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A Note on a Theorem of N. Jacobson

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In [1] N. Jacobson proved that a Lie triple system T can be imbedded in a Lie algebra L in such a way that T becomes a subspace of L and that $[abc]=[[ab]c]$. The purpose of this note is to show that this theorem is still valid for the homogeneous systems (h. systems) defined below.

From this fact we have as a conclusion the following relationship between the h. system, general L.t.s., L.t.s. and the Lie algebra.

(i) H. system characterizes the structure of a subspace A of a Lie algebra L such that L is the direct sum of two subspaces A and B, B being a subalgebra of L (see Remark 2).

(ii) General L.t.s. characterizes the structure of the subspace A of a Lie algebra L such that L is the direct sum of two subspaces A and B, B being a subalgebra of L and $[A, B]\subseteqq A$.[1]

(iii) L.t.s. characterizes the structure of the subspace A of a Lie algebra L such that L is the direct sum of two subspaces A and B, B being a subalgebra of L and $[A, B]\subseteqq A$, $[A, A]\subseteqq B$.

1. Let V be a vector space over a field Φ.[2] Suppose that there exist the multilinear compositions $a\circ b$, $[a_1\cdots a_k]$ $(k=3, 4, \cdots)$ in V and they satisfy the following axioms:

(I) $$a\circ a=0,$$

(II) $$[a a a_1\cdots a_k]=0,$$

(III) $$(a\circ b)\circ c+(b\circ c)\circ a+(c\circ a)\circ b+[abc]+[bca]+[cab]=0,$$

(IV) $$[a\circ bcd]+[b\circ cad]+[c\circ abd]+[abcd]+[bcad]+[cabd]=0,$$

(V) $$[a_1\cdots a_k b\circ c]-[a_1\cdots a_k b]\circ c+[a_1\cdots a_k c]\circ b -[a_1\cdots a_k bc]+[a_1\cdots a_k cb]=0,$$

(VI) $$[D_{(a_1\cdots a_k)}, D_{(bc)}]=D_{([a_1\cdots a_k b]c)}-D_{([a_1\cdots a_k c]b)} +D_{(a_1\cdots a_k bc)}-D_{(a_1\cdots a_k cb)}-D_{(a_1\cdots a_k b\circ c)},$$

(VII) $$[D_{(a_1\cdots a_k)}, D_{(b_1\cdots b_l c)}]=\mathfrak{D}_{([D_{(a_1\cdots a_k)}, D_{(b_1\cdots b_l)}], c)}-[D_{(a_1\cdots a_k c)}, D_{(b_1\cdots b_l)}] -D_{(a_1\cdots a_k[b_1\cdots b_l c])}+D_{(b_1\cdots b_l[a_1\cdots a_k c])},$$

where $D_{(a_1\cdots a_k)}$ $(k=2, 3, \cdots)$ is a linear transformation: $x\rightarrow[a_1\cdots a_k x]$ of V into itself, $[D_{(a_1\cdots a_k)}, D_{(b_1\cdots b_l)}]$ means $D_{(a_1\cdots a_k)}D_{(b_1\cdots b_l)}-D_{(b_1\cdots b_l)}D_{(a_1\cdots a_k)}$ and

1) P. K. Rachevsky obtained the algebraic relations between the structure constants in such a Lie algebra L ([2], § 5, (28), $\cdots$, (33)).

2) Throughout this note we shall assume that the characteristic of Φ is 0.

$\mathfrak{D}$ is defined by $\mathfrak{D}_{(D_{(a_1\cdots a_k)},c)}=D_{(a_1\cdots a_k c)}$ with $\mathfrak{D}_{(\alpha+\beta,c)}=\mathfrak{D}_{(\alpha,c)}+\mathfrak{D}_{(\beta,c)}$ for $\alpha, \beta \in D$, D being the vector space spanned by the linear transformations $D_{(a_1\cdots a_k)}$. Then we call the system V, equipped with these laws of compositions, a *homogeneous system* (*h. system*) over Φ.

Remark 1. It follows from (VI) and (VII) that $\mathfrak{D}_{([D_{(a_1\cdots a_k)}, D_{(b_1\cdots b_l)}], c)}$ is well defined and any commutator product $[D_{(a_1\cdots a_k)}, D_{(b_1\cdots b_l)}]$ belongs to D. Hence D is a Lie algebra.

The h. system, in which $[a_1\cdots a_k]=0$ $(k=3, 4, \cdots)$, is a Lie algebra with respect to the composition $a\circ b$. The h. system with $a\circ b=0$ and $[a_1\cdots a_k]=0$ $(k=4, 5, \cdots)$ is a L.t.s. with respect to the ternary composition $[abc]$ [1, 3], and if $[a_1\cdots a_k]=0$ $(k=4, 5, \cdots)$, then the axioms stated above reduce to the axioms of general L.t.s. [4][3]. In this sense, the h. system is a more general concept than those of the Lie algebra, L.t.s. and general L.t.s..

