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On the Definition of Convolutions for Distributions

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The main purpose of this paper is to show the equivalence of the definitions of convolutions available in the theory of distributions.

Let S and T be two distributions on R^n, n-dimensional Euclidean space. L. Schwartz ([12], exposé 21) defined the convolution $S*T$ by the relation

$$<S*T, \varphi> = \iint (S_x \otimes T_y)\varphi(x+y)dx\,dy \quad \text{for any} \quad \varphi \in (\mathscr{D}),$$

if the following condition is satisfied:

$$(*) \qquad (S_x \otimes T_y)\varphi(\dot{x}+\dot{y}) \in (\mathscr{D}'_{L^1}) \quad \text{for any} \quad \varphi \in (\mathscr{D}).$$

In his lecture notes [4], C. Chevalley gave two definitions of convolutions. His first definition (in more precise form) is: $S*T$ is defined as

$$(1) \qquad \int_{R^n} S(y)T(\dot{x}-y)\,dy,$$

when this makes sense. This last phrase is interpreted as in the case of integration of vector-valued functions, that is, (1) has the meanings if and only if

$$(\overline{*})' \qquad S(\check{T}*\varphi) \in (\mathscr{D}'_{L^1}) \quad \text{for any} \quad \varphi \in (\mathscr{D}).$$

And

$$<\int_{R^n} S(y)T(\dot{x}-y)dy, \varphi> = \int_{R^n} S(y)(\check{T}*\varphi)(y)dy.$$

Then in the terminology of L. Schwartz ([14], p. 130), the definition is equivalent to saying that $S*T$ is the integral (1) when the integrand $S(y)T(x-y)$ is partially summable with respect to y. The second definition (generalized convolution in his sense, [4], p. 112) is:

$S*T$ is defined when the condition

$$(\overline{*}) \qquad (S*\varphi)(\check{T}*\psi) \in L^1 \quad \text{for any} \quad \varphi, \psi \in (\mathscr{D})$$

is satisfied, and $S*T$ is given by

$$<(S*T)*\varphi, \psi> = \int_{R^n} (S*\varphi)(x)(\check{T}*\psi)(x)dx.$$

In sec. 3, we show that these definitions of convolutions are equivalent, and furthermore that it remains valid that the definitions obtained by replacing $(\mathscr{D})$ by $(\mathscr{S})$ in the above discussions are also equivalent. After Hirata and Ogata [8] we say that the $(\mathscr{S}')$-convolution $S*T$ is defined when

$$(S_x \otimes T_y)\varphi(\hat{x}+\hat{y}) \in (\mathscr{D}'_{L^1})_{x,y} \quad \text{for any} \quad \varphi \in (\mathscr{S}).$$

$S * T$ then belongs to $(\mathscr{S}')$. It is my open question whether the convolution $S * T$ belongs to $(\mathscr{S}')$ or not whenever $S * T$ is defined for any $S, T \in (\mathscr{S}')$. If the question is positively settled, then it may be superfluous to introduce the concept of $(\mathscr{S}')$-convolution because then we can show that the $(\mathscr{S}')$-convolution is defined if and only if the convolution is defined for the two given distributions $\in (\mathscr{S}')$.

In Sec. 4 we are concerned with the simultaneous convolutions. We show that the following two conditions

$$(*)_3 \qquad (S_x \otimes T_y \otimes U_z)\varphi(\hat{x}+\hat{y}+\hat{z}) \in (\mathscr{D}'_{L^1}) \quad \text{for any} \quad \varphi \in (\mathscr{D})$$

and

$$(\bar{*})_3 \qquad (S * \varphi)(\hat{x})(T * \psi)(\hat{y})(\check{U} * \chi)(\hat{x}+\hat{y}) \in L^1 \quad \text{for any} \quad \varphi, \psi, \chi \in (\mathscr{D})$$

are equivalent. We also show that if $S \neq 0$, $T * U$ is defined under any one of these equivalent conditions. We define $S * T * U$ by the relation

$$< S * T * U, \varphi > = \iiint (S_x \otimes T_y \otimes U_z)\varphi(x+y+z)dx\,dy\,dz \quad \text{for any} \quad \varphi \in (\mathscr{D}),$$

when the right hand side makes sense, that is, $(*)_3$ holds. If $S * T * U$ is defined for $S, T, U \neq 0$, then it will be seen that

$$S * T * U = S * (T * U) = T * (U * S) = U * (S * T).$$

We show that for the simultaneous convolution of more than three distributions the same is true.

Secs. 1 and 2 are devoted to the preliminary discussions. We close the last section 5 with treatment of simple consequences of the preceding sections.

1. Let E, F, G be locally convex spaces such that $F \subset G$ and the injection j: $F \to G$ is continuous. Let $\mathscr{L}$ be any linear application of E into F such that $j \circ \mathscr{L}$ is continuous from E into G. Under certain conditions we can infer that $\mathscr{L}$ is continuous. In the treatment of the theory of distributions, we often encounter the cases where F is a space of distributions and $G = (\mathscr{D}')$ (a locally convex space is said to be a space of distributions if it is algebraically a subspace of $(\mathscr{D}')$ and the injection into $(\mathscr{D}')$ is continuous). For a barrelled space E, the closed graph theorems often meet our requirments, e.g. (1) E is a space of type (β) and F has an (LF)-topology stronger than its initial one (Grothendieck, [6]). (2) F is a space of type (F) (Robertson, [9]). Another sort of results concerning this continuity was also given by Yoshinaga-Ogata [15] and Hirata [7]; applying Banach-Steinhaus theorem in Bourbaki's form ([2], Chap III).

We shall here give a theorem of this sort.

THEOREM 1. *Let E be a barrelled locally convex space, F a convex space, and $\mathscr{L}$ a linear application of E into F. Suppose that the identical application I of*

F into itself is strictly adherent to a subset A of $L_s(F;F)$ such that $u\circ\mathscr{L}$, for each $u\in A$, is continuous from E into F. Then $\mathscr{L}$ is continuous.

PROOF. Let $\hat{F}$ denote the quasi-completion of F. Let $\varLambda_s(E;\hat{F})$ be the linear space of linear applications of E into $\hat{F}$ with the topology of simple convergence. Then $L_s(E;\hat{F})$ is a quasi-complete subspace of $\varLambda_s(E;\hat{F})$. This is obtained by applying Banach-Steinhaus theorem cited above as shown in the proof of Proposition 27 of L. Schwartz [14]. The mapping $v\rightarrow v\circ\mathscr{L}$ of $L_s(F;F)$ into $\varLambda(E;\hat{F})$ is clearly continuous and $\mathscr{L}$ is strictly adherent to the subset $\{u\circ\mathscr{L};u\in A\}$ of $\varLambda_s(E;\hat{F})$ which is also a subset of $L_s(E;\hat{F})$, as clear from our hypothesis. Since $L_s(E;\hat{F})$ is, as stated above, a quasi-complete subspace of $\varLambda_s(E;\hat{F})$, therefore $\mathscr{L}\in L_s(E;\hat{F})$. As $\mathscr{L}(E)\subset F$, $\mathscr{L}$ is continuous from E into F. This completes the proof.

