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Decomposition of General Lattices into Direct Summands of Types I, II and III

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In the demension theory, the decomposition of lattices into direct summands of types I, II and III is an important result. The basic concepts used in this decomposition are the minimality and the finiteness of the dimension values. Therefore the decomposition theorem is obtained after the dimension is introduced. But Kaplansky [4] decomposed a Baer ring where the demension is not introduced. A Bear ring A is a ring in which every annihilator is generated by an idempotent. An idempotent e in A is said to be abelian in case the idempotents of eAe mutually commute, and an idempotent e in A is said to be finite, if $xy=e$, $x,y \in eAe$ imply $yx=e$. Using these two conceptions, Kaplansky have a decomposition theorem of a Bear ring. But these two basic conceptions are far from the dimension. Corresponding to this decomposition of a Baer ring A, there must be a decomposition of the lattice of annihilators in A. This motivates me to consider the decomposition theorem of general lattices where the dimension is not introduced.

In § 1, I give the decomposition theorem of lattices in a most abstract form. A Z-lattice L is a complete lattice whose center Z is a complete Boolean sublattice of L with some infinite distributivity. In a Z-lattice L, we introduce P-property which satisfies the following conditions:

(P 1) If a non-zero element a has P-property, then $z\frown a$ has P-prorerty for every $z \in Z$ such that $z\frown a\neq 0$.

(P 2) Let $\{a_\alpha;\ \alpha \in I\}$ be a family of elements with P-property, and $\perp(e(a_\alpha);\ \alpha \in I)$, then $\bigvee_{\alpha\in I} a_\alpha$ has P-property. ($e(a_\alpha)$ means the central cover of a_α.)

I show that taking any two P-properties for defining the minimal element and the finite element, we can decompose the Z-lattice L into direct summands of types I, II and III.

By Kaplansky [3, p. 538], an element a is called a D-element in case $L(0, a)$ is distributive, and he decomposed a complete complemented modular lattice using this element. Since a D-element has the P-property, we can use it for the decomposition of Z-lattices. But the minimal property of the D-element is not clear. Therefore, in §2 I introduce the lowest element. I write $a\ll b$ when either $z\frown a< z\frown b$ or $z\frown a=z\frown b=0$ for every $z \in Z$, and say that a non-zero element a is a lowest element if $x\ll a$ implies $x=0$. A lowest element has the P-property, and its geometrical meaning is clear. The properties of the lowest element and its relation to the D-element are obtained.

To obtain the element of finite property, in §3 I consider a Z-lattice L with an equivalence relation "$\overset{e}{\sim}$". This relation "$\overset{e}{\sim}$" may satisfy the weaker conditions than those which introduce the dimension. The relative center $Z_0=\{z \in Z;\ a \overset{e}{\sim} b \leqq z$ implies $a \leqq z\}$ is a complete Boolean sublattice of L. we write $a \overset{e}{\lesssim} b$ when there exists b_1, such that $a \overset{e}{\sim} b_1 < b$, and $a \overset{e}{\ll} b$ when either $z \cap a \overset{e}{\lesssim} z \cap b$ or $z \cap a = z \cap b = 0$ for every $z \in Z_0$. A nonzero element a is called e-minimal if $x \overset{e}{\ll} a$ implies $x=0$, and an element a is called e-finite if $a \overset{e}{\sim} b \leqq a$ implies $b=a$. Then e-minimal elements and e-finite elements have P-property. Hence with these conceptions we have a decomposition of the Z-lattice, which may be called an analytic decomposition. Of course this decomposition depends on the equivalence relation "$\overset{e}{\sim}$".

S. Maeda [9] gave a theory of dimension functions in a most general lattice, which we may call a generalized relatively orthocomplemented lattice. Since this lattice is a Z-lattice, S. Maeda's decomposition of this lattice is an analytic decomposition.

In §4, I consider a relatively orthocomplemented complete lattice. This lattice, being a special case of S. Maeda's general lattice, is a Z-lattice. In this section, I prove this fact directly using the compatible elements. Loomis's [5] dimension lattice is a relatively orthocomplemented complete lattice on which the dimension is introduced. His decomposition of this lattice coincides with the analytic decomposition.

In §5, I investigate the lattice $\mathring{R}_A$ of right-annihilators in Baer ring A. $\mathring{R}_A$ is composed of all principal right ideals $(e)_r$ generated by idempotents e. The set R_A of all right ideals in A is a complete modular lattice ([8] Kapitel VI, Sat 1.1), but $\mathring{R}_A$ is not a sublattice of R_A, g.l.b. of these lattices coinside, but l.u.b. may be different. I prove that $\mathring{R}_A$ is a Z-lattice. By Kaplansky [4], two idempotents e, f in A are said to be equivalent, when there exists $x \in eAf$, $y \in fAe$ with $xy=e$, $yx=f$. This equivalence induces an equivalence $(e)_r \overset{a}{\sim} (f)_r$ in $\mathring{R}_A$. With this equivalence relation "$\overset{a}{\sim}$" we have an analytic decomposition of $\mathring{R}_A$. When A has no nilpotent ideals, this analytic decomposition of $\mathring{R}_A$ corresponds to Kaplansky's decomposition of A.

When A is a Baer*-ring, $\mathring{R}_A$ is isomorphic to the lattice of projections in A, which is investigated in detail by S. Maeda [10]. In this lattice another equivalence relation "$\overset{*}{\sim}$" is also introduced, and a non-zero abelian projection, a non-zero D-element, a lowest element, an a-minimal element and a $*$-minimal element all coincide.

§1. General Decomposition of Z-lattices

DEFINITION 1.1. Let L be a complete lattice whose center Z is a complete Boolean sublattice of L, and when $a_\alpha \in Z$ for all $\alpha \in I$ or $b \in Z$

$$(1) \qquad \bigvee_{\alpha \in I} a_\alpha \cap b = \bigvee_{\alpha \in I} (a_\alpha \cap b)$$

holds. For brevity, we call such a lattice L a *Z-lattice.*

The most simple example of a Z-lattice is a continuous lattice. (Cf. [8] Kapitel I. Satz 3.7 and Hilfssatz 3.6.)

DEFINITION 1.2. A lattice L with 0 is called *relatively complemented* (in a weak sense), if, given $b \leqq a$, an element c exists such that $a = b \dot{\cup} c$.[1)]

Remark 1.1. In definition 1.1, if L is moreover relatively complemented, the infinite distributivity (1) for $b \in Z$ is superfluous by the following Lemma 1.1.

LEMMA 1.1. *In a relatively complemented complete lattice with the center* Z, *we have*, for $b \in Z$,

$$\bigvee_{\alpha \in I} a_\alpha \cap b = \bigvee_{\alpha \in I} (a_\alpha \cap b).$$

PROOF. Let $a_\alpha = (a_\alpha \cap b) \dot{\cup} c_\alpha$ $(\alpha \in I)$ and $1 = b \dot{\cup} b'$. Then $c_\alpha \cap b = c_\alpha \cap a_\alpha \cap b = 0$. Since $b \in Z$ we have $c_\alpha = c_\alpha \cap (b \cup b') = c_\alpha \cap b'$, that is, $c_\alpha \leqq b'$. Hence $\bigvee_{\alpha \in I} c_\alpha \leqq b'$, and $\bigvee_{\alpha \in I} c_\alpha \cap b = 0$. Now, we have

$$\bigvee_{\alpha \in I} a_\alpha \cap b = \{\bigvee_{\alpha \in I} (a_\alpha \cap b) \cap \bigvee_{\alpha \in I} c_\alpha\} \cap b = \bigvee_{\alpha \in I} (a_\alpha \cap b).$$

Remark 1.2. By [3] Theorem 5, the center Z of a complete complemented modular lattice L is a complete Boolean sublattice of L, in which (1) of Definition 1.1 holds for $a_\alpha \in Z(\alpha \in I)$. But since L is relatively complemented, (1) holds for $b \in Z$. Hence *a complete complemented modular lattice is a Z-lattice.*

Since the center Z of a Z-lattice L is complete, we can define the central cover $e(a)$ of a as the least central element z such that $a \leqq z$, and we can prove the following Lemma as in [8] Kapitel II, Hilfssatz 4.7.

LEMMA 1.2. *Let L be a Z-lattice, and Z be its center.*

(i) *When* $z \in Z$, *then* $e(z \cap a) = z \cap e(a)$.

(ii) $e(\bigvee_{\alpha \in I} a_\alpha) = \bigvee_{\alpha \in I} e(a_\alpha)$.

Remark 1.3. Let z_0 be a central element of a Z-lattice L. Then it is evident that $L(0, z_0)$ is also a Z-lattice, and the central cover of an element a in $L(0, z_0)$ is the same as the central cover of a in L.

LEMMA 1.3. *Let* $a_\alpha (\alpha \in I)$ *be elements of a Z-lattice L, such that* $\perp (e(a_\alpha); \alpha \in I)$. *Set* $a = \bigvee_{\alpha \in I} a_\alpha$, *then* $e(a_\alpha) \cap a = a_\alpha$ *for all* $\alpha \in I$.

PROOF. $e(a_\alpha) \cap a = e(a_\alpha) \cap \bigvee_{\beta \in I} a_\beta = \bigvee_{\beta \in I} (e(a_\alpha) \cap a_\beta)$.

