\hat{R}^n$ with $\prod_{\nu=1}^{n}(N(\nu-1)+1)$ sheets is obtained, we connect each one of $N(n)\prod_{\nu=1}^{n}(N(\nu-1)+1)$ replicas of R crosswise across each slit $S_{n+1,k}$ and obtain $\hat{R}^{n+1}$. The limiting surface $\hat{R}$ of $\hat{R}^n$ as $n\to\infty$ is a covering surface of planar character. From our construction, the set of all points, which are covered by $\hat{R}$ infinitely often, is just R.

We define $\{\hat{R}_n\}_{n=0,1,2,\dots}$ as follows:

$\hat{R}_0$: the part of the first replica lying over the subdomain R_0 of R,
$\hat{R}_1$: the part of $\hat{R}^1$ lying over the subdomain R_1 of R,
....,
$\hat{R}_n$: the part of $\hat{R}^n$ lying over the subdomain R_n of R,
....

Then it is easy to see that $\{\hat{R}_n\}_{n=0,1,2,\dots}$ form an exhaustion of R. Let $\hat{\omega}_n$ be the harmonic measure of $\partial\hat{R}_n$ with respect to the domain $\hat{R}_n-\bar{\hat{R}}_0$ and let u_n be the subharmonic function of R_n such that $u_n=\omega_n$ on $R_n-\bar{R}_0$ and $u_n=0$ on $\bar{R}_0$. By the Dirichlet principle, we have

$$D(\hat{\omega}_n)=D_{\hat{R}_n-\hat{R}_0}(u_n\circ\varphi)=\left(\prod_{\nu=1}^{n}(N(\nu-1)+1)\right)D(\omega_n),$$

because $\hat{R}_n$ covers each point of R_n just $\prod_{\nu=1}^{n}(N(\nu-1)+1)$ times. Here we denote by φ the projection of $\hat{R}$ on the w-plane. Let $\hat{N}(n)$ be the number of closed curves which are the components of $\partial\hat{R}_n$. Then

$$\hat{N}(n)=N(n)\prod_{\nu=1}^{n}(N(\nu-1)+1)$$

and hence from the condition (*)

$$(**)\qquad \varliminf_{n\to\infty} N(n)D(\hat{\omega}_n)\leq d.$$

Since $\hat{R}$ is of planar character, we can map $\hat{R}$ one-to-one conformally onto a domain Ω on the z-plane which is the complement of a compact set E. We denote by $\hat{f}$ this mapping function of $\hat{R}$ onto Ω. From the relation (**) and Theorem 2, we see that E has a logarithmic measure not greater than $\frac{d}{\pi}$. By the same reasoning as in the proof of Theorem 1, we see that the function $f(z)=\varphi\circ\hat{f}^{-1}(z)$ has an essential singularity at each point of E and has E_w as the set of its exceptional values at each singularity. Thus our theorem is established.

REMARK. When E_w is a compact set of capacity zero, then there is always an exhaustion with the condition that

$$\lim_{n\to\infty}\prod_{\nu=1}^{n}(N(\nu-1)+1)D(\omega_n)=0.$$

For the number $\prod_{\nu=1}^{n}(N(\nu-1)+1)$ is determined by only $R_0, R_1, \cdots, R_{n-1}$ and we can take R_n so that $\prod_{\nu=1}^{n}(N(\nu-1)+1)D(\omega_n)<\frac{1}{n}$. Hence, if we construct $\hat{R}$ in the same manner as above, then $\lim_{n\to\infty} D(\hat{\omega}_n)=0$ and hence E is of capacity zero. Thus we have Theorem 1 in the particular case that the set K itself is compact.

6. In this section, we shall give an example of compact sets in the w-plane which are not enumerable and satisfy the condition (*) in Theorem 3. From this example, we see that the case stated in Theorem 3 is not empty.

Let S^1 be a closed segment on the real axis in the nuit disc $|w|<1$ such that its harmonic measure $\omega^{(1)}$ with respect to the unit disc has a positive Dirchlet integral less than $1/(1\cdot 2(2^0+1)^2)$. Next we exclude an open segment from the middle of S^1 so that, for the remaining segments being denoted by S_1^2 and S_2^2, the harmonic measure $\omega^{(2)}$ of their union S^2 with respect to the unit disc has a positive Dirichlet integral less than $1/(2\cdot 2^2(2^0+1)^2(2+1)^2)$. Continue these constructions inductively; that is, if after n times of these constructions, we have 2^{n-1} subsegments S_k^n $(k=1, 2, \cdots, 2^{n-1})$ of S^1 of equal length such that the harmonic measure $\omega^{(n)}$ of their union S^n with respect to the unit disc has a positive Dirichlet integral less than $1/(n\cdot 2^n(\prod_{\nu=1}^{n}(2^{\nu-1}+1))^2)$, then we exclude an open segment from the middle of every $S_k^n (k=1, 2, \cdots, 2^{n-1})$ so that the harmonic measure $\omega^{(n+1)}$ of the remaining part S^{n+1} with respect to the unit disc has a positive Dirichlet integral less than $1/((n+1)\cdot 2^{n+1}(\prod_{\nu=1}^{n+1}(2^{\nu-1}+1))^2)$.

Now, we put as follows:

$$E_w=\bigcap_{n=1}^{\infty} S^n,$$

$$R_0=\{w;\ |w|>1\}$$

and

$$R_n=\left\{w;\ \omega^{(n)}(z)<\frac{1}{2}\right\}\cup \bar{R}_0.$$

Evidently E_w is a non-enumerable compact set.

We shall show that the sequence $\{R_n\}_{n=0,1,2,\dots}$ just obtained is an exhaustion which satisfies the condition (*) in Theorem 3. Since $\omega^{(n)}(w)$ tends to zero as $n\to\infty$ at every point w of R, ∂R_n tends to E_w and hence the sequence $\{R_n\}$ is an exhaustion of R. Let ω_n be the harmonic measure of ∂R_n with respect to the domain $R_n-\bar{R}_0$. Then $\omega_n=2\omega^{(n)}$ there and hence we have the following:

$$D(\omega_n)=4D_{R_n-\bar{R}_0}(\omega^{(n)})=2D(\omega^{(n)})\leq 2/(n\cdot 2^n(\prod_{\nu=1}^{n}(2^{\nu-1}+1))^2).$$

Since the number $N(n)$ of closed curves which are components of the relative boundary ∂R_n is not greater than 2^n, we have

$$\varlimsup_{n\to\infty} N(n)(\prod_{\nu=1}^{n}(N(\nu-1)+1))^2 D(\omega_n)$$

$$\leqq \varlimsup_{n\to\infty} N(n)(\prod_{\nu=1}^{n}(N(\nu-1)+1))^2 \cdot 2/(n\cdot 2^n(\prod_{\nu=1}^{n}(2^{\nu-1}+1))^2) \leqq \lim_{n\to\infty} 2/n = 0.$$

Thus the condition (*) is satisfied. Our proof is now complete.

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