As an immediate consequence of this theorem we have

COROLLARY. *Let F, G be locally convex spaces such that F is algebraically a subspace of G and has a continuous injection $j:F\rightarrow G$. We suppose that $I\in L_s(F;F)$ is strictly adherent to a subset A of $L_s(F;F)$ such that each $u\in A$ is a restriction of a continuous linear application of G into F, and $\mathscr{L}$ is any linear application of a barrelled space E into F such that $j\circ\mathscr{L}$ is continuous from E into G. Then $\mathscr{L}$ is continuous.*

In this Corollary, if we let F be a permitted[1)] space or its dual and let $G=(\mathscr{D}')$, the result is a part of the theorem obtained by Yoshinaga and Ogata [15]. If F is a space of type $\mathbf{H}^m$ or its dual (L. Schwartz [13]), then, by taking A to be any seqeunce of multiplicators $\{\alpha_k\}$ and $G=(\mathscr{E}^m)$ and using the closed graph theorem stated in (2), we have a result: Let $\mathscr{L}$ be any linear application of a barrelled space E into a space of type $\mathbf{H}^m$ or its dual, then $\mathscr{L}$ is continuous if $j\circ\mathscr{L}$ is continuous from E into $(\mathscr{D}')$.

Let F, G be locally convex spaces such that $F\subset G$, F has G-closed fundamental neighbourhood system of zero and the injection $j:F\rightarrow G$ is continuous. Then, for any barrelled locally convex space E, the linear application $\mathscr{L}$ of E into F is continuous if $j\circ\mathscr{L}$ is continuous from E into G. Indeed, let V be any G-closed, absolutely convex neighbourhood of zero in F. It follows since $\mathscr{L}$ is continuous from E into G that $\mathscr{L}^{-1}(V)$ is a barrel in E, and therefore

1) A space of distributions F is said to be an *admissible* space if and only if F contains $(\mathscr{D})$ such that the injection $(\mathscr{D})\rightarrow F$ is continuous and $(\mathscr{D})$ is dense in F (*normal* in Schwartz's terminologies). An admissible space F is said to be *permitted* if the following condition is satisfied: $\alpha_k(T*\rho_k)\rightarrow T$ and $(\alpha_kT)*\rho_k\rightarrow T$ in F for any $T\in F$ as $k\rightarrow\infty$, where $\{\alpha_k\}$ is a sequence of multiplicators and $\{\rho_k\}$ is a sequence of regularizations (Yoshinaga and Ogata [15]). We note that if F is a barrelled admissible space, then it is permitted if and only if $\{(\alpha_kT)*\rho_k\}$ is bounded for any $T\in F$. Indeed, the "only if" part is evident. Conversely, let $\{(\alpha_kT)*\rho_k\}$ be bounded for any $T\in F$. As F is admissible, so the application $T\in F\rightarrow(\alpha_kT*\rho_k)$ of F into itself is continuous for each k, and the sequence of the applications $T\rightarrow(\alpha_kT)*\rho_k$ is equi-continuous since F is barrelled. For any $\varphi\in(\mathscr{D})$, $(\alpha_k\varphi)*\rho_k\rightarrow\varphi$ in $(\mathscr{D})$ as $k\rightarrow\infty$, and *a fortiori* $(\alpha_k\varphi)*\rho_k\rightarrow\varphi$ in F as $k\rightarrow\infty$. Then it follows from Banach-Steinhaus theorem that $(\alpha_kT)*\rho_k\rightarrow T$ in F as $k\rightarrow\infty$. Similarly $\alpha_k(T*\rho_k)\rightarrow T$ in F as $k\rightarrow\infty$. Therefore F is permitted.

$\mathscr{L}^{-1}(V)$ is a neighbourhood of zero in E, because E is a barrelled space. Hence $\mathscr{L}$ is continuous from E into F.

A space of distributions F is said to possess the *property* (C) when any linear application $\mathscr{L}$ of any barrelled space E into F is continuous if $\mathscr{L}$, considered as an application of E into $(\mathscr{D}')$, is continuous.

According to L. Schwartz ([14], p. 53), we say that a space of distributions possesses the *property* (ε) if, for any quasi-complete locally convex space E, any linear application $\mathscr{L}$ of E'_c into F is continuous whenever $j\circ\mathscr{L}$ is continuous, j being the injection $F\rightarrow(\mathscr{D}')$.

The property (ε) implies the property (C), but not always the converse. Indeed, let E be any barrelled space, then E'_c is quasi-complete and E has the γ-topology, that is, $E=(E'_c)'_c$. The property (ε) implies that any linear application $\mathscr{L}$ of $E=(E'_c)'_c$ into F is continuous if $j\circ\mathscr{L}$ is continuous. This means that F possesses the property (C). $\mathbf{H}^m$, m being finite, possesses the property (C), as stated above, but not the property (ε) (L. Schwartz [14], p. 56).

Let F be an admissible space. The following two conditions are equivalence; i) F'_c possesses the property (C), ii) F' (strong dual of F) possesses the property (C). Indeed, ii)$\rightarrow$i) is clear. Conversely, let E be a barrelled space and $\mathscr{L}$ be any application of E into F' such that $j\circ\mathscr{L}$ is continuous from E into $(\mathscr{D}')$. By the hypothesis $\mathscr{L}$ is continuous from E into F'_c, and its transposed ${}^t\mathscr{L}$: $F\rightarrow E'$ is weakly continuous. Then, for any bounded subset B of F, ${}^t\mathscr{L}(B)$ is weakly bounded in E', and therefore strongly bounded, since E is barrelled. It follows that $\mathscr{L}$ is continuous from E into F', as desired.

If F is an admissible metrisable space, then F'_c possesses the property (ε) (L. Schwartz [14], p. 53) therefore F'_c and F' possess the property (C).

From the preceding discussions we see that permitted spaces, space of type $\mathbf{H}^m$ and their duals have the property (C).

Examples: $(\mathscr{D}^m)$, $(\mathscr{D}'^m)$, $(\mathscr{B})$, $(\mathscr{D}'_{L^p})$, $(\mathscr{E}^m)$, $(\mathscr{E}'^m)$, $(\mathscr{S})$, $(\mathscr{S}')$, $(\mathscr{O}'_C)$, $(\mathscr{O}_M)$, (L^p) etc.

2. If T is a summable distribution of R^n, that is, $T\in(\mathscr{D}'_{L^1})$, T is a continuous linear form on $(\dot{\mathscr{B}}_c)$. $T(1)$ is called the integral of T over R^n and denoted by $\int_{R^n}T(x)dx$ or $<T, 1>$.

For $T\in(\mathscr{D}')$ and $f\in(\mathscr{E})$, $<T, f>$ stands for $\int_{R^n}T(x)f(x)dx$, provided this has a meaning, that is, $Tf\in(\mathscr{D}'_{L^1})$.