If $\alpha \neq \beta$, by Lemma 1.2 we have $e(e(a_\alpha) \cap a_\beta) = e(a_\alpha) \cap e(a_\beta) = 0$. Hence $e(a_\alpha) \cap a_\beta = 0$. Therefore $e(a_\alpha) \cap a = e(a_\alpha) \cap a_\alpha = a_\alpha$.

DEFINITION 1.3. In a Z-lattice L with the center Z, we introduce *P-pro-*

1) $b \dot{\cup} c$ means $b \cup c$ with $b \cap c = 0$. $\dot{\bigvee}_{\alpha \in I} a_\alpha$ means $\bigvee_{\alpha \in I} a_\alpha$ with $\perp (a_\alpha ; \alpha \in I)$. Cf. [8] Kapitel I, Definition 1. 15.

perty which satisfies the following condition:

(P1) If a non-zero element a has P-property, then $z \cap a$ has P-property for every $z \in Z$ such that $z \cap a \neq 0$.

(P2) Let $\{a_\alpha; \alpha \in I\}$ be a family of elements with P-property, and $\perp (e(a_\alpha); \alpha \in I)$, then $_{\alpha \in I}\bigcup a_\alpha$ is an element with P-property.

DEFINITION 1.4. In a Z-lattice L, let P_m and P_f are two P-properties. We say that a non-zero element a is *minimal* (with respect to P_m) if a has P_m, and an element a is *finite* (with respect to P_f) if a has P_f. An element a is *infinite* (with respect to P_f) when a is not finite, and a non-zero element a is *properly infinite* (with respect to P_f) if $z \cap a$ is infinite or zero for any $z \in Z$.

DEFINITION 1.5. Using the two P-properties P_m and P_f, we classify the Z-lattice L as follows: L is said to be of *type* I if it has a finite minimal element a such that $e(a) = 1$; *type* II if it has no finite minimal element and has a finite element b such that $e(b) = 1$; *type* III if all non-zero elements are infinite. L is said to be of *type* I_1, (resp. II_1) if it is of type I (resp. II) and 1 is finite; *type* I_∞ (resp. II_∞) if it is of type I (resp. II) and 1 is properly infinite.

LEMMA 1.4. *For any element a of a Z-lattice L, with respect to the given P-properties P_m and P_f, there exist $e^f(a)$, $e^i(a) \in Z$ which have the following properties*:

(1°) $e^f(a) \dot{\cup} e^i(a) = e(a)$,

(2°) $e^f(a) \cap a$ *is finite,*

(3°) *if* $e^i(a) \neq 0$, *then* $e^i(a) \cap a$ *is properly infinite.*

Then $e^f(a)$ and $e^i(a)$ are uniquely determined.

PROOF.[1] Let $e^f(a) = \bigcup (z \in Z; z \leqq e(a),\ z \cap a \text{ is finite})$, $e^i(a) = e(a) - e^f(a)$. Since Z is a complete Boolean lattice, using (P1) we may consider $e^f(a)$ as a join of an independent system of central elements $z_\alpha (\alpha \in I)$, such that $z_\alpha \leqq e(a)$ and $z_\alpha \cap a$ are finite, then

$$e^f(a) \cap a = {}_{\alpha \in I}\dot{\bigcup} (z_\alpha \cap a).$$

By (P2) $e^f(a) \cap a$ is finite. If $e^i(a) \neq 0$, then since $e(e^i(a) \cap a) = e^i(a) \cap e(a) = e^i(a)$, we have $e^i(a) \cap a \neq 0$. Assume that $z \cap e^i(a) \cap a$ is finite for $z \in Z$. Then $z \cap e^i(a) \leqq e^f(a)$, so $z \cap e^i(a) = 0$. Hence $e^i(a) \cap a$ is properly infinite. Next $e^f_1(a)$, $e^i_1(a) \in Z$ have also properties (1°), (2°) and (3°). Since $e^f_1(a) \cap e^i(a) \cap a$ is finite by (2°) and (P1), we have $e^f_1(a) \cap e^i(a) \cap a = 0$ by (3°), whence $e^f_1(a) \cap e^i(a) = e(e^f_1(a) \cap e^i(a) \cap a) = 0$. Hence $e^f_1(a) \leqq e^f(a)$. Similarly $e^f(a) \leqq e^f_1(a)$, so $e^f(a) = e^f_1(a)$ and $e^i(a) = e^i_1(a)$.

THEOREM 1.1. *Let L be a Z-lattice, in which two P-properties P_m and P_f are given. Then L is decomposed into a direct sum of sublattices of types* I, II *and* III *such that* $L = L_{\mathrm{I}} \dot{\cup} L_{\mathrm{II}} \dot{\cup} L_{\mathrm{III}}$.

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1) This method of proof is due to [9] Theorem 2. 1.

PROOF.[1] Let $z_{\mathrm{I}} = \bigcup (e(d); d$ is finite and minimal). Since Z is a complete Boolean lattice, using (P 1), we may write $z_{\mathrm{I}} = {}_{\alpha \in I}\dot{\bigcup} z_\alpha$, where $z_\alpha = e(d_\alpha)$ and d_α is finite and minimal. By (P 2) $h = {}_{\alpha \in I}\dot{\bigcup} d_\alpha$ is finite and minimal, and $e(h) = z_{\mathrm{I}}$. Next, let $z^* = \bigcup (e(a); a$ is finite). As above, we have $z^* = e(b)$, where b is finite. Since $z^* \geqq z_{\mathrm{I}}$, we put $z_{\mathrm{II}} = z^* - z_{\mathrm{I}}$, $z_{\mathrm{III}} = 1 - z^*$. Then $b_o = z_{\mathrm{II}} \cap b$ is finite and $e(b_o) = z_{\mathrm{II}} \cap e(b) = z_{\mathrm{II}}$. Hence $L_{\mathrm{I}} = L(0, z_{\mathrm{I}})$, $L_{\mathrm{II}} = L(0, z_{\mathrm{II}})$, $L_{\mathrm{III}} = L(0, z_{\mathrm{III}})$ are of type I, type II and type III respectively, and $L = L_{\mathrm{I}} \dot{\cup} L_{\mathrm{II}} \dot{\cup} L_{\mathrm{III}}$.

Remark 1.4. From Lemma 1.4, we have $e^f(1) \dot{\cup} e^i(1) = 1$. Since $z_{\mathrm{III}} \cap z^f(1) = 0$, we have a decomposition

$$1 = z_{\mathrm{I}_1} \dot{\cup} z_{\mathrm{I}_\infty} \dot{\cup} z_{\mathrm{II}_1} \dot{\cup} z_{\mathrm{II}_\infty} \dot{\cup} z_{\mathrm{III}},$$

where $z_{\mathrm{I}_1} = z_{\mathrm{I}} \cap e^f(1)$, $z_{\mathrm{I}_\infty} = z_{\mathrm{I}} \cap e^i(1)$, $z_{\mathrm{II}_1} = z_{\mathrm{II}} \cap e^f(1)$, $z_{\mathrm{II}_\infty} = z_{\mathrm{II}} \cap e^i(1)$. Thus *a Z-lattice is a direct sum of five lattices*

$$L = L_{\mathrm{I}_1} \dot{\cup} L_{\mathrm{I}_\infty} \dot{\cup} L_{\mathrm{II}_1} \dot{\cup} L_{\mathrm{II}_\infty} \dot{\cup} L_{\mathrm{III}},$$

where $L_{\mathrm{I}_1} = L(0, z_{\mathrm{I}_1})$ is type I_1, and so on.

Remark 1.5. In order to investigate the properties of the center of a lattice L, the following theorem is useful.

Here $(a, b)D$ means $(a \cup b) \cap x = (a \cap x) \cup (b \cap x)$ for all $x \in L$,

$(a, b)M$ means $(c \cup a) \cap b = c \cup (a \cap b)$ when $c \leqq b$.

THEOREM 1.2.[2] *In a lattice L with 0 and 1, $a \in L$ is a central element of L if and only if, a has a complement a' such that $(a, a')D$, $(a, a')M$ and (a', a) M.*

PROOF. The necessity is evident. Setting $S = L(0, a)$, $T = L(0, a')$, it is sufficient to show $L \cong ST$, by the correspondence $x \rightarrow [a \cap x, a' \cap x]$. Let $[s, t]$ be any element of ST, and set $x = s \cup t$. By $(a', a)M$, we have $s \leqq x \cap a = (s \cup t) \cap a \leqq (s \cup a') \cap a = s$. That is, $s = a \cap x$, similarly $t = a' \cap x$. Thus the correspondence is onto. Next assume that $a \cap x = a \cap y$, $a' \cap x = a' \cap y$. By $(a, a')D$, we have

$$x = (a \cup a') \cap x = (a \cap x) \cup (a' \cap x) = (a \cap y) \cup (a' \cap y) = y.$$

Therefore, the correspondence $x \rightarrow [a \cap x, a' \cap x]$ is a one-one correspondence between L and ST, preserving the lattice-order. Hence $L \cong ST$.

§ 2. D-elements and Lowest Elements in Z-lattices

In this section, we introduce concrete elements which have the minimal property.

An element a in a lattice L with 0 is called by Kaplansky [3, p. 538] a *D-element* if $L(0, a)$ is distributive, and he decomposed a complete complemented modular lattice. Since we can easily verily that the non-zero D-element

1) This method of proof is the same as [9] Theorem 2. 2.

