JOURNAL OF SCIENCE OF THE HIROSHIMA UNIVERSITY, SER. A, VOL. 22, No. 3, DECEMBER 1958

On the Nonlinear Autonomous System Admitting of a Family of Periodic Solutions near its Certain Periodic Solution

By

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(Received September, 20, 1958)

1. Introduction

Let

$$\frac{dy}{dt}=Y(y) \tag{1.1}$$

be a given nonlinear autonomous system, where $Y(y)\in C_y^2$ in a domain G of the phase $(n+1)$-space R^{n+1}. Here, of course, y and Y are the $(n+1)$-dimensional vectors. In the sequel, let us call the independent variable t the *time*.

In this paper, we consider the case where, in a domain G, the system (1.1) admits of a family $\mathfrak{F}$ of closed paths in the neighborhood of its certain closed path

$$C_0: y=\phi(t).$$

According to the nomenclature of the preceding paper [5][1], the system (1.1) is called respectively the *fully* or the *partially oscillatory* system according as the family $\mathfrak{F}$ consists of whole paths or of a portion of them lying near C_0. In either case, according to [5], we assume that, when $n\geqq 2$, the periods of the closed paths belonging to $\mathfrak{F}$ are bounded. Then, by **4.1** of [5], there exist the *universal periods* for paths belonging to $\mathfrak{F}$ such that they are continuous at C_0. Let us denote such a universal period of C_0 by ω_0.

According to [5], we make use of a moving orthonormal system along C_0. Let ξ_i's $(i=1, 2, \cdots, n)$ be the normal unit vectors of a moving orthonormal system along C_0 such that $\xi_i(t)\in C_t^2$. Then, with respect to this moving orthonormal system, any path C lying near C_0 is represented as

$$y=\phi(t)+\sum_{i=1}^{n} x^i\xi_i \tag{1.2}$$

and the time variable τ along C and the n-dimensional vector $x=(x^i)$ are determined respectively by the differential equations

$$\frac{d\tau}{dt}=\frac{\|Y\|^2+\sum_{i=1}^{n} x^i Y^*\dot{\xi}_i}{Y^*Y'}\ \text{[2]}, \tag{1.3}$$

1) Numbers in brackets refer to the references listed at the end of the paper.

2) $\|Y\|$ denotes the Euclidean norm of Y. The dots above the letters denote differentiation with respect to t. The symbol * denotes the transposed.

$$\text{(1.4)} \qquad \frac{dx^i}{dt}=X^i(x,t)=\frac{\|Y\|^2+\sum_{j=1}^{n}x^jY^*\dot{\xi}_j}{Y^*Y'}\cdot\xi_i^*Y'-\sum_{j=1}^{n}x^j\xi_i^*\dot{\xi}_j \qquad (i=1,2,\cdots,n),$$

where

$$Y'=Y(\psi+\sum_{i=1}^{n}x^i\xi_i).$$

Since Y' is written as

$$Y'=Y+\sum_{i=1}^{n}x^iV(t)\xi_i+o(\|x\|)$$

where $V(t)$ is a matrix whose elements are

$$\left.\frac{\partial Y^k}{\partial y^l}\right|_{y=\phi(t)} \qquad (k,l=1,2,\cdots,n+1),$$

the equations (1.4) can be written in the vector form as follows:

$$\text{(1.5)} \qquad \frac{dx}{dt}=X(x,t)=A(t)x+o(\|x\|);$$

where $A(t)$ is a periodic matrix whose elements are

$$\text{(1.6)} \qquad A_j^i(t)=\xi_i^*V(t)\xi_j-\xi_i^*\dot{\xi}_j \qquad (i,j=1,2,\cdots,n).$$

From (1.4) and (1.6), it is evident that $X(x,t)$ and $A(t)$ are continuous and periodic in t with period ω_0, and $X(x,t)\in C_x^2$ for sufficiently small $\|x\|$. The equation (1.5) is the equation for the normal deviation of C from C_0, or, in other words, the relative equation of C in the normal hyperplane of C_0.

The equation (1.3) enables us to calculate the universal period ω of C belonging to $\mathfrak{F}$ as follows:

$$\omega=\int_0^{\omega_0}\frac{\|Y\|^2+\sum_{i=1}^{n}x^iY^*\dot{\xi}_i}{Y^*Y'}dt,$$

because the universal periods are continuous at C_0.

In this paper, first, the equation (1.5) is reduced to that of the form as simple as possible and thereby the behavior of paths of the initial system (1.1) lying near C_0 is made clear. Next, making use of the reduced form obtained there, the perturbation problems are discussed and the criteria of stability of the perturbed oscillation are found for both fully and partially oscillatory systems.

As is proved in their paper [1], the system considered by Diliberto and Hufford—the system which admits of regular integrals through C_0—is a special one of the partially oscillatory systems. Our results show that their reduced form can be reduced further and that they are of the special form of ours.

The perturbation problems were discussed already for both fully and partially oscillatory systems in the preceeding paper [5], but there the criteria of stability have not yet been found for the partially oscillatory system. In this paper, making use of the reduced form of (1.5), the criteria of stability have been found also for the partially oscillatory system.