DEFINITION 1. (*Schwartz [12], exposé 21*). *Let S and T be two distributions. The convolution $S*T$ is defined by*

$$(1)\qquad <S*T, \varphi>=\iint(S_x\otimes T_y)\varphi(x+y)dx\,dy,\ \varphi\in(\mathscr{D}),$$

when the right-hand side makes sense, that is,

$(*)$ $\qquad (S_x \otimes T_y)\varphi(\dot{x}+\dot{y}) \in (\mathscr{D}'_{L^1})_{x,y}$ *for any* $\varphi \in (\mathscr{D})$.

If $(*)$ is satisfied, the application $\varphi \in (\mathscr{D}) \to (S_x \otimes T_y)\ \varphi(\dot{x}+\dot{y}) \in (\mathscr{D}'_{L^1})_{x,y}$ is continuous since $(\mathscr{D}'_{L^1})_{x,y}$ has the property (*C*). Then $S*T$ is well defined by (1).

Let $\vec{T}(x)=\tau(x)T$, where $\tau(x)$ denotes the translation: $<\vec{T}(x), \varphi> = <T_y, \varphi(x+\dot{y})>$. Then $\vec{T}(x)$ is an indefinitely differentiable function with values in $(\mathscr{D}')$, that is, $\vec{T}(\dot{x}) \in \mathscr{E}(\mathscr{D}')$ (Schwartz [13]). Noting that $<\vec{T}(x), \varphi> = (\check{T} * \varphi)(x)$, $<S_x, \vec{T}(\dot{x})>$ has a meaning, by definition, provided

$(\overline{*})'$ $\qquad S(\check{T} * \varphi) \in (\mathscr{D}'_{L^1})$ *for any* $\varphi \in (\mathscr{D})$.

Then the application $\varphi \in (\mathscr{D}) \to S(\check{T} * \varphi) \in (\mathscr{D}'_{L^1})$ is continuous, and $<S_x, \vec{T}(\dot{x})>$ is defined by

$$(2) \qquad <<S_x, \vec{T}(x)>, \varphi> = \int_{R^n} S(x)(\check{T} * \varphi)(x)dx.$$

The condition $(\overline{*})'$ means in Schwartz's terminology (Schwartz [14], p. 130) that the kernel distribution $S(\dot{x})T(\dot{y}-\dot{x})$ is partially summable with respect to x and $<S_x, \vec{T}(\dot{x})>$ is defined as $\int_{R^n} S(x)T(\dot{y}-x)dx$. C. Chevalley's first definition of $S*T$ (Chevalley [4], p. 67) is stated as follows.

DEFINITION 2. *Let S and T be two distributions, the convolution $S*T$ is defined by*

$$(3) \qquad <S*T, \varphi> = \int_{R^n} S(x)(\check{T} * \varphi)(x)dx, \quad \varphi \in (\mathscr{D}),$$

when the right-hand side makes sense, that is, $(\overline{*})'$ holds. In this case $S*T$ also is written as $\int_{R^n} S(x)T(\dot{y}-x)dx$.

If S or T has a compact support, $(S_x \otimes T_y)\varphi(\dot{x}+\dot{y})$ has also a compact support for any $\varphi \in (\mathscr{D})$, so that $(*)$ holds. And $(\overline{*})'$ also is clearly satisfied. Fubini's theorem for tensor products (Schwartz [10], p. 109) shows us that the two definitions coincide in this case. But if we prefer the definition of Schwartz, the convolution is automatically commutative. It will be shown later that this also holds for Def. 2, though it is not clear at once from the definition itself (Schwartz [12], exposé 22). Hence, for the sake of convenience, we give

DEFINITION 3. *Let S and T be two distributions. The convolution $S*T$ is defined by*

$$(4) \qquad <S*T, \varphi> = \int_{R^n} (\check{S} * \varphi)(x)(T)(x)dx, \quad \varphi \in (\mathscr{D}),$$

when the right-hand side makes sense, that is,

$(\overline{*})''$ $\qquad (\check{S} * \varphi)T \in (\mathscr{D}'_{L^1})$ *for any* $\varphi \in (\mathscr{D})$.

Then $S*T$ *is written as* $\int_{R^n} S(\hat{y}-x)T(x)dx$.

On the other hand, Chevalley gave a second definition on convolution (named generalized convolution) (Chevalley [4], p. 112).

DEFINITION 4. *Let* S *and* T *be two distributions. The convolution* $S*T$ *is defined by*

$$(5)\qquad <(S*T)*\varphi, \psi> = \int_{R^n}(S*\varphi)(x)(\check{T}*\psi)(x)dx,\ \varphi, \psi \in (\mathscr{D}),$$

when the right-hand side makes sense, that is,

$$(\overline{*})\qquad (S*\varphi)(\check{T}*\psi)\in L^1 \quad \textit{for any} \quad \varphi, \psi \in (\mathscr{D}).$$

A simple existence proof of $S*T$ in this case was given by K. Yoshinaga and H. Ogata [15].

3. We shall show the equivalence of four definitions on convolution given in the preceding section. For the distingisuhed convolutions in four definitions, we use, if necessary, the notations $S*T, S\overline{*}'T, S\overline{*}''T$ and $S\overline{*}T$ according to the cases.

THEOREM 2. *Four definitions of convolutions in section 2 are equivalent to each other.*

PROOF. We first show that the conditions $(*)$, $(\overline{*})'$, $(\overline{*})''$ and $(\overline{*})$ are equivalent to each other.

ad $(\overline{*})\to(*)$. As L^1 possesses the property (C), the application $\psi\in(\mathscr{D})\to (S*\varphi)(\check{T}*\psi)\in L^1$ is continuous. Let $\vec{A}(y)=(S*\varphi)\tau(y)(T*\psi) =(S*\varphi)(T*\tau(y)\psi)\in L^1$. Then $\vec{A}(y)$ will become continuous, which implies the continuity of $\int_{R^n}(S*\varphi)(x)(\check{T}*\psi)(x-y)dx$. Therefore $(S*\varphi)(\hat{x})(\check{T}*\psi)(\hat{x}-\hat{y})\alpha(\hat{y}) \in (L^1)_{x,y}$ for any $\varphi, \psi, \alpha\in(\mathscr{D})$. By change of the variables it follows that

$$(1)\qquad (S*\varphi)(\hat{x})(T*\check{\psi})(\hat{y})\alpha(\hat{x}+\hat{y})\in(L^1)_{x,y} \text{ for any } \varphi, \psi, \alpha\in(\mathscr{D}).$$