2) This is already proved in F. Maeda [6].

has the P-property, we can use this element for the decomposition of Z-lattices. But the minimal property of the non-zero D-element is not clear, we introduce the lowest element as follows.

DEFINITION 2.1. Let L be a lattice with 0 and 1, and Z be the center of L. For $a, b \in L$, $a \ll b$ means that for every $z \in Z$, either $z \cap a < z \cap b$ or $z \cap a = z \cap b = 0$.

In this definition, let $z=1$, then $a \ll b$ implies $a < b$ or $a = b = 0$. $a \ll b$ and $b < c$ imply $a \ll c$.

LEMMA 2.1. *For $z \in Z$, $a \ll b$ implies $z \cap a \ll z \cap b$.*

PROOF. For any $y \in Z$, we have $y \cap z \in Z$. Hence $a \ll b$ implies either $y \cap z \cap a < y \cap z \cap b$ or $y \cap z \cap a = y \cap z \cap b = 0$, that is $z \cap a \ll z \cap b$.

DEFINITION 2.2. Let a be a non-zero element of a lattice L with 0 and 1. If $x \ll a$ implies $x = 0$, we say that a is a *lowest element* of L.

Remark 2.1. If a is a lowest element of L and $0 < b < a$, then b is also a lowest element, since $x \ll b$ implies $x \ll a$.

When L is irreducible, Z consists of 0 and 1 only. Hence a lowest element coincides with an atomic element, that is, a point.

Let z_0 be a central element of L. For $a \in L(0, z_0)$, it is evident that a is a lowest element of $L(0, z_0)$ if and only if a is a lowest element of L.

LEMMA 2.2. *Let L be a Z-lattice with the center Z.*

(i) *If a is a lowest element of L, then $z \cap a$ is a lowest element of L for every $z \in Z$ such that $z \cap a \neq 0$.*

(ii) *Let $\{a_\alpha;\ \alpha \in I\}$ be a family of lowest elements in L, and $\perp (e(a_\alpha);\ \alpha \in I)$, then ${}_{\alpha \in I}\bigcup a_\alpha$ is a lowest element of L.*

PROOF. (i) By Remark 2.1.

(ii) Set $a = {}_{\alpha \in I}\bigcup a_\alpha$. When $x \ll a$, by Lemma 2.1 and Lemma 1.3 we have $e(a_\alpha) \cap x \ll e(a_\alpha) \cap a = a_\alpha$. Hence $e(a_\alpha) \cap x = 0$. Since $x < a \leqq e(a)$, we have by Lemma 1.2 $x = e(a) \cap x = {}_{\alpha \in I}\bigcup e(a_\alpha) \cap x = {}_{\alpha \in I}\bigcup (e(a_\alpha) \cap x) = 0$. Therefore a is a lowest element of L.

By Lemma 2.2, the lowest elements have the P-property, and their minimal property is evident. Hence we may use the lowest elements for the decomposition. Next we shall investigate the further properties of the lowest elements and compare with the D-elements.

LEMMA 2. 3. *Let a be an element of a Z-lattice L and $a = a_1 \dot{\cup} a_2$. Then $a_1 \ll a$ if and only if $e(a_2) = e(a)$.*

PROOF. $a_1 \ll a$ means that either $z \cap a_1 < z \cap a$ or $z \cap a_1 = z \cap a = 0$ for every $z \in Z$. Since $z \cap a = (z \cap a_1) \dot{\cup} (z \cap a_2)$, this means that for any $z \in Z$, either $z \cap a_2 \neq 0$ or $z \cap a = 0$; that is, $z \cap a_2 = 0$ implies $z \cap a = 0$; z' being the complement of z, this means that $a_2 \leqq z'$ implies $a \leqq z'$. Hence $a_1 \ll a$ is equivalent to $e(a) \leqq e(a_2)$.

Since $e(a) \geqq e(a_2)$, $a_1 \ll a$ is equivalent to $e(a) = e(a_2)$.

THEOREM 2.1. *Let L be a relatively complemented Z-lattice. For a non-zero element a of L, the following four statements are equivalent.*

(α) a *is a lowest element of L.*

(β) $b < a$ *implies* $e(b) < e(a)$.

(γ) *If* $b < a$, *then* $b = e(b) \cap a$.

(δ) *If* $b < a$, $c < a$, $b \cap c = 0$, then $e(b) \cap e(c) = 0$.

PROOF. $(\alpha) \to (\beta)$. Let $a = b \dot{\cup} b_1$, then $b_1 > 0$. Assume $e(b) = e(a)$, then by Lemma 2.3 we have $b_1 \ll a$. Since a is a lowest element, it must be that $b_1 = 0$, which is absurd.

$(\beta) \to (\gamma)$. When $b < a$, assume $b < e(b) \cap a$, and let $e(b) \cap a = b \dot{\cup} c$, then $c > 0$. Since $e(e(b) \cap a) = e(b) \cap e(a) = e(b)$, by Lemma 2.3 we have $c \ll e(b) \cap a$, and hence $c \ll a$. Consequently, when $a = c \dot{\cup} d$, then $d < a$ and by Lemma 2.3 we have $e(a) = e(d)$, which contradicts (β).

$(\gamma) \to (\delta)$. Since $b = e(b) \cap a$, $c = e(c) \cap a$, we have $b \cap c = e(b) \cap e(c) \cap a$, and $e(b \cap c) = e(b) \cap e(c) \cap e(a) = e(b) \cap e(c)$ by Lemma 1.2. Since $b \cap c = 0$, we have $e(b) \cap e(c) = 0$.

$(\delta) \to (\alpha)$. When $x \ll a$, let $a = x \dot{\cup} c$. By Lemma 2.3 we have $e(a) = e(c)$. Now $e(x) \leqq e(a)$, and $e(x) \cap e(c) = 0$ from (δ). Therefore $e(x) = 0$ and $x = 0$. That is, a is a lowest element.

DEFINITION 2.3. If a Z-lattice L has the following property (α), we say that L is a Z_α*-lattice.*

(α) For any element a of L, the center of $L(0, a)$ consists exactly of all elements of the form $z \cap a$ with $z \in Z$.

THEOREM 2.2. *In a relatively complemented Z_α-lattice L, the lowest element and the non-zero D-element coincide.*

PROOF. Let a be a lowest element of L, from $(\alpha) \to (\beta)$ of Theorem 2.1, by $b \to e(b)$, $L(0, a)$ and $\{e(b);\, b \in L(0, a)\}$ have a one-one correspondence preserving the lattice-order. Hence $L(0, a)$ is distributive, and a is a D-element. Next, let a be a nonzero D-element of L, and $b < a$. Since b is a central element of $L(0, a)$, by (α) of Definition 2.3 there exists $z \in Z$ such that $b = z \cap a$. Since $e(b) = z \cap e(a)$, we have $e(b) \cap a = z \cap e(a) \cap a = b$. Therefore by ($\gamma$) of Theorem 2.1, a is a lowest element of L.

Remark 2.2. By Remark 1.2 and [3] Theorem 4, a complete complemented modular lattice is a relatively complemented Z_α-lattice. Hence, by Theorem 2.2, in this lattice the non-zero D-element and the lowest element coincide.

Thus we can use D-elements and lowest elements for the elements of minimal property. But to obtain the elements of finite property we must introduce an equivalence relation.

§ 3. Analytic Decompositions of Z-lattices

In this section, L is a Z-lattice with the center Z. Let $a \overset{e}{\sim} b$ be an *equivalence relation* in L (i.e. reflexive, symmetric and transitive). An element $z \in Z$, such that $a \overset{e}{\sim} b \leqq z$ implies $a \leqq z$, is called a *relative central element* of L (with respect to "$\overset{e}{\sim}$"), and denote by Z_0 the set of all relative central elements of L. Z_0 is called the *relative center* of L (with respect to "$\overset{e}{\sim}$").

Now assume that "$\overset{e}{\sim}$" have the following properties:

(*e.* 1) $a \overset{e}{\sim} 0$ implies $a=0$.

(*e.* 2) If $0 \neq a_1 \leqq a \overset{e}{\sim} b$, then there exists $b_1 \neq 0$ such that $a_1 \overset{e}{\sim} b_1 \leqq b$.

(*e.* 3) $a \overset{e}{\sim} b$ implies $z \cap a \overset{e}{\sim} z \cap b$ for all $z \in Z_0$.

Using (*e.*2) we can prove that Z_0 is a complete Boolean sublattice of Z, as [9] Lemma 2.1. Since Z_0 is complete, for any $a \in L$ there is the smallest element $z \in Z_0$ such that $a \leqq z$. We shall denote it by $e_0(a)$. Of course $e(a) \leqq e_0(a)$.

Now we add the following assumption:

(*e.* 4) If $a_1 \overset{e}{\sim} b_1$, $a_2 \overset{e}{\sim} b_2$ and $e_0(a_1) \cap e_0(a_2) = 0$, then $a_1 \cup a_2 \overset{e}{\sim} b_1 \cup b_2$.

We can prove the following Lemma as in [8] Kapitel II, Hilfssatz 4.7.