2. We shall extend the theorem of N. Jacobson by the following:

THEOREM 1. *Any h. system V over a field Φ can be imbedded in a Lie algebra L in such a way that L is the direct sum of V and the Lie algebra of some linear transformations acting on V.*

PROOF. We use the same notations as in Section 1. Let L be the direct sum of V and Lie algebra D. To show that L becomes a Lie algebra, we define the product for the elements of L as follows:

$$[a_1, a_2]=a_1\circ a_2+D_{(a_1 a_2)},$$

$$[D_{(a_1\cdots a_k)}, a_{k+1}]=-[a_{k+1}, D_{(a_1\cdots a_k)}]=[a_1\cdots a_k a_{k+1}]+D_{(a_1\cdots a_k a_{k+1})},$$

$$[D_{(a_1\cdots a_k)}, D_{(b_1\cdots b_l)}] \text{ as above, } a_i, b_j \in V,$$

and in general for $x=a+\Sigma D_{(a_1\cdots a_k)}$, $y=b+\Sigma D_{(b_1\cdots b_l)}$

$$[x, y]=[a, b]+\Sigma[a, D_{(b_1\cdots b_l)}]+\Sigma[D_{(a_1\cdots a_k)}, b]+\Sigma[D_{(a_1\cdots a_k)}, D_{(b_1\cdots b_l)}].$$

We shall prove that these skew-symmetric bilinear products satisfy the Jacobi identity. To do this it suffices to prove the following relations:

$$[[ab]c]+[[bc]a]+[[ca]b]=0,$$

$$[[D_{(a_1\cdots a_k)}, b]c]+[[bc], D_{(a_1\cdots a_k)}]+[[c, D_{(a_1\cdots a_k)}]b]=0,$$

$$[[D_{(a_1\cdots a_k)}, D_{(b_1\cdots b_l)}]c]+[[D_{(b_1\cdots b_l)}, c]D_{(a_1\cdots a_k)}]+[[c, D_{(a_1\cdots a_k)}]D_{(b_1\cdots b_l)}]=0.$$

In fact, the first of these follows from (III) and (IV), the second follows from (V) and (VI), and the last may be obtained by using (VII).

3. PROPOSITION 1. *Let L be a Lie algebra and L have a direct sum decomposition $L=A\oplus B$ into two subspaces A and B. Assume that B is a subalgebra of L. Then we can define the multilinear compositions $a\circ b$, $[a_1\cdots a_k]$ $(k=3, 4, \cdots)$ in A so that A can be made into an h. system with respect to these products.*

PROOF. For an element x in L, x_A (resp. x_B) denotes the A-component (resp. the B-component) of x with respect to the decomposition $L=A\oplus B$.

3) We have denoted in [4] $a\circ b$ by $-a\circ b$.

$[x, y]$ means the Lie product of the elements x and y in L. For the elements $a_1, \cdots, a_k$ in A, we define the multilinear compositions in A as follows:

$$a_1 \circ a_2 = [a_1, a_2]_A,$$
$$[a_1 a_2 \cdots a_k] = \langle\langle \cdots \langle\langle a_1, a_2\rangle a_3\rangle \cdots \rangle a_{k-1}\rangle \circ a_k \quad (k=3, 4, \cdots),$$

where $\langle a_i, a_j\rangle$ denotes $[a_i, a_j]_B$. Hereafter $\langle a_1 a_2 \cdots a_k\rangle$ means $\langle\langle \cdots \langle a_1, a_2\rangle \cdots \rangle a_k\rangle$ $(k=2, 3, \cdots)$ for the sake of simplicity. We show that these products satisfy the conditions (III), $\cdots$, (VII). Making use of the Jacobi identity for the elements a, b, c in A and of the fact that L is a direct sum of A and B, we obtain (III) and

$$\langle a \circ b, c\rangle + \langle b \circ c, a\rangle + \langle c \circ a, b\rangle + \langle abc\rangle + \langle bca\rangle + \langle cab\rangle = 0,$$

which implies immediately (IV). (V) follows by taking the A-component of both sides of the identity:

$$[\langle a_1 \cdots a_k\rangle, [bc]] + [[\langle a_1 \cdots a_k\rangle, c]b] - [[\langle a_1 \cdots a_k\rangle, b]c] = 0$$

and by taking B-component we have

$$(1) \qquad [\langle a_1 \cdots a_k\rangle, \langle bc\rangle] - \langle [a_1 \cdots a_k b]c\rangle + \langle [a_1 \cdots a_k c]b\rangle - \langle a_1 \cdots a_k bc\rangle + \langle a_1 \cdots a_k cb\rangle + \langle a_1 \cdots a_k b \circ c\rangle = 0.$$