The application $(\varphi, \psi, \alpha)\in(\mathscr{D}_K)\times(\mathscr{D}_K)\times(\mathscr{D}_K)\to(S*\varphi)(\hat{x})(T*\psi)(\hat{y})\alpha(\hat{x}+\hat{y})\in (L^1)_{x,y}$ is continuous (K is a compact set in R^n: $\{|x|\leq r\}$), and hence it is continuous in the topology induced by $(\mathscr{D}_K^k)\times(\mathscr{D}_K^k)\times(\mathscr{D}_K^k)$ for some positive integer k. We can take a positive integer m such that some $u\in(\mathscr{D}_K^k)$ is a parametrix of an iterated Laplacian Δ^m ([4], p. 76):

$$(2)\qquad \delta=\Delta^m u+\xi,\ \xi\in(\mathscr{D}_K).$$

$$\begin{aligned}(3)\qquad (S_x\otimes T_y)\alpha(\hat{x}+\hat{y}) &= \{(S_x*\Delta^m u+S_x*\xi)\otimes(T_y*\Delta^m u+T_y*\xi)\}\alpha(\hat{x}+\hat{y})\\ &=\{(S_x*\Delta^m u)\otimes(T_y*\Delta^m u)\}\alpha(\hat{x}+\hat{y})\\ &+\{(S_x*\xi)\otimes(T_y*\xi)\}\alpha(\hat{x}+\hat{y})\\ &+\{(S_x*\Delta^m u)\otimes(T_y*\xi)\}\alpha(\hat{x}+\hat{y})\\ &+\{(S_x*\xi)\otimes(T_y*\Delta^m u)\}\alpha(\hat{x}+\hat{y}).\end{aligned}$$

Since $u \in (\mathscr{D}_K^k)$, we can choose a sequence $\{u_j\}$ such that $u_j \in (\mathscr{D}_K)$ and $u_j \to u$ in $(\mathscr{D}_K^k)$. Then $\{(S_x * u)\otimes(T * u)\}\alpha(\hat{x}+\hat{y})$, $\{(S_x * u)\otimes(T_y * \xi)\}\alpha(\hat{x}+\hat{y})$, $\{(S_x * \xi)\otimes(T_y * u)\}\alpha(\hat{x}+\hat{y})$ and $\{(S_x * \xi)\otimes(T_y * \xi)\}\alpha(\hat{x}+\hat{y})$ belong to L^1.

$$\left\{(S_x * \frac{\partial}{\partial x_i} u)\otimes(T_y * u)\right\}\alpha(\hat{x}+\hat{y})$$

$$=\frac{\partial}{\partial x_i}\left\{S_x * u)\otimes(T_y * u)\alpha(\hat{x}+\hat{y})\right\}-\left\{(S_x * u)\otimes(T_y * u)\right\}\frac{\partial}{\partial x_i}\alpha(\hat{x}+\hat{y}) \in (\mathscr{D}'_{L^1})_{x,y}.$$

Repeating this and analogous process, we see that $(S_x\otimes T_y)\alpha(\hat{x}+\hat{y}) \in (\mathscr{D}'_L)_{x,y}$ for any $\alpha \in (\mathscr{D})$.

ad $(*)\to(\overline{*})'$. $(S_x\otimes T_y)\varphi(\hat{x}+\hat{y})$ is partially summable with respect to y (Schwartz [14]). This implies $S_x \int_{R^n} T_y\varphi(\hat{x}+y)dy = S_x(\check{T} * \varphi) \in (\mathscr{D}'_{L^1})$.

ad $(*)\to(\overline{*})''$. The proof is very similar to the preceding case.

ad $(\overline{*})'\to(\overline{*})$. Let $\vec{A}(y)=(\tau(y)S)(\check{T} * \psi)$. Then $\vec{A}(y)$ is a continuous function of y with values in $(\mathscr{D}'_{L^1})$. This is because the application $y \in R^n \to S(\check{T} * \tau(-y)\psi) \in (\mathscr{D}'_{L^1})$ is continuous since the application $\psi \in (\mathscr{D}')\to S(\check{T} * \psi) \in (\mathscr{D}'_{L^1})$ is continuous and so is the application $(y, U) \in R^n\times(\mathscr{D}'_{L^1})\to\tau(y)U \in (\mathscr{D}'_{L^1})$. Then for any $\varphi \in (\mathscr{D})$, $\varphi(y)\vec{A}(y)$ is of compact support, so that integral $\int\varphi(y)\vec{A}(y)dy \in (\mathscr{D}'_{L^1})$ (Bourbaki [3]). For any $\alpha \in (\mathscr{D})$, we have

$$(4)\qquad \begin{aligned} <\int\varphi(y)\vec{A}(y)dy, \alpha> &= \int\varphi(y)<\vec{A}(y), \alpha>dy \\ &= \int\varphi(y)<\tau(y)S, (\check{T} * \psi)\alpha>dy \\ &= <\int\varphi(y)\tau(y)Sdy, (\check{T} * \psi)\alpha> \\ &= <(S * \varphi), (\check{T} * \psi)\alpha> \\ &= <(S * \varphi)(\check{T} * \psi), \alpha> \end{aligned}$$

This yields

$$(5)\qquad (S * \varphi)(\check{T} * \psi) \in (\mathscr{D}'_{L^1}) \quad \text{for any} \quad \varphi, \psi \in (\mathscr{D}).$$

Using the fact that $U \in (\mathscr{D}'_{L^1})$ if and only if $U * \alpha \in L^1$ for any $\alpha \in (\mathscr{D})$, it follows from (4) that

$$(6)\qquad (S * \varphi)(\check{T} * \psi) * \alpha \in L^1.$$

The mapping $(\varphi, \psi, \alpha) \in (\mathscr{D}_K)\times(\mathscr{D}_K)\times(\mathscr{D}_K)\to(S * \varphi)(\check{T} * \psi) * \alpha \in L^1$ is continuous. Considering a parametrix u for a certain iterated Laplacian Δ^m, we have $(S * \varphi)(\check{T} * \psi)=(S * \varphi)(\check{T} * \psi) * \Delta^m u+(S * \varphi)(\check{T} * \psi) * \xi$ By a similar reasoning as in the proof of $(\overline{*})\to(*)$, we can infer that $(S * \varphi)(\check{T} * \psi) \in L^1$ for any $\varphi, \psi \in (\mathscr{D})$ which is the condition $(\overline{*})$.

ad $(\overline{*})''\to(\overline{*})$. We can carry out the proof in a similar manner as in the preceding case, so the proof is omitted.

Now we show that the four definitions concerning convolutions yield the same distribution for S and T.

Owing to Fubini's theorem (Schwartz [14], p. 132), we have

$$<S * T, \varphi> = \iint (S_x \otimes T_y)\varphi(x+y)dx\,dy$$
$$= \int\left\{\int (S_x \otimes T_y)\varphi(x+y)dy\right\}dx$$
$$= \int S(x)(\check{T} * \varphi)(x)dx = <S \,\bar{*}\,'T, \varphi>.$$

and similarly we have

$$<S * T, \varphi> = <S \,\bar{*}\,''T, \varphi>.$$

By Definition 4,

$$<(S \,\bar{*}\, T) * \varphi, \psi> = \int_{R^n} (S_x * \varphi)(x)(\check{T} * \psi)(x)dx.$$

As proved in $(\bar{*})' \to (\bar{*})$, $(S_x * \Delta^m u_j)(\check{T} * \psi) + (S_x * \xi)(\check{T} * \psi) \to S_x(\check{T} * \psi)$ in $(\mathscr{D}'_{L^1})$ as $j \to \infty$. Then

$$<S \,\bar{*}\,'T, \psi> = \lim_j \left\{\int (S_x * \Delta^m u_j)(x)(\check{T} * \psi)(x)dx + (S_x * \xi)(x)(\check{T} * \psi)(x)dx\right\}$$
$$= \lim_j <(S \,\bar{*}\, T) * (\Delta^m u_j + \xi), \psi>$$
$$= <S \,\bar{*}\, T, \psi>.$$

From these facts we can infer that $S * T = S \,\bar{*}\,'T = S \,\bar{*}\,''T = S \,\bar{*}\, T$, which established the proof.