LEMMA 3.1. (i) *$a \overset{e}{\sim} b$ implies $e_0(a) = e_0(b)$.*

(ii) *If $z \in Z_0$, then $e_0(z \cap a) = z \cap e_0(a)$.*

(iii) $e_0(\bigvee_{\alpha \in I} a_\alpha) = \bigvee_{\alpha \in I} e_0(a_\alpha)$.

As in the continuous geometry, we shall write $a \overset{e}{\prec} b$ if there exists b_1 such that $a \overset{e}{\sim} b_1 < b$, and write $a \overset{e}{\ll} b$ if for any $z \in Z_0$ either $z \cap a \overset{e}{\prec} z \cap b$ or $z \cap a = z \cap b = 0$.

Clearly $a \overset{e}{\ll} b$ implies $a \overset{e}{\prec} b$ or $a=b=0$, and $a \overset{e}{\prec} b$ implies $e_0(a) \leqq e_0(b)$ from Lemma 3.1 (i).

Then the following Lemma is evident.

LEMMA 3.2. (i) *$a \overset{e}{\ll} b$ and $b < c$ imply $a \overset{e}{\ll} c$*

(ii) *$a \overset{e}{\ll} b$ implies $z \cap a \overset{e}{\ll} z \cap b$ for every $z \in Z_0$.*

(iii) *$a \overset{e}{\ll} b$ implies $a \leqq e_0(b)$.*

DEFINITION 3.1. For $0 \neq a \in L$, if $x \overset{e}{\ll} a$ implies $x=0$, we say that a is an *e-minimal element* of L.

If a is an e-minimal element of L, and $0 \neq b < a$, then b is also an e-minimal element of L by Lemma 3.2 (i).

LEMMA 3.3. (i) *If a is an e-minimal element of L, then $z \cap a$ is an e-minimal element of L for every $z \in Z_0$ such that $z \cap a \neq 0$.*

(ii) *Let $\{a_\alpha; \alpha \in I\}$ be a family of e-minimal elements in L, and $\perp(e_0(a_\alpha); \alpha \in I)$, then $\bigvee_{\alpha \in I} a_\alpha$ is an e-minimal element of L.*

PROOF. Using Lemma 3.2, we can prove as Lemma 2.2.

DEFINITION 3.2. $a \in L$ is called *e-infinite* if $a \overset{e}{\prec} a$ holds, and otherwise *e-finite*. And $a \in L$ is called *properly e-infinite* if $a \neq 0$ and $a \overset{e}{\ll} a$ holds, that is, for any $z \in Z_0$, $z \frown a$ is e-infinite or zero.

LEMMA 3.4. (i) *If a is e-finite, then $z \frown a$ is e-finite for every $z \in Z_0$.*

(ii) *Let $\{a_\alpha;\ \alpha \in \mathrm{I}\}$ be a family of e-finite elements in L, and $\perp (e_0(a_\alpha);\ \alpha \in \mathrm{I})$ then ${}_{\alpha\in\mathrm{I}}\bigcup a_\alpha$ is e-finite.*

PROOF. (i) Assume $z \frown a$ is e-infinite, then there exists b such that $z \frown a \overset{e}{\precsim} b < z \frown a$. Since $e_0(z \frown a) \frown e_0((1-z) \frown a) = 0$, by ($e$. 4) we have

$$(z \frown a) \smile ((1-z) \frown a) \overset{e}{\precsim} b \smile ((1-z) \frown a) < (z \frown a) \smile ((1-z) \frown a)^{1)},$$

that is $a \overset{e}{\lessdot} a$, which is absurd.

(ii) Set $a = {}_{\alpha\in\mathrm{I}}\bigcup a_\alpha$. When $a \overset{e}{\precsim} b \leqq a$, then by Lemma 1.3 and (e. 3)

$$a_\alpha = e_0(a_\alpha) \frown a \overset{e}{\precsim} e_0(a_\alpha) \frown b \leqq e_0(a_\alpha) \frown a = a_\alpha.$$

Since a_α is e-finite, we have $e_0(a_\alpha) \frown b = a_\alpha$ for all $\alpha \in \mathrm{I}$. Consequently

$$b = e_0(a) \frown b = {}_{\alpha\in\mathrm{I}}\bigcup e_0(a_\alpha) \frown b = {}_{\alpha\in\mathrm{I}}\bigcup (e_0(a_\alpha) \frown b) = {}_{\alpha\in\mathrm{I}}\bigcup a_\alpha = a.$$

Therefore a is e-finite.

LEMMA 3.5. *An e-minimal element a is e-finite.*

PROOF. By the corresponding Lemma to Lemma 1.4, using Z_0 instead of Z, if $e_0^i(a) \neq 0$ then since $e_0^i(a) \frown a$ is properly e-infinite we have $e_0^i(a) \frown a \overset{e}{\ll} e_0^i(a) \frown a$. But by Lemma 3.3 (i) $e_0^i(a) \frown a$ is e-minimal, which is absurd. Hence $e_0^i(a) = 0$, and $a = e_0^f(a) \frown a$ is e-finite.

By Lemma 3.3 and Lemma 3.4, e-minimality and e-finiteness are P-properties, where Z_0 is used instead of Z. Hence as in § 1 we have a decomposition of a Z-lattice L using these conceptions. We may call this decomposition an *analytic decomposition of a Z-lattice* with respect to "$\overset{e}{\precsim}$".

Remark 3.1. In the theory of dimension functions in general lattices, S. Maeda [9] considered a complete lattice L with a binary relation "$\perp$" which satisfies some conditions. We may call this lattice a *generalized relatively orthocomplemented lattice*. This lattice L is a Z-lattice by [9] Theorem 1.3 and Lemma 1.3. S. Maeda introduced in this lattice a dimension by defining a binary relation "$\sim$" which satisfies some conditions. Since this relation "$\sim$" satisfies (e. 1)—(e. 4), S. Maeda's decomposition of a generalized relatively orthocomplmented lattice is an analytic decomposition. In this case, the meanings of e-minimality and e-finiteness are evident by their dimension values.

In order to give the geometrical meaning of e-minimal elements of a Z-lattice L in Definition 3.1, we give the following definitions similar to Definition 2.1 and Definition 2.2.

DEFINITION 3.3 For $a \in L$, if either $z \frown a < z \frown b$ or $z \frown a = z \frown b = 0$ for every $z \in Z_0$, we write $a \overset{e}{\ll} b$. A non-zero element a is called an *e-lowest element*

1) If $b \smile ((1-z) \frown a) = (z \frown a) \smile ((1-z) \frown a)$, then $z \frown \{b \smile ((1-z) \frown c)\} = z \frown \{(z \frown a) \smile ((1-z) \frown a)\}$, that is $b = z \frown a$, which is absurd.

if $x\overset{e}{\ll}a$ implies $x=0$.

The geometrical meaning of an e-lowest element is evident.

LEMMA 3.6. *When b is e-finite, $a\overset{e}{\ll}b$ if and only if there exists b_1 such that $a\overset{e}{\sim}b_1\overset{e}{\ll}b$.*

PROOF. When $a\overset{e}{\ll}b$, since $a\overset{e}{<}b$ or $a=b=0$, there exists b_1 such that $a\overset{e}{\sim}b_1<b$ or $a=b=0$. When $a\overset{e}{\sim}b_1<b$, we have $z\cap a\overset{e}{\sim}z\cap b_1\leqq z\cap b$ for $z\in Z_0$. Since $a\overset{e}{\ll}b$, $z\cap a\overset{e}{<}z\cap b$ or $z\cap a=z\cap b=0$. Hence if $z\cap a=z\cap b=0$ does not hold, then it must be that $z\cap b_1<z\cap b$, since $z\cap b$ is e-finite. Hence $z\cap b_1<z\cap b$ or $z\cap b_1=z\cap b=0$. Therefore $b_1\overset{e}{\ll}b$, and b_1 is the required element. When $a=b=0$, let $b_1=0$.

Conversely when $a\overset{e}{\sim}b_1\ll b$, we have either $z\cap a\overset{e}{\sim}z\cap b_1<z\cap b$ or $z\cap b_1=z\cap b=0$ for every $z\in Z_0$, This means that either $z\cap a\overset{e}{<}z\cap b$ or $z\cap a=z\cap b=0$. Therefore $a\overset{e}{\ll}b$.

THEOREM 3.1. *In a Z-lattice L, an element a is e-minimal if and only if a is an e-finite and e-lowest element. Especially when L is a relatively complemented Z-lattice, for a non-zero element a of L, the following five statements are equivalent.*

(α) *a is an e-lowest element of L.*

(β) *$b<a$ implies $e_0(b)<e_0(a)$.*

(γ) *If $b<a$, then $b=e_0(b)\cap a$.*

(δ) *If $b<a$, $c<a$, $b\cap c=0$, then $e_0(b)\cap e_0(c)=0$.*

(ε) *a is an e-minimal element of L.*

PROOF. First part of the theorem is evident from Lemma 3.5 and Lemma 3.6. When L is a relatively complemented Z-lattice, we can prove the equivalence of (α), (β), (γ) and (δ) as Theorem 2.1, using Z_0 instead of Z. To prove the equivalence of (ε), it is sufficient to show that an e-lowest element a is e-finite. But this follows from the fact that if $a\overset{e}{\sim}b<a$, then by Lemma 3.1 (i) $e_0(a)=e_0(b)$, which contradicts (β).