Using the identity

$$[\langle a_1 \cdots a_k\rangle [\langle b_1 \cdots b_l\rangle, c]] - [\langle b_1 \cdots b_l\rangle [\langle a_1 \cdots a_k\rangle, c]] - [[\langle a_1 \cdots a_k\rangle, \langle b_1 \cdots b_l\rangle]c] = 0,$$

we obtain the following two relations:

$$(2) \quad [\langle a_1 \cdots a_k\rangle, \langle b_1 \cdots b_l c\rangle] - \langle [\langle a_1 \cdots a_k\rangle, \langle b_1 \cdots b_l\rangle]c\rangle + [\langle a_1 \cdots a_k c\rangle, \langle b_1 \cdots b_l\rangle] + \langle a_1 \cdots a_k [b_1 \cdots b_l]c\rangle - \langle b_1 \cdots b_l [a_1 \cdots a_k c]\rangle = 0,$$

$$(3) \quad [a_1 \cdots a_k [b_1 \cdots b_l c]] - [b_1 \cdots b_l [a_1 \cdots a_k c]] - [\langle a_1 \cdots a_k\rangle, \langle b_1 \cdots b_l\rangle] \circ c = 0.$$

From (1) and (2) we see that any $[\langle a_1 \cdots a_k\rangle, \langle b_1 \cdots b_l\rangle]$ is a linear combination of elements of the form $\langle d_1 \cdots d_m\rangle$. If we denote by $D_{\langle a_1 \cdots a_k\rangle}$ the linear mapping $x \to [a_1 \cdots a_k x]$ in V, then from (3), we have $[\langle a_1 \cdots a_k\rangle, \langle b_1 \cdots b_l\rangle] \circ x = [D_{\langle a_1 \cdots a_k\rangle}, D_{\langle b_1 \cdots b_l\rangle}](x)$. And by using (1) it follows (VI). If we put $\mathfrak{D}_{\langle D_{\langle a_1 \cdots a_k\rangle} + D_{\langle b_1 \cdots b_l\rangle}, c\rangle} = D_{\langle a_1 \cdots a_k c\rangle} + D_{\langle b_1 \cdots b_l c\rangle}$, the identity (2) implies (VII). Hence the proposition is proved.

Remark 2. Let G/H be a homogeneous space of a Lie group G by a closed subgroup H of G. Let $\mathfrak{G}$ be its Lie algebra, which may be identified with the tangent space at the identity of G. Then there exists a vector subspace V of $\mathfrak{G}$ such that $\mathfrak{G} = V \oplus \mathfrak{H}$, where $\mathfrak{H}$ is the subalgebra of $\mathfrak{G}$ corresponding to H. By virtue of the results in 2 and 3 it follows that h. system completely characterizes the structure of V.

4. PROPOSITION 2. *Let a Lie algebra L have a direct sum decomposition $L = A \oplus B$ into two subspaces A and B. Suppose that B is a subalgebra of L and have a finite dimension. Then B cannot contain a non-zero ideal of L if and only if* $b \circ A = (0)$, $\langle b, \underbrace{A \cdots A}_{k\ times}\rangle \circ A = (0)$ $(k=1, 2, \cdots)$ *for*

an element b of B implies b=0.

PROOF. Let us assume that we have a non-zero element b of B such that $b\circ A=(0)$, $\langle bA\cdots A\rangle\circ A=(0)$. Denote by $B_{0,k}$ the subspace: $\Phi b+[b, B]+\cdots+[b, B^k]$, where Φb is the vector space spanned by b and $[b, B^k]=[\cdots[b, \underbrace{B], \cdots, B}_{k\ times}]$, then $[b, B^{n_0+1}]$ is contained in B_{0,n_0} for some integer n_0, since B has a finite dimension. Next, there is an integer n_1 ($\geqq n_0$) such that $[[b, A]B^{n_1+1}]$ is contained in the subspace $B_{1,n_1}=[b, A]+[[b, A]B]+\cdots+[[b, A]B^{n_1}]$. Thus we obtain the series of subspaces $B_{0,n_0}, B_{1,n_1}, \cdots, B_{i,n_i}, \cdots$ ($n_0\leqq n_1\leqq\cdots\leqq n_i\leqq\cdots$), such that $[[b, A^i]B^{n_i+1}]$ is contained in B_{i,n_i}. It follows that $[[[b, A^i]B^k]A]\subseteqq[[b, A^i]B]+\cdots+[[b, A^i]B^k]+[b, A^{i+1}]+[[b, A^{i+1}]B]+\cdots+[[b, A^{i+1}]B^k]$ by induction, hence it holds $[B_{i,n_i}, A]\subseteqq B_{i,n_i}+B_{i+1,n_{i+1}}$. Let $\mathfrak{B}_i$ be the subspace $B_{0,n_0}+B_{1,n_1}+\cdots+B_{i,n_i}$ ($i=0, 1, 2, \cdots$). If $[B_{i,n_i}, A]$ is not contained in $\mathfrak{B}_i$, then we construct $\mathfrak{B}_{i+1}$ from $\mathfrak{B}_i$. Therefore we have the relation $[\mathfrak{B}_m, A]\subseteqq\mathfrak{B}_m$ for some integer m, because of the finite dimensionality of B. Hence we have a non-zero ideal $\mathfrak{B}_m$ of L included in B. Conversely, it is clear that if B contains an ideal C ($\neq 0$) of L, then there exists an element b ($\neq 0$) of C such that $b\circ A=(0)$, $\langle bA\cdots A\rangle\circ A=0$.

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