We shall introduce the notation of the $(\mathscr{S}')$-convolution. To this end we show the following

THEOREM 3. *The following conditions are equivalent for two distributions* $S, T \in (\mathscr{S}')$.

(i) $(S_x \otimes T_y)\varphi(\hat{x}+\hat{y}) \in (\mathscr{D}'_{L^1})$ *for any* $\varphi \in (\mathscr{S})$;

$(\text{ii})_1$ $S(\check{T} * \varphi) \in (\mathscr{D}'_{L^1})$ *for any* $\varphi \in (\mathscr{S})$;

$(\text{ii})_2$ $(\check{S} * \varphi)T \in (\mathscr{D}'_{L^1})$ *for any* $\varphi \in (\mathscr{S})$;

$(\text{iii})_1$ $(S * \varphi)(\check{T} * \psi) \in L^1$ *for any* $\varphi \in (\mathscr{D})$ *and* $\psi \in (\mathscr{S})$;

$(\text{iii})_2$ $(S * \varphi)(\check{T} * \psi) \in L^1$ *for any* $\varphi \in (\mathscr{S})$ *and* $\psi \in (\mathscr{D})$;

(iv) $(S * \varphi)(\check{T} * \psi) \in L^1$ *for any* $\varphi, \psi \in (\mathscr{S})$.

PROOF. ad $(\text{i}) \to (\text{ii})_1$. $(S_x \otimes T_y)\varphi(\hat{x}+\hat{y})$ is partially summable with respect to y and its partial integral over R^n belongs to $(\mathscr{D}'_{L^1})$ (Schwartz [14], p. 132). Namely

$$\int_{R^n} (S_x \otimes T_y)\varphi(\hat{x}+\hat{y})dy = S_x(\check{T} * \varphi) \in (\mathscr{D}'_{L^1}).$$

ad $(\text{ii})_1 \to (\text{iii})_1$. The proof is carried out in a similar manner as in the proof of $(\bar{*})' \to (\bar{*})$ in Theorem 2, so the proof is omitted.

ad $(\text{iii})_1 \to (\text{i})$. Consider the kernel distribution

$$A(\hat{x}, \hat{y}) = (S * \varphi)(\hat{x})T(\hat{y} - \hat{x})\psi(\hat{y}), \varphi \in (\mathscr{D}) \quad \text{and} \quad \psi \in (\mathscr{S}).$$

Now, for any $\alpha \in (\mathscr{D})$, it follows since $\psi\alpha \in (\mathscr{D})$ that

$$A\cdot\alpha = (S * \varphi)(\check{T} * \psi\alpha) \in L^1.$$

L^1 has the property (C) and the application $\psi \in (\mathscr{S}) \to (S * \varphi)(\check{T} * \psi) \in (\mathscr{D}'_{L^1})$ is continuous, so that the mapping $\psi \in (\mathscr{S}) \to (S * T)(\check{T} * \psi) \in L^1$ also is continuous. The application $\alpha \in (\mathscr{D}) \to (S * \varphi)(\check{T} * \alpha\psi) \in L^1$, then, can be continuously extended to a linear application of $(\mathscr{O}_C)$ (the dual of $(\mathscr{O}'_C)$) to L^1, since the injection $(\mathscr{O}_C) \to (\mathscr{O}_M)$ and the application $\alpha \in (\mathscr{O}_M) \to \alpha\psi \in (\mathscr{S})$ are continuous. By using the result of Schwartz ([14], p. 135) and observing that $(\mathscr{O}'_C)$ is nuclear,

$$\begin{aligned} A(\dot{x}, \dot{y}) &\in (\mathscr{D}'_{L^1})\hat{\otimes}_\varepsilon(\mathscr{O}'_C) \\ &= (\mathscr{D}'_{L^1})\hat{\otimes}_\pi(\mathscr{O}'_C) \subset (\mathscr{D}'_{L^1})\hat{\otimes}_\pi(\mathscr{D}'_{L^1}) \\ &= (\mathscr{D}'_{L^1})_{x,y}. \end{aligned}$$

By change of the variables, we obtain

$$(S * \varphi)(\dot{x})T(\dot{y})\psi(\dot{x}+\dot{y}) \in (\mathscr{D}'_{L^1})$$

Any linear continuous application of a space of type (F) into any space of type (DF) is bounded (Bourbaki [2]), so that, as is usually done in the theory of distributions, we can apply the parametrix method to infer the relation

$$(S(\dot{x})\otimes T(\dot{y}))\psi(\dot{x}+\dot{y}) \in (\mathscr{D}'_{L^1}).$$

The implications (i)$\to$(ii)$_2\to$(iii)$_2\to$(i) will be proved in a simillar way, so the proof is omitted.

Finally, (iv) $\to$(iii)$_1$ or (iii)$_2$ is clear. Conversely, if (iii)$_1$ is holds, then $(S * \alpha)(\check{T} * \psi * \chi) = (S * \alpha)((T * \check{\psi})^\vee * \chi) \in L^1$ for any $\alpha \in (\mathscr{D})$, $\psi, \chi \in (\mathscr{S})$. Hence S, $T * \check{\psi}$ satisfy the condition (iii)$_1$, so that $(S * \varphi)((T * \check{\psi})^\vee * \beta) \in L^1$ for any $\beta \in (\mathscr{D})$, $\varphi, \psi \in (\mathscr{S})$ by (iii)$_2$. Therefore $(S * \varphi)(\check{T} * \psi) \in (\mathscr{D}'_{L^1})$ by (ii)$_2$, and so by the parametrix method $(S * \alpha)(\check{T} * \psi) \in L^1$ for any $\varphi, \psi \in (\mathscr{S})$. This completes the proof.

REMARK 1. Let S and T be two distributions satisfying the condition

(i) $\qquad (S_x\otimes T_y)\varphi(\dot{x}+\dot{y}) \in (\mathscr{D}'_{L^1})_{x,y}$ for any $\varphi \in (\mathscr{S})$.

If one of these distributions is not zero, the other belongs to $(\mathscr{S}')$, so that if both are not zero, then they belong to $(\mathscr{S}')$. Indeed, let $S \neq 0$. For any $\alpha, \beta \in (\mathscr{D})$, the condition (i) implies and is implied by

(v) $\qquad (S * \alpha)(\dot{x})(T * \beta)(\dot{y})\varphi(\dot{x}+\dot{y}) \in (L^1)_{x,y}$ for any $\alpha, \beta \in (\mathscr{D})$ and $\varphi \in (\mathscr{S})$.