Remark 3.2. When L is a *continuous geometry*, that is, a continuous complemented modular lattice, L is a Z_α-lattice by [8] Kapitel I, Satz 3.7 and Kapitel IV, Satz 1.4, and the perspecivity "$\sim$" is an equivalence relation which satisfies (e. 1)—(e. 4) and $Z_0=Z$. (Cf. [8] Kapitel I Hilfssatz 3.8, Kapitel II Satz 3.1, Kapitel IV Satz 2.2 and Hilfssatz 2.2.) Hence von Neumann's decomposition of the continuous geometry ([8] Kapitel IV Satz 4.9) is an analytic decomposition with respect to the perspectivity, and in this lattice the D-element, the lowest element and the e-minimal element coincide by Remark 2.2 and Theorem 3.1.

In a continuous geometry, another equivalence relations are introduced. Hence corresponding to these equivalence relations we have different decompositions of a continuous geometry. Cf. Halperin [1], Iwamura [2] and F. Maeda [7].

§ 4. Decomposition of Relatively Orthocomplemented Complete Lattices

DEFINITION 4.1. A lattice L with 0 and 1 is called *orthocomplemented* if it admits a dual automorphism $a \to a^{\perp}$ satisfying (α) $a^{\perp\perp} = a$ and (β) $a \leqq a^{\perp}$ implies $a = 0$.

In this case, of course, $a \cap a^{\perp} = 0$ and $a \cup a^{\perp} = 1$.

We write $a \perp b$ when $a \leqq b^{\perp}$, and we say that a and b are *orthogonal.*

An orthocomplemented lattice L is called *relatively orthocomplemented* if, given $b \leqq a$, an element c exists such that $a = b \cup c$ and $b \perp c$. And c is called *a relative orthocomplement* of b in a.

THEOREM 4.1. *Let L be an orthocomplemented lattice. Then the following three statements are equivalent.*

(α) *L is a relatively orthocomplemented lattice.*

(β) *$a \perp b$ implies $(a, b)M$.*

(γ) *For any $a \in L$, $(a, a^{\perp})M$.*

In (α), when $b \leqq a$ the relative orthocomplement of b in a is $a \cap b^{\perp}$. That is

(1) $$a = b \cup (a \cap b^{\perp}.)$$

PROOF.[1] (α) $\to$ (β). Let $a \perp b$ and $c \leqq b$. Since $(c \cup a) \cap b \geqq c$, by ($\alpha$) there exists d such that $(c \cup a) \cap b = c \cup d$, $c \perp d$. From $d \leqq b$ and $c \perp d$, we have $c \cup b^{\perp} \leqq d^{\perp}$, and from $a \perp b$ and $c \cup a \geqq d$, we have $c \cup b^{\perp} \geqq c \cup a \geqq d$. Therefore $d = 0$, and we have $(c \cup a) \cap b = c = c \cup (a \cap b)$, that is (a, b) M.

(β) $\to$ (γ). It is evident.

(γ) $\to$ (α). If $b \leqq a$, then $a^{\perp} \leqq b^{\perp}$. Hence by ($\gamma$) $(a^{\perp} \cup b) \cap b^{\perp} = a^{\perp}$. By duality, $a = (a^{\perp} \cup b)^{\perp} \cup b = b \cup (a \cap b^{\perp})$, and $a \cap b^{\perp}$ is the relative orthocomplement of b in a.

Remark 4.1. Since (1) in Theorem 4.1 is the axiom *(M)* in Loomis [5], the dimension theory of Loomis is a theory on a relatively orthocomplemented complete lattice.

From (α) $\rightleftarrows$ (β) in Theorem 4.1, a relatively orthocomplemented lattice may be called an *orthomodular lattice.*

Remark 4.2. Since a relatively orthocomplemented complete lattice L is a special case of a generalized relatively orthocomplemented lattice, by Remark 3.1 L is a Z-lattice. But in what follows, I shall give a direct proof using the compatible elements, since these elements have an interest of its own.

LEMMA 4. 1. *In a relatively orthocomplemented lattice, if $a = (a \cap b) \cup (a \cap b^{\perp})$ then $b = (b \cap a) \cup (b \cap a^{\perp})$.*

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1) (α) $\rightleftarrows$ (γ) is already proved in [9] 216.

PROOF. Since $b\cap a^{\perp}=b\cap(a\cap b)^{\perp}\cap(a^{\perp}\cup b)=b\cap(a\cap b)^{\perp}$, we have $(b\cap a^{\perp})^{\perp}=b^{\perp}\cup(a\cap b)$. Therefore, by $(a\cap b, (a\cap b)^{\perp})M$,

$$(b\cap a^{\perp})^{\perp}\cap(a\cap b)^{\perp}=\{b^{\perp}\cup(a\cap b)\}\cap(a\cap b)^{\perp}=b^{\perp},$$

that is, $(b\cup a^{\perp})\cup(a\cap b)=b$.

DEFINITION 4.2. In a relatively orthocomplemented lattice, when $a=(a\cap b)\cup(a\cap b^{\perp})$, then of course $b=(b\cap a)\cup(b\cap a^{\perp})$ by Lemma 4.1, we say that a and b are *compatible.*[1]

From this definition, it is evident that when $a\cap b=0$, a and b are compatible if and only if a and b are orthogonal.

LEMMA 4.2. *In a relatively orthocomplemented lattice L, an element z is a central element of L if and only if z is compatible with any element of L*[2].

PROOF. If z is a central element of L, it is evident that $a=(a\cap z)\cup(a\cap z^{\perp})$ for any $a\in L$. Next assume that $a=(a\cap z)\cup(a\cap z^{\perp})$ for any $a\in L$. This means that $(z, z^{\perp})D$. But by Theorem 4.1, $(z, z^{\perp})M$ and $(z^{\perp}, z)M$. Hence by Theorem 1.2, z is a central element of L.

LEMMA 4.3. *Let $\{a_\delta;\ \delta\in D\}$ be a directed set of a relatively orthocomplemented complete lattice L. If a_δ and b are compatible for any $\delta\in D$, then*

$$a_\delta\uparrow a \qquad \textit{implies} \qquad a_\delta\cap b\uparrow a\cap b.\text{[3]}$$

PROOF. Since $a_\delta=(a_\delta\cap b)\cup(a_\delta\cap b^{\perp})$ for any $\delta\in D$, and $({}_{\delta\in D}\bigvee(a_\delta\cap b^{\perp}), b)M$, we have

$$a\cap b={}_{\delta\in D}\bigvee a_\delta\cap b=\{{}_{\delta\in D}\bigvee(a_\delta\cap b)\cup{}_{\delta\in D}\bigvee(a_\delta\cap b^{\perp})\}\cap b={}_{\delta\in D}\bigvee(a_\delta\cap b).$$

That is, $a_\delta\cap b\uparrow a\cap b$.

THEOREM 4.2. *The center Z of a relatively orthocomplemented complete lattice L is a complete Boolean sublattice of L, and when $a_\alpha\in Z$ for all $\alpha\in I$ or $b\in Z$,*

$$(1)\qquad {}_{\alpha\in I}\bigvee a_\alpha\cap b={}_{\alpha\in I}\bigvee(a_\alpha\cap b).$$

That is, L is a Z-lattice.

PROOF. By [8] Kapitel I Satz 3.4, Z is a Boolean sublattice of L. Let S be any subset of Z, and T be any finite subset of S. Since ${}_{z\in T}\bigvee z$ is a central element, by Lemma 4.2 we have

$${}_{z\in T}\bigvee z=({}_{z\in T}\bigvee z\cap a)\cup({}_{z\in T}\bigvee z\cap a^{\perp}).$$

Hence by Lemma 4.3 we have

$${}_{z\in S}\bigvee z=({}_{z\in S}\bigvee z\cap a)\cup({}_{z\in S}\bigvee z\cap a^{\perp}).$$

Therefore ${}_{z\in S}\bigvee z\in Z$. Since $z^{\perp}\in Z$, we have as above ${}_{z\in S}\bigvee z^{\perp}\in Z$. That is ${}_{z\in S}\bigwedge z\in Z$. Consequently Z is a complete sublattice of L.

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1) Cf. [8] Kapitel XII Definition 1.3.
2) Cf. [11] 5.
3) Cf. [11] 4.

To prove (1), let N be any finite subset of I, and put

$$s_N = {}_{\alpha\in N}\bigcup a_\alpha, \quad t_N = {}_{\alpha\in N}\bigcup (a_\alpha \cap b).$$

Then $s_N \in Z$ or $b \in Z$[1], and $s_N \cap b = t_N$. Since s_N and b are compatible and $s_N \uparrow {}_{\alpha\in I}\bigcup a_\alpha$, $t_N \uparrow {}_{\alpha\in I}\bigcup (a_\alpha \cap b)$, we have (1).

Remark 4.3. By Theorem 4.2 or Remark 4.2, a relatively orthocomplemented complete lattice is a Z-lattice. Hence we can apply the decomposition theory of §1 and § 2. The *dimension lattice* L of Loomis [5] is a relatively orthocomplemented complete lattice with an equivalence relation "$\overset{e}{\sim}$" satisfying some properties which are similar to S. Maeda [9]. In this dimension lattice L, an element e is *invariant* if and only if $x \overset{e}{\lesssim} e$ implies $x \leq e$, and it is proved that the invariant elements of L form a complete Boolean sublattice B and if $e \in B$ then

$$(1) \qquad a = (a \cap e) \cup (a \cap e^{\perp})$$

for any $a \in L$. ([5] Lemma 21 and Theorem 2.) But (1) means that e is compatible with any $a \in L$, hence by Lemma 4.2 $e \in Z$. Therefore B is nothing more than the relative center of L defined in § 3. Hence, in the dimension lattice L, Theorem 3.1 holds, and (α), (β), (γ), (δ), (ε) in this theorem are equivalent. But (γ) corresponds to the definition of a *simple element* in Loomis [5, p. 2]. Consequently Loomis's decomposition of a dimension lattice coincides with the analytic decomposition.

§ 5. Lattice of Right Annihilators in a Baer Ring

A ring A with unity is a *Baer ring* if the right (or left) annihilator of any subset S of A is a principal right (or left) ideal generated by an idempotent $e \in A$. (Kaplansky [4] Chapter I, Definition 1.)

In this paper, as in [8] Kapitel VI, Definitions 1.5 and 1.8, the principal right (or left) ideal generated by e is denoted by $(e)_r$ (or $(e)_l$), and the right (or left) annihilator of S is denoted by S^r (or S^l).

Denote the set of all right-annihilators of a Baer ring A by $\mathring{R}_A$. Since $(e)_r = (1-e)_l^r$ for any idempotent e of A, $\mathring{R}_A$ is the set of all principal right ideals of the form $(e)_r$ with an idempotent $e \in A$.

In this section A is a Baer ring.

THEOREM 5.1. *$\mathring{R}_A$ is a complete lattice ordered by set-inclusion.*

PROOF. Let $\{S_i^r; i \in I\}$ be any subset of $\mathring{R}_A$. If we denote the set-sum and set-intersection by $\sum$ and Π respectively, then $(\sum_{i\in I} S_i)^r = \prod_{i\in I} S_i^r$. Hence $\prod_{i\in I} S_i^r$ belongs to $\mathring{R}_A$ and it is the g.l.b. of S_i^r $(i \in I)$. Hence $\mathring{R}_A$ is a complete lattice ordered by set-inclusion.

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1) We may use Lemma 1. 1.

Remark 5.1. The set R_A of all right ideals of A is a complete lattice, where l.u.b. is denoted by $\cup$ and $\bigcup$, and g. l. b. is denoted by $\cap$ and $\bigcap$. (Cf. [8] Kapitel VI Satz 1.1.) But $\mathring{R}_A$ is not a sublattice of R_A. In $\mathring{R}_A$, the g.l.b. is set intersection as in R_A, hence we may use the same symbol $\cap$ and $\bigcap$. But for the l. u. b. in $\mathring{R}_A$ we use $\vee$ and $\bigvee$.

LEMMA 5.1. *Let $\{e_i;\, i \in I\}$ be a family of idempotents in A. Then there exists an idempotent e such that $\{e_i;\, i \in I\}^l = (1-e)_l$. In this case ${}_{i\in I}\bigvee(e_i)_r = (e)_r$ in $\mathring{R}_A$.*

PROOF. The existence of e is evident from the definition of the Baer ring. For all $i \in I$, we have $(1-e)e_i = 0$, that is, $e_i = ee_i$ and $(e_i)_r \leqq (e)_r$. Next, let $(e_i)_r \leqq (f)_r$ for all $i \in I$, where f is an idempotent in A, then $(e_i)_r{}^l \geqq (f)_r{}^l$. Since $\{e_i;\, i \in I\}^l = \{(e_i)_r;\, i \in I\}^l = {}_{i\in I}\bigcap(e_i)_r{}^l$, we have $(1-e)_l \geqq (f)_r{}^l$. Hence $(1-e)_l{}^r \leqq (f)_r{}^{lr}$ and $(e)_r \leqq (f)_r$. Therefore $(e)_r$ is the l.u.b. of $(e_i)_r$ $(i \in I)$ in $\mathring{R}_A$.

Remark 5.2. The idempotent e in Lemma 5.1 is characterized by the following conditions:

(α) $ee_i = e_i$ for all $i \in I$.

(β) If $xe_i = 0$ for all $i \in I$, then $xe = 0$.

For, when $\{e_i;\, i \in I\}^l = (1-e)_l$, $(1-e)e_i = 0$ for all $i \in I$, and we have (α). If $xe_i = 0$ for all $i \in I$, then $x \in (1-e)_l$, that is, $x = x(1-e)$ and we have (β).

Conversely, from (α) we have $(1-e)e_i = 0$ for all $i \in I$. Hence $\{e_i;\, i \in I\}^l \geqq (1-e)_l$. Next, if $xe_i = 0$ for all i $\in I$, from (β) we have $x \in (1-e)_l$. Hence $\{e_i;\, i \in I\}^l = (1-e)_l$.

Remark 5.3. Between the l.u.b. ${}_{i\in I}\bigcup(e_i)_r$ in R_A and the l.u.b. ${}_{i\in I}\bigvee(e_i)_r$ in $\mathring{R}_A$, there are the following relations (1°) and (2°).

Since $\{e_i;\, i \in I\}^l = ({}_{i\in I}\bigcup(e_i)_r)^l$, $(1-e)_l = (e)_r{}^l$, we have from Lemma 5.1

(1°) $$({}_{i\in I}\bigcup(e_i)_r)^l = ({}_{i\in I}\bigvee(e_i)_r)^l.$$

And since $S \leqq S^{lr}$, we have

(2°) $${}_{i\in I}\bigcup(e_i)_r \leqq (e)_r{}^{lr} = (e)_r = {}_{i\in I}\bigvee(e_i)_r.$$

LEMMA 5.3. *Let e, f be idempotents in A such that $(e)_r \leqq (f)_r$. Then*

(1) $$(f)_r = (e)_r \vee \{(f)_r \cap (1-e)_r\}$$

PROOF. From [8] Kapitel VI, Hilfssatz 1.6, there exists a right ideal $\mathfrak{a}$ such that

$$(f)_r = (e)_r \cup \mathfrak{a},\quad (e)_r \cap \mathfrak{a} = (0)_r,$$

and $\mathfrak{a} = (f)_r \cap (e)^r$. Since $(e)^r = (1-e)_r$, we have

$$(f)_r = (e)_r \cup \{(f)_r \cap (1-e)_r\} \leqq (e)_r \vee \{(f)_r \cap (1-e)_r\} \leqq (f)_r.$$

Hence (1) holds.

LEMMA 5.4. *Let e, f be idempotents in A such that $(e_r) \leqq (f)_r$. Then there exists an idempotent e_0 such that $(e)_r = (e_0)_r$ and $e_0 = fe_0 = e_0 f$.*

PROOF. Since $(e)_r \leqq (f)_r$, we have $e = fe$. Let $e_0 = ef$, then e_0 is an idempotent and $e_0 = fe_0 = e_0 f$. Since $e_0 = ee_0$ and $e = e_0 e$, we have $(e)_r = (e_0)_r$.

LEMMA 5.5 *Let e, f be idempotents in A.*

(i) $(e)_r \cap (f)_r = (ef)_r$ *when* $ef = fe$.

(ii) $(e)_r \cup (f)_r = (e)_r \vee (f)_r = (e+f)_r$ *when* $ef = fe = 0$.

PROOF. (i) When $ef = fe$, $g = ef$ is an idempotent. Since $g \in (e)_r \cap (f)_r$ we have $(g)_r \leqq (e)_r \cap (f)_r$. Next, if $x \in (e)_r \cap (f)_r$, then $x = ex$ and $x = fx$. Hence $x = efx = gx$, that is, $x \in (g)_r$. Therefore $(g)_r = (e)_r \cap (f)_r$.

(ii) When $ef = fe = 0$, $e + f$ is an idempotent and by [8] Kapitel VI, Satz 1.3, $(e)_r \cup (f)_r = (e+f)_r$. Since $(e)_r \cup (f)_r \leqq (e)_r \vee (f)_r \leqq (e+f)_r$, (ii) holds.

LEMMA 5.6. *Let e be an idempotent in A. Then $(1-e)_r$ is a complement of $(e)_r$ in $\mathring{R}_A$ and $((e)_r, (1-e)_r)M$.*

PROOF. Since $(1)_r = (e)_r \cup (1-e)_r \leqq (e)_r \vee (1-e)_r \leqq (1)_r$, $(1-e)_r$ is a complement of $(e)_r$ in $\mathring{R}_A$. By Lemma 5.4, any element of $\mathring{R}_A$ contained in $(1-e)_r$ is expressed as $(f)_r$, where f is an idempotent such that $f = (1-e)f = f(1-e)$, that is $ef = fe = 0$. Therefore by Lemma 5.5 (ii) $(e)_r \cup (f)_r = (e)_r \vee (f)_r$. Since R_A is a modular lattice ([8] Kapitel VI, Satz 1.1), we have

$$\{(e)_r \vee (f)_r\} \cap (1-e)_r = \{(e)_r \cup (f)_r\} \cap (1-e)_r = (f)_r.$$

That is, $((e)_r, (1-e)_r)M$ in $\mathring{R}_A$.