This is shown by the parametrix method as in the proof of Theorem 2. For any given point $x_0 \in R^n$, we can choose $\alpha \in (\mathscr{D})$ so that $(S * \alpha)(x_0) \neq 0$. If we take any $\gamma \in (\mathscr{D})$ with support contained in a sufficiently small neighbourhood of x_0, then

$$|\gamma(\dot{x})(T * \beta)(\dot{y})\varphi(\dot{x}+\dot{y})| \leq M|(S * \alpha)(\dot{x})(T * \beta)(\dot{y})\varphi(\dot{x}+\dot{y})|,$$

where M is a constant depending on γ. This yields

(7) $\qquad \gamma(\dot{x})(T * \beta)(\dot{y})\varphi(\dot{x}+\dot{y}) \in (L^1)_{x,y}.$

Using a decomposition of unity, we can conclude that (7) holds for any $\gamma, \beta \in (\mathscr{D})$ and $\varphi \in (\mathscr{S})$. By change of the variables we obtain

$$(8) \qquad \gamma(\dot{x})(T * \beta)(\dot{y}-\dot{x})\varphi(\dot{y}) \in (L^1)_{x,y} \quad \text{for any } \gamma, \beta \in (\mathscr{D}) \text{ and } \varphi \in (\mathscr{S}).$$

Integrating (8) with respect to x, we obtain

$$(T * \beta * \gamma)\varphi \in L^1 \quad \text{for any} \quad \gamma, \beta \quad \text{and} \quad \varphi \in (\mathscr{S}).$$

This implies that $T * \beta * \gamma \in (\mathscr{S}')$ for any $\beta, \gamma \in (\mathscr{D})$, and then in turn $T \in (\mathscr{S}')$.

We shall say that two distributions $S, T \in (\mathscr{S}')$ have the $(\mathscr{S}')$-convolution if the condition (i) holds (Hirata and Ogata [8]). Then the application $\varphi \in (\mathscr{S}) \rightarrow (S_x \otimes T_y)\varphi(\dot{x}+\dot{y}) \in (\mathscr{D}'_{L^1})_{x,y}$ is continuous and therefore the application $\varphi \rightarrow \iint (S_x \otimes T_y)\varphi(x+y)dx\,dy = \langle S * T, \varphi \rangle$ is continuous, hence $S * T \in (\mathscr{S}')$.

REMARK 2. If the convolutions $S * T$ is defined for given distributions S, T and $\alpha S * T$ belongs to $(\mathscr{S}')$ for each $\alpha \in (\mathscr{B})$, then S, T have the $(\mathscr{S}')$-convolution. Indeed, let $A(\dot{x}, \dot{y}) = S(\dot{y})T(\dot{x}-\dot{y})$. Then $\varphi \cdot A = S(\check{T} * \varphi) \in (\mathscr{D}'_{L^1})$ for any $\varphi \in (\mathscr{D})$. For any $\alpha \in (\mathscr{D}'_{L^1})'_c = (\mathscr{B}_c)$, $\langle \varphi \cdot A, \alpha \rangle = \int \alpha(x)S(x)(\check{T} * \varphi)(x)\,dx = \iint \alpha(x)S(x)T(y-x)\varphi(y)dx\,dy = \int (\alpha S * T)(y)\varphi(y)dy = \langle \varphi, \alpha S * T \rangle$, so that $A \cdot \alpha = \alpha S * T \in (\mathscr{S}')$. The application $\alpha \in (\mathscr{B}_c) \rightarrow A \cdot \alpha = \alpha S * T \in (\mathscr{S}')$ is continuous, because $(\mathscr{S}')$ possesses the property (ε) (L. Schwartz [14], p. 59). Therefore, the transposed application $\varphi \in (\mathscr{D}) \rightarrow S(\check{T} * \varphi) \in (\mathscr{D}'_{L^1})$ is continuous with respect to the topology induced by that of $(\mathscr{S})$. Hence, for any $\psi \in (\mathscr{S})$, $S(\check{T} * \psi) \in (\mathscr{D}'_{L^1})$, since $(\mathscr{D})$ is dense in $(\mathscr{S})$ and $(\mathscr{D}'_{L^1})$ is complete. This means that S, T have the $(\mathscr{S}')$-convolution.

If the convolution of two temperate distributions is temperate whenever its convolution is defined, then the ordinary convolution in $(\mathscr{S}')$ implies the $(\mathscr{S}')$-convolution. For $S \in (\mathscr{S}')$, αS belongs to $(\mathscr{S}')$ for each $\alpha \in (\mathscr{B})$ and if $S * T$ is defined for $S, T \in (\mathscr{S}')$, so is $\alpha S * T$. Then by the above remark 2, S, T have the $(\mathscr{S}')$-convolution.

4. In this section we shall be concerned with a simultaneous convolution of three or more distributions. We begin with the following lemma:

LEMMA 1. *Suppose that three distributions S, T, U satisfy the condition:*

$$(\ast)_3 \qquad (S * \varphi)(\dot{x})(T * \psi)(\dot{y})(\check{U} * \chi)(\dot{x}+\dot{y}) \in (L^1)_{x,y} \quad \textit{for any} \quad \varphi, \psi, \chi \in (\mathscr{D}).$$

If any one of the convolutions S, T, U is not zero, then the other two distributions have the convolutions.

PROOF. It follows by a suitable change of variables that the condition $(\ast)_3$ is symmetric for three distributions S, T, U. Suppose, therefore, $S \neq 0$.

The same reasoning as done in the remark 1 after Theorem 3 yields that

(1) $\alpha(x)(T*\psi)(\hat{y})(\check{U}*\chi)(\hat{x}+\hat{y})\in(L^1)_{x,y}$ for any $\alpha,\psi,\chi\in(\mathscr{D})$.

Integrating (1) with respect to x, we obtain

$$(T*\psi)(\check{U}*\check{\alpha}*\chi)\in L^1 \text{ for any } \alpha,\psi,\chi\in(\mathscr{D}),$$

which implies that $T*U$ is defined. The proof is completed.

Now, suppose $S, T, U\neq 0$, and assume that $(\bar{*})_3$ holds. Integrating the expression of $(\bar{*})_3$ with respect to y, we have

(2) $(S*\varphi)((\check{T}*\check{U})*\check{\psi}*\chi)\in L^1,$

which shows us that $S*(T*U)$ is defined. Further, integrating the expression of (2) we have

(3) $\langle S*(T*U),\ \check{\varphi}*\check{\psi}*\chi\rangle.$

In this process let the order of integrations convert. We then see that $T*(S*U)$ is defined and the repeated integrations give

(4) $\langle T*(S*U),\ \check{\varphi}*\check{\psi}*\chi\rangle.$

It follows by Fubini's theorem of the classical calculus that (3) and (4) are equals, so that $T*(S*U)=S*(T*U)$. By a suitable change of the variables we have

$$S*(T*U)=T*(S*U)=U*(S*T).$$

THEOREM 4. *For the three distributions S, T, U, the following conditions are equivalent*:

$(\bar{*})_3$ $(S*\varphi)(\hat{x})(T*\psi)(\hat{y})(\check{U}*\chi)(\hat{x}+\hat{y})\in(L^1)_{x,y}$ *for any* $\varphi,\psi,\chi\in(\mathscr{D})$;

$(*)_3$ $(S_x\otimes T_y\otimes U_z)\varphi(\hat{x}+\hat{y}+\hat{z})\in(\mathscr{D}'_{L^1})_{x,y,z}$ *for any* $\varphi\in(\mathscr{D})$.

PROOF. ad $(\bar{*})_3\rightarrow(*)_3$. By a similar reasoning as in the implication $(\bar{*})\rightarrow(*)$ in the proof of Theorem 2, we can infer that

(5) $(S*\varphi)(\hat{x})(T*\psi)(\hat{y})(U*\chi)(\hat{z})\alpha(\hat{x}+\hat{y}+\hat{z})\in(L^1)_{x,y,z}$
for any $\varphi,\psi,\chi,\alpha\in(\mathscr{D})$.

Then by a familiar reasoning using a parametrix, we see that $(\bar{*})_3$ implies $(*)_3$.

ad $(*)_3\rightarrow(\bar{*})_3$. The proof is carried out by making use of the vector-valued integration as in the proof of $(\bar{*})'\rightarrow(\bar{*})$ and a parametrix in a familiar way. The proof is omitted.

From the proof of this theorem it is easy to show

COROLLARY. *The following two conditions are equivalent for distributions* $T_1, T_2,\cdots, T_m$:

$(\bar{*})_m$ $(T_1*\varphi_1)(\hat{z}_1)(T_2*\varphi_2)(\hat{z}_2)\cdots\cdots\cdots$
$(T_{m-1}*\varphi_{m-1})(\hat{z}_{m-1})(\check{T}_m*\varphi_m)(\hat{z}_1+\hat{z}_2+\cdots+\hat{z}_{m-1})\in(L^1)_{z_1,z_2,\cdots,z_{m-1}}$

for any $\varphi_1, \varphi_2, \cdots, \varphi_m \in (\mathscr{D})$;

$(*)_m \qquad \{T_1(\hat{z}_1)\otimes T_2(\hat{z}_2)\otimes\cdots\otimes T_m(\hat{z}_m)\}\varphi(\hat{z}_1+\hat{z}_2+\cdots+\hat{z}_m) \in (\mathscr{D}'_{L^1})_{z_1,z_2,\cdots,z_m}$

for any $\varphi \in (\mathscr{D})$.

Now, we shall define the simultaneous convolution for a finite set of distributions $\{T_j\}_{j=1,2,\cdots,m}$. We say that the simultaneous convolution $T_1 * T_2 * \cdots * T_m$ or briefly $\prod_{j=1}^{m} * T_j$ is defined provided the remaining set obtained from $\{T_j\}$ after excluding the zero distributions satisfies any one of $(\bar{*})_m$ and $(*)_m$. $T_1 * T_2 * \cdots * T_m$ is given by

$$(6) \qquad <T_1 * T_2 * \cdots * T_m, \varphi> = \int\cdots\int T_1(z_1)T_2(z_1)\cdots T_m(z_m)\varphi(z_1+\cdots+z_m)dz_1\cdots dz_m.$$

The right-hand side is a continuous linear form of $\varphi \in (\mathscr{D})$, so that (6) has a meaning.

From this definition we can conclude that if the union of two sets $\{S_i\}_{i=1,2,\cdots,l}$ and $\{T_j\}_{j=1,2,\cdots,m}$ of distributions has the simultaneous convolution, then each of $\{S_i\}$ and $\{T_j\}$ has also the simultaneous convolution, and

$$S_1 * S_2 * \cdots * S_l * T_1 * T_2 * \cdots * T_m = (S_1 * S_2 * \cdots * S_l) * (T_1 * T_2 * \cdots * T_m).$$

For example, suppose that S and T have the $(\mathscr{S}')$-convolution, then for any $U \in (\mathscr{O}'_C)$ the three distributions S, T, and U have the simultaneous convolution $S * T * U$, which is equal to any one of the following

$$S * (T * U), \quad T * (S * U), \quad U * (S * U).$$

This is because $U \in (\mathscr{O}'_C)$ is characterized by the property

$$U * \varphi \in (\mathscr{S}) \text{ for any } \varphi \in (\mathscr{D}).$$

The notion of $(\mathscr{S}')$-simultaneous convolution is defined in a natural way. In this case we note that the convolution must be an element of $(\mathscr{S}')$.

5. This section is devoted to simple consequences of the preceding discussions. Let E be a space of distributions on R^n, that is, $E \subset (\mathscr{D}')$ and the injection $E \to (\mathscr{D}')$ is continuous. We write $T \in E^*$ if the convolution of T and any distribution of E is defined. E^* consists of such distributions T and is called a c-dual of E (K. Yoshinaga and H. Ogata [15]). $T \in E^*$ is equivalent to the condition:

$$(1) \qquad S(\check{T} * \varphi) \in (\mathscr{D}'_{L^1}) \quad \text{for any } S \in E \text{ and } \varphi \in (\mathscr{D}).$$

Under certain conditions the application $S \in E \to S(\check{T} * \varphi) \in (\mathscr{D}'_{L^1})$ will be continuous. For example, if E is a barrelled admissible space, then the application is continuous because $(\mathscr{D}'_{L^1})$ has the property (C), and the bilinear form $B(S, \varphi) = \int_{R^n} S(x)(\check{T} * \varphi)(x)\,dx = <S * T, \varphi>$ is hypo-continuous. This is because any separately continuous bilinear application on the product space of two barrelled spaces is hypo-continuous. This shows that if we let E be any admissible barrelled space and $T \in E^*$, then $\check{T} * \varphi \in E'$ for any $\varphi \in (\mathscr{D})$ and the application $S \in E \rightarrow S * T \in (\mathscr{D}')$ is continuous (K. Yoshinaga and H. Ogata [15], Theorem 3).