THEOREM 5.2. *Let e be an idempotent in A. Then $(e)_r$ is a central element of $\mathring{R}_A$, if and only if e is a central element of A.*

PROOF. (i) When $(e)_r$ is a central element of $\mathring{R}_A$, $(e)_r$ has a unique complement in $\mathring{R}_A$. Since $(e)_r \cup (f)_r \leqq (e)_r \vee (f)_r$ by Remark 5.3, $(e)_r$ must have a unique complement in R_A. Hence by [8] Kapitel VI, Satz 1.5, $(e)_r$ is expressed with a unique idempotent e. Therefore by ibid., Hilfssatz 1.5, $ex(1-e) = 0$, that is, $ex = exe$ for all $x \in R$. Since $(1-e)_r$ is a complement of $(e)_r$ in $\mathring{R}_A$, $(1-e)_r$ is also a central element in $\mathring{R}_A$. Therefore, as above we have $(1-e)xe = 0$, that is, $xe = exe$. Therefore $ex = xe$ for all $x \in A$, and e is a central element of A.

(ii) Conversely, let e be a central element in A. By Lemma 5.6, $(1-e)_r$ is a complement of $(e)_r$ in $\mathring{R}_A$ and $((e)_r, (1-e)_r)M$, $((1-e)_r, (e)_r)M$. By [8] Kapitel VI, Satz 1.8, $(e)_r$ is a central element in R_A. Hence for any idempotent f in A, we have

$$\begin{aligned}(f)_r = \{(e)_r \cup (1-e)_r\} \cap (f)_r &= \{(e)_r \cap (f)_r\} \cup \{(1-e)_r \cap (f)r\} \\ &\leqq \{(e)_r \cap (f)_r\} \vee \{(1-e)_r \cap (f)_r\} \leqq (f)_r.\end{aligned}$$

Therefore, we have

$$\{(e)_k \cap (f)_r\} \vee \{(1-e)_r \cap (f)_r\} = (f)_r = \{(e)_r \vee (1-e)_r\} \cap (f)_r.$$

That is, $((e)_r, (1-e)_r)D$ in $\mathring{R}_A$. Consequently, be Theorem 1.2, $(e)_r$ is a central element in $\mathring{R}_A$.

THEOREM 5.3. *The center of $\mathring{R}_A$ is a complete Boolean sublattice of $\mathring{R}_A$, and* *if either* $(e_i)_r$ $(i \in I)$ *or* $(f)_r$ *are central elements of* $\mathring{R}_A$, *then*

$$\bigvee_{i\in I} (e_i)_r \cap (f)_r = \bigvee_{i\in I} ((e_i)_r \cap (f)_r).$$

Therefore, $\mathring{R}_A$ *is a relatively complemented Z-lattice.*

PROOF. (i) Let $e_i (i \in I)$ be central idempotents in A. Then by [4] Chapter I, Theorem 3, there exists a central idmpotents e such that $\{e_i;\ i \in I\}^l = (1-e)_l$. Then by Lemma 5.1.

(1) $$\bigvee_{i\in I} (e_i)_r = (e)_r.$$

(ii) Since $\bigwedge_{i\in I} (e_i)_r = \prod_{i\in I} (1-e_i)_l{}^r = \{1-e_i;\ i \in I\}^r$, as (i) there exists a central idempotent g such that

(2) $$\bigwedge_{i\in I} (e_i)_r = (g)_r.$$

By Theorem 5.2, (1) and (2) mean that the center of $\mathring{R}_A$ is a complete sublattice of $\mathring{R}_A$.

(iii) Next, when either $e_i (i \in I)$ or f[1] are central idempotents of A, take an idempotent e such that $\bigvee_{i\in I} (e_i)_r = (e)_r$. Then by Remark 5.2 e is characterized by the following conditions:

(1°) $ee_i = e_i$ for all $i \in I$.

(2°) If $xe_i = 0$ for all $i \in I$, then $xe = 0$.

And by (i) e is a central idempotent when $e_i (i \in I)$ are central idempotents. Hence by Lemma 5.5 (i), we have

(3) $$\bigvee_{i\in I} (e_i)_r \cap (f)_r = (ef)_r.$$

From (1°) and (2°), we have

(1′) $ef \cdot e_i f = e_i f$ for all $i \in I$.

(2′) If $x \cdot e_i f = 0$ for all $i \in I$, then $x \cdot ef = 0$.

Therefore, we have

(4) $$\bigvee_{i\in I} (e_i f)_r = (ef)_r.$$

Since $(e_i f)_r = (e_i) \cap (f)_r$, from (3) and (4), we have

$$\bigvee_{i\in I} (e_i)_r \cap (f)_r = \bigvee_{i\in I} ((e_i)_r \cap (f)_r).$$

Therefore, $\mathring{R}_A$ is Z-lattice. The relatively complementedness of $\mathring{R}_A$ follows from Lemma 5.3.

Kaplansky [4, Chapter II, Definition 1 and Lemma 1] defined the equivalence of idempotents as follows: Idempotents e, f in A are said to be *equivalent*, when there exist $x \in eAf$, $y \in fAe$ with $xy = e$, $yx = f$, and proved that e and f are equivalent if and only if $(e)_r$ and $(f)_r$ are isomorphic right A-modules. Hence we may give the following definition.

DEFINITION 5.1. In $\mathring{R}_A$, $(e)_r$ and $(f)_r$ are said to be *a-equivalent*, written $(e)_r \overset{a}{\sim} (f)_r$, if there exist $x \in eAf$, $y \in fAe$ with $xy = e$, $yx = f$.

1) We may use Lemma 1.1.

LEMMA 5.7. *When* $(e)_r \overset{a}{\sim} (f)_r$, *then* $L((0)_r, (e)_r)$ *and* $L((0)_r, (f)_r)$ *are isomorphic in* $\mathring{R}_A$.

PROOF. When $(e)_r \overset{a}{\sim} (f)_r$, then $(e)_r$ and $(f)_r$ are isomorphic as right A-modules. Since any element in $L((0)_r, (e)_r)$ is a right A-submodule of $(e)_r$, this lemma is evident.

LEMMA 5.8. *Let* e, f *be idempotents and* h *be a central idempotent* in A. *If* $(e)_r \overset{a}{\sim} (f)_r \leqq (h)_r$, *then* $(e)_r \leqq (h)_r$. *That is, the relative center of* $\mathring{R}_A$ *with respect to* "$\overset{a}{\sim}$" *coincides with the center of* $\mathring{R}_A$. *And* $(e)_r \overset{a}{\sim} (f)_r$ *implies* $e((e)_r) = e((f)_r)$.

PROOF. Since $(e)_r \overset{a}{\sim} (f)_r$, there exist $x \in eAf$, $y \in fAe$ with $xy = e$, $yx = f$. Then, since $y = fy$ and $f = hf$, we have $hy = y$. Hence $he = hxy = xhy = xy = e$. Therefore $(e)_r \leqq (h)_r$. When $(e)_r \overset{a}{\sim} (f)_r$, put $e((f)_r)$ instead of $(h)_r$, then $(e)_r \leqq e((f)_r)$. Therefore $e((e)_r) \leqq e((f)_r)$. Similarly $e((f)_r) \leqq e((e)_r)$, and we have $e((e)_r) = e((f)_r)$.

Remark 5.4. It is evident that the central cover $e((e)_r)$ of $(e)_r$ in $\mathring{R}_A$ is expressed as $(h)_r$ where h is the central cover $C(e)$ of e in A.[1]

LEMMA 5.9. *The relation* "$\overset{a}{\sim}$" *in* $\mathring{R}_A$ *is an equivalence relation which satisfy* (e. 1)—(e. 4) *in* § 3.

PROOF. Since $(e)_r \overset{a}{\sim} (f)_r$ means the isomorphism as right A-modules, "$\overset{a}{\sim}$" is an equivalence relation.

(e. 1) When $(e)_r \overset{a}{\sim} (0)_r$, let $f = 0$ in Definition 5.1, then $x = y = 0$ and $e = xy = 0$.

(e. 2) This follows from Lemma 5.7.

(e. 3) Let $(e)_r \overset{a}{\sim} (f)_r$, and h be a central idempotent in A. Then by Lemma 5.5 (i), $(h)_r \cap (e)_r = (he)_r$ and $(h)_r \cap (f)_r = (hf)_r$. From Definition 5.1, we have $hx \in heAhf$, $hy \in hfAhe$ and $hx \cdot hy = he$, $hy \cdot hx = hf$. Hence $(he)_r \overset{a}{\sim} (hf)_r$, that is, $(h)_r \cap (e)_r \overset{a}{\sim} (h)_r \cap (f)_r$.

(e. 4) Let $(e_1)_r \overset{a}{\sim} (f_1)_r$, $(e_2)_r \overset{a}{\sim} (f_2)_r$ and $e((e_1)_r) \cap e((e_2)_r) = (0)_r$. Then there exist $x_1 \in e_1Af_1$, $y_1 \in f_1Ae_1$, $x_2 \in e_2Af_2$, $y_2 \in f_2Ae_2$ with $x_1y_1 = e_1$, $y_1x_1 = f_1$, $x_2y_2 = e_2$, $y_2x_2 = f_2$. It we denote $h_1 = C(e_1) = C(f_1)$, $h_2 = C(e_2) = C(f_2)$, then $h_1h_2 = 0$. Set $e = e_1 + e_2$, $f = f_1 + f_2$, $x = x_1 + x_2$, $y = y_1 + y_2$. Since $e_1, f_1, x_1, y_1 \in h_1A$ and e_2, $f_2, x_2, y_2 \in h_2A$, it is evident that e, f are idempotents such that $(e_1)_r \vee (e_2)_r = (e)_r$, $(f_1)_r \vee (f_2)_r = (f)_r$ and $x \in eAf$, $y \in fAe$, $xy = e$, $yx = f$. Hence $(e)_r \overset{a}{\sim} (f)_r$.