As a second example, let $E = \mathbf{H}'^m$, where $\mathbf{H}'^m$ is the dual of a space of type $\mathbf{H}^m$ (Schwartz [13], p. 118). An $f \in (\mathscr{E}^m)$ belongs to $\mathbf{H}^m$ if and only if $Sf \in (\mathscr{D}'_{L^1})$ for every $S \in \mathbf{H}'^m$. The "only if" part is shown in Schwartz ([13], p. 121). Suppose $Sf \in (\mathscr{D}'_{L^1})$ for every $S \in \mathbf{H}'^m$. Let $\{\alpha_j\}$ be a sequence of multiplicators. Then $\alpha_j f \in (\mathscr{D}^m)$ and $<S, \alpha_j f> = \int \alpha_j(x) S(x) f(x) dx \rightarrow \int S(x) f(x) dx$ as $j \rightarrow \infty$, so that $\{\alpha_j f\}$ is weakly bounded, and therefore bounded in $\mathbf{H}^m$. By definition ([13], p. 98), on any bounded subset of $\mathbf{H}^m$, the induced topology by $\mathbf{H}^m$ coincides with that induced by $(\mathscr{E}^m)$. Then it follows since $\mathbf{H}^m$ is quasi-complete that $f \in \mathbf{H}^m$ and $\alpha_j f \rightarrow f$ in $\mathbf{H}^m$, as desired. Now suppose $T \in (\mathbf{H}'^m)^*$. Then $S(\check{T} * \varphi) \in (\mathscr{D}'_{L^1})$ for any $S \in \mathbf{H}'^m$ and any $\varphi \in (\mathscr{D})$. It follows that $\check{T} * \varphi \in \mathbf{H}^m$ for any $\varphi \in (\mathscr{D})$. The application $S \in \mathbf{H}'^m \rightarrow S(\check{T} * \varphi) \in (\mathscr{D}'_{L^1})$ is continuous. This is because it is the transposed application of $\mathscr{L} : \alpha \in (\mathscr{B}_c) \rightarrow \alpha(\check{T} * \varphi) \in \mathbf{H}^m$ which is continuous. In fact, the continuity of the latter is evident, and for any $S \in \mathbf{H}'^m$, $<S, \mathscr{L}(\alpha)> = \int S\alpha(\check{T} * \varphi) dx$ and therefore ${}^t\mathscr{L}(S) = S(\check{T} * \varphi)$. Furthermore, if φ runs through a bounded subset B of $(\mathscr{D})$, then the image $\{T * \varphi;\ \varphi \in B\}$ is relaively compact in $\mathbf{H}^m$ since the application $\varphi \in (\mathscr{D}) \rightarrow T * \varphi \in \mathbf{H}^m$ is continuous and B is relatively compact in $(\mathscr{D})$. On the other hand, the set of polars of compact subsete C of $(\mathscr{B}_c)$ forms a fundamental system of neighbourhoods of zero in $(\mathscr{D}'_{L^1})$. C is bounded in $(\mathscr{B})$ and is compact in the topology induced by $(\mathscr{E})$. Now the set $\{S;\ |<S(\check{T} * \varphi), \alpha>| = |<S, (\check{T} * \varphi)\alpha>| \leq 1$ for any $\varphi \in B$ and any $\alpha \in C\}$ will be a neighbourhood of zero in $\mathbf{H}'^m_c$ where $\mathbf{H}'^m_c$ is the dual of $\mathbf{H}^m$ which is equipped with the topology of uniform convergence on every compact subset of $\mathbf{H}^m$. This is because the set $\{(T * \varphi)\alpha;\ \varphi \in B$ and $\alpha \in C\}$ is relatively compact in $\mathbf{H}^m$. Thus we have proved

LEMMA 2. (1) *A distribution T has the convolution with any element of $\mathbf{H}'^m_c$ if and only if $\check{T} * \varphi \in \mathbf{H}^m$ for any $\varphi \in (\mathscr{D})$. In this case,*

(2) *the application $S \in \mathbf{H}'^m_c \rightarrow S(\check{T} * \varphi) \in (\mathscr{D}'_{L^1})$ is equi-continuous if φ runs through any bounded subset of $(\mathscr{D})$. Therefore the application $S \in \mathbf{H}'^m \rightarrow S * T \in (\mathscr{D}')$ is continuous.*

Let E be an admissible space and F a space of distributions. A continuous linear application $\mathscr{L}$ of E into F is called an *operator of composition* of E into F, if its restriction on $(\mathscr{D})$ is of the form $\mathscr{L}(\varphi) = T * \varphi$ for any $\varphi \in (\mathscr{D})$ and a fixed T. Now let E be a space $\mathbf{H}'^m_c$. Under somewhst different definition of operator of composition Schwartz ([13], p. 132) shows that $\mathscr{L}$ is an operator of composition of $\mathbf{H}'^m_c$ into F if and only if $\vec{A}(y) = \tau(y)T$ belongs to $\mathbf{H}^m(F)$. This result is also true for the above definition of operator of composition if F is quasi-complate. We omit the proof since this follows by slight modifications of his proof.

THEOREM 5. *Let $\mathscr{L}$ be an operator of composition of $\mathbf{H}_c'^m$ into a space of distributions F, then there exists a unique distribution $T \in (\mathbf{H}_c'^m)^*$ such that $\mathscr{L}(S) = S * T$ for any $S \in \mathbf{H}_c'^m$.*

Conversely, if T is any distribution which belongs to $(\mathbf{H}_c'^m)^*$ such that $S * T \in F$ for any $S \in \mathbf{H}_c'^m$, then $S \in \mathbf{H}_c'^m \rightarrow S * T \in F$ is continuous provided F is an admissible space with the property $(\mathcal{E})$.

PROOF. We first show ${}^t\mathscr{L}(\varphi) = \check{T} * \varphi \in \mathbf{H}^m$ for any $\varphi \in (\mathscr{D})$. φ is an element of F' so that for any $\alpha \in (\mathscr{D})$, $<{}^t\mathscr{L}(\varphi), \alpha> = <\mathscr{L}(\alpha), \varphi> = <T * \alpha, \varphi> = <\alpha, \check{T} * \varphi>$. This implies that ${}^t\mathscr{L}(\varphi) = \check{T} * \varphi \in \mathbf{H}^m$. Hence $T \in (\mathbf{H}_c'^m)^*$. By Lemma 2, $S \in \mathbf{H}_c'^m \rightarrow S * T \in (\mathscr{D}')$ is continuous. On the other hand, let j be the injection $F \rightarrow (\mathscr{D}')$. Then $j \circ \mathscr{L}$ is an operator of composition of $\mathbf{H}_c'^m$ into $(\mathscr{D}')$. These two applications coincide on $(\mathscr{D})$ by our definition, so that $j \circ \mathscr{L}(S) = S * T$ for every $S \in \mathbf{H}_c'^m$, hence $\mathscr{L}(S) = S * T$.

Now let $T \in (\mathbf{H}_c'^m)^*$ and let F be an admissible space with the property $(\mathcal{E})$. Let $A(\hat{x}, \hat{y})$ be the kernel distribution defined by $\tau(y)T$. $\varphi \cdot A = \check{T} * \varphi \in \mathbf{H}^m$, because $T \in (\mathbf{H}_c')^*$. For any $S \in \mathbf{H}_c'^m$, $<S, \varphi \cdot A> = \int S(\check{T} * \varphi) dx = <S * T, \varphi>$. This implies that the transposed of the application $\varphi \in (\mathscr{D}) \rightarrow \varphi \cdot A \in \mathbf{H}^m$ is the application $S \in \mathbf{H}'^m \rightarrow S * T \in F$. It follows since F has the poperty $(\mathcal{E})$ that this application is continuous, completing the proof.

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