If e is an idempotent in a Baer ring A, then eAe is also a Baer ring. (Cf. [4] Chapter I, Theorem 2.) Now we have the following lemma.

LEMMA 5.10. $\mathring{R}_{eAe}$ *is isomorphic to the sublattice* $L((0)_r, (e)_r)$ *of* $\mathring{R}_A$.

PROOF. By Lemma 5.4, any element in $L((0)_r, (e)_r)$ is denoted by $(f)_r$ with an idempotent f in eAe. We denote the element of $\mathring{R}_{eAe}$ by $(f)_{r'}$, f being an idempotent in eAe.

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1) $C(e)$ is the smallest central idempotent h such that $he = e$. ([4] Chapter I Definition 3.)

Let f_1 and f_2 be idempotents in eAe. If $(f_1)_{r'}=(f_2)_{r'}$, then by [8] Kapitel VI, Hilfssatz 1.5, there exists $x \in eAe$ such that $f_2=f_1+f_1x(1-f_1)$. Therefore by the same Hilfssatz, we have $(f_1)_r=(f_2)_r$.

Conversely if $(f_1)_r=(f_2)_r$, $f_1, f_2 \in eAe$, then there exists $x \in A$ such that $f_2=f_1+f_1x(1-f_1)$. Since $f_1e=ef_1=f_1$, $f_2e=f_2$, we have

$$f_2=f_2e=f_1e+f_1x(1-f_1)e=f_1+f_1exe(1-f_1).$$

Therefore $$(f_1)_{r'}=(f_2)_{r'}.$$

Hence by $(f)_r \leftrightarrow (f)_{r'}$, f being an idempotent in eAe, there exists one-one correspondence between $L((0)_r, (e)_r)$ and $\mathring{R}_{eAe}$, preserving the lattice-order. Hence $L((0)_r,(e)_r)$ and $\mathring{R}_{eAe}$ are isomorphic.

THEOREM 5.4. *Let e be an idempotent in a Baer ring A without nilpotent ideals. Then the center of $L((0)_r,(e)_r)$ consists exactly of all elements of the form $(h)_r \cap (e)_r$ with central idempotents h of A. That is, $\mathring{R}_A$ is a Z_α-lattice.*

PROOF. By Lemma 5.10 $L((0)_r,(e)_r)$ is isomorphic to $\mathring{R}_{eAe}$. But by [4] Chapter III, Exercise 2, c) the central idempotents of eAe are the idempotents he with h a central element of A. Hence by Theorem 5.2 and Lemma 5.5 (i), the central elements of $L((0)_r, (e)_r)$ are $(he)_r=(h)_r \cap (e)_r$ with $(h)_r$ a central element of $\mathring{R}_A$. Since the inverse statement is evident, $\mathring{R}_A$ is a Z_α-lattice.

When A is a Baer ring, by Theorem 5.3 $\mathring{R}_A$ is a Z-lattice, and by Lemma 5.9 the equivalence relation "$\overset{a}{\sim}$" satisfies (e. 1)—(e. 4). Hence we can apply the analytic decompositions of Z-lattices to $\mathring{R}_A$, where we say "a-minimal" and "a-finite" instead of "e-minimal" and "e-finite". Especially when A has no nilpotent ideal $\mathring{R}_A$ is a Z_α-lattice, and we can apply Theorem 2. 2. Now we shall compare these decompositions of $\mathring{R}_A$ with Kaplansky's decomposition of A itself. For the decomposition of A, Kaplansky gave the following definitions. An idempotent e in A is said to be *abelian* in case the idempotents of eAe mutually commute. ([4] Chapter I, Definition 4.) And e is said to be *finite* if $xy=e$, $x, y \in eAe$ imply $yx=e$. ([4] Chapter I, Definition 6.)

THEOREM 5.5. *Let e be a non-zero idempotent in a Baer ring A. Consider the following statements.*

(α) *e is an abelian idempotent in A.*

(β) *$(e)_r$ is a D-element in $\mathring{R}_A$.*

(γ) *$(e)_r$ is a lowest element in $\mathring{R}_A$.*

(δ) *$(e)_r$ is an a-minimal element in $\mathring{R}_A$.*

Then $(\alpha) \leftrightarrows (\beta)$ *and* $(\gamma) \leftrightarrows (\delta)$. *Especially when A has no nilpotent ideals,* $(\alpha) \leftrightarrows (\beta) \leftrightarrows (\gamma) \leftrightarrows (\delta)$.

PROOF. $(\alpha) \leftrightarrows (\beta)$. By [4] Chapter I, Exercise, (α) is equivalent to the fact that all idempotents in eAe are central in eAe, that is, $\mathring{R}_{eAe}$ is a Boolean lattice, by Theorem 5.2. Since, by Lemma 5.3, $\mathring{R}_A$ is relatively complemented, (β) is equivalent to the fact that $L((0)_r, (e)_r)$ is a Boolean lattice. Hence, by

Lemma 5.10, (α) and (β) are equivalent.

$(\gamma) \leftrightarrows (\delta)$. This follows from Theorem 3.1, since by Theorem 5.3 $\mathring{R}_A$ is a relatively complemented Z-lattice, and by Lemma 5.8 the relative center of $\mathring{R}_A$ with respect to "$\overset{a}{\sim}$" coincides with the center of $\mathring{R}_A$.

Especially when A has no nilpotent ideals, by Theorem 5.4 $\mathring{R}_A$ is a Z_α-lattice. Hence by Theorem 2.2, $(\beta) \leftrightarrows (\gamma)$.

THEOREM 5.6. *Let e be an idempotent in a Baer ring A. Then the following two statements are equivalent.*

(α) *e is finite in A.*

(β) *$(e)_r$ is a-finite in $\mathring{R}_A$.*

PROOF. $(\alpha) \to (\beta)$. Let $(e)_r \overset{a}{\sim} (f)_r \leqq (e)_r$. By Lemma 5.4, we may assume that $f = ef = fe$. Since $(e)_r \overset{a}{\sim} (f)_r$, there exist $x \in eAf$, $y \in fAe$ with $xy = e$, $yx = f$. Then $x \in eAfe \leqq eAe$ and $y \in efAe \leqq eAe$. Since e is finite, we have $yx = e$. Therefore $f = e$, and $(e)_r$ is a-finite.

$(\beta) \to (\alpha)$. When $xy = e$, $x, y \in eAe$, set $f = yx$. Then f is an idempotent and $ef = eyx = yx = f$. Hence $(f)_r \leqq (e)_r$. Since $x = ex = xyx = xf$, $y = ye = yxy = fy$, we have $x \in eAf$ and $y \in fAe$. Therefore $(e)_r \overset{a}{\sim} (f)_r \leqq (e)_r$. Since $(e)_r$ is a-finite, we have $(f)_r = (e)_r$, and $e = fe$. Since $fe = yxe = yx = f$, we have $e = f$, and f is finite in A.

Theorems 5.5 and 5.6 show the relation between the decomposition of a Baer ring A and the decomposition of $\mathring{R}_A$. Especially *when A has no nilpotent ideals, the decomposition of A corresponds to the analytic decomposition of $\mathring{R}_A$.*

Remark 5.4. By Kaplansky [4, Chapter III, Definition 2], a ring A with involution * is called a *Baer*-ring* if there exists a projection e such that $S^r = (e)_r$ for any subset S of A. A is of course a Baer ring and the projection e such that $S^r = (e)_r$ is uniquely determined. Hence we may use the lattice of projections which is isomorphic to $\mathring{R}_A$.[1)] Since a Baer*-ring has no nilpotent ideals, by Theorem 5.4 $\mathring{R}_A$ is a Z_α-lattice. (Or, since the central projections of eAe, e being a projection, are the projections he with h central projections ([4] Chapter VI, Lemma 1), we can prove as Theorem 5.4). And (α), (β), (γ) and (δ) in Theorem 5.5 for a projection e are equivalent.

But in a Baer*-ring A, we have another equivalence relation "$\overset{*}{\sim}$" which is defined as follows. Let e and f be two projections of A, $e \overset{*}{\sim} f$ if and only if there exists $x \in A$ with $xx^* = e$, $x^*x = f$. ([4] Chapter III, Definition 5. In this case, $x \in eAf$ holds). We can treat "$\overset{*}{\sim}$" as "$\overset{a}{\sim}$" and we have the same Theorem as Theorem 5.5. Hence combining these theorems, we have the following result. *In the lattice of projections in a Baer*-ring, a non-zero abelian*

1) For the lattice of projections in a Baer*-ring, cf. S. Maeda [10]. This is a relatively orthocomplemented complete lattice by Lemma 5.3 or Lemma 5.6 and Theorem 4.1, where $e^\perp = 1 - e$.

*projection, a non-zero D-element, a lowest element, an a-minimal element and a *-minimal element all coincide.*

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