2. Fully oscillatory system

Let

(2.1) $$x=\varphi(t, c)$$

be the solution of (1.5) such that

$$\varphi(0, c)=c.$$

Then evidently $\varphi(t, c)\in C_c^2$ for any finite t and sufficiently small $\|c\|$. Consequently $\varphi(t, c)$ is written in the form as follows:

(2.2) $$\varphi(t, c)=G(t)c+o(\|c\|)$$

and $G(t)$ becomes a fundamental matrix of the linear homogeneous differential equation

$$\frac{dx}{dt}=A(t)x$$

such that

$$G(0)=E,$$

where E is a unit matrix.

In the present case, from the continuity of universal periods, it holds that

(2.3) $$\varphi(\omega_0, c)=c$$

for any c such that $\|c\|$ is sufficiently small. Consequently, if, according to the previous paper [3], we consider the transformation

(2.4) $$x'=\varphi(t, x)$$

depending on a parameter t[1)], this transformation becomes an identical transformation for $t=\omega_0$. Since this identical transformation is of course imbedded in a one-parameter group of transformations

$$x'=e^{tB}x,$$

where B is any matrix such that

(2.5) $$\exp(\omega_0 B)=E,$$

by the results of the paper [3], the equation (1.5) is transformed to the equation

(2.6) $$\frac{dx}{dt}=Bx$$

by the periodic transformation

1) Since, from (2.2),

$$\left.\frac{\partial(\varphi^1, \varphi^2, \cdots, \varphi^n)}{\partial(x^1, x^2, \cdots, x^n)}\right|_{x=0}\neq 0$$

for any t, (2.4) expresses surely a transformation depending on a parameter t for sufficiently small $\|x\|$.

$$x=\varphi(t, e^{-tB}x'). \tag{2.7}$$

But the above discussion evidently prevails whenever $Y(y)$ of the right-hand side of (1.1) is once continuously differentiable. Thus we have

Theorem 1. *For the once continuously differentiable fully oscillatory system* (1.1), *the relative equation of any path C lying near C_0 in the normal hyperplane of C_0 is transformed to the equation* (2.6) *by the periodic transformation* (2.7). *Here B is any matrix such that* (2.5) *holds.*

When B is chosen so that $B=0$, the equation (2.6) *becomes*

$$\frac{dx}{dt}=0,$$

consequently all paths lying near C_0 becomes the fixed points in the normal hyperplane of C_0.

When the system (1.1) is perturbed as

$$\frac{dy}{dt}=Y(y)+\varepsilon H(y;\varepsilon) \tag{2.8}$$

where $|\varepsilon|$ is small and $H(y;\varepsilon)\in C^1_{y;\varepsilon}$, from (1.4), it is seen that the equation (1.5) is perturbed as follows:

$$\frac{dx}{dt}=X(x, t)+\varepsilon K(x, t;\varepsilon), \tag{2.9}$$

where $K(x, t;\varepsilon)$ is continuous and periodic in t with period ω_0 and $K(x, t;\varepsilon)\in C^1_{x;\varepsilon}$. Then, by the transformation (2.7) where B is chosen so that $B=0$, the equation (2.9) is transformed to the equation of the form

$$\frac{dx}{dt}=\varepsilon F(x, t;\varepsilon) \tag{2.10}$$

and $F(x, t;\varepsilon)$ becomes a function of the same character as that of $K(x, t;\varepsilon)$, because $\partial\varphi/\partial c^i\in C^1_{c,t}$.

Thus the problem to seek for a periodic solution of the perturbed system (2.8) and to decide its orbital stability is reduced to the same problem for the system (2.10). This latter problem is already solved by the so-called *stroboscopic method* [3][1)], namely by seeking for a critical point of the stroboscopic image

$$\frac{dx}{dt}=\varepsilon F_0(x)=\frac{\varepsilon}{\omega_0}\int_0^{\omega_0}F(x, t;0)\,dt \tag{2.11}$$

and by deciding its stability.

Thus, by the results of [3], we have

Theorem 2. *When the stroboscopic image* (2.11) *has no critical point, there exists no periodic solution of the perturbed system* (2.8). *When* (2.11)

1) In [3], there is given a proof for mathematical legality of the heuristic stroboscopic method of N. Minorsky.

has some simple critical points[1), *there exist periodic solutions of* (2.8) *corresponding to each simple critical point of* (2.11), *and the orbital stability of these periodic solutions of* (2.8) *becomes the same as that of simple critical points of* (2.11), *so long as the stability is decided according to the signs of the real parts of the characteristic exponents of the equation of first variation.*

The method stated in this theorem is a direct generalization of the idea of the method proposed by Kryloff and Bogoliuboff [2], though the methods seem to be entirely different in appearance.

3. Reduced form of the partially oscillatory system

In the present case, we assume that the initial system (1.1) admits of an m-parameter family $\mathfrak{F}$ of closed paths. When $m=n$, $\mathfrak{F}$ consists of all paths, therefore the system becomes a fully oscillatory system. This case is already discussed in the preceding paragraph, therefore, in this paragraph, we assume that $1 \leqq m \leqq n-1$.

Since the intersection of closed paths belonging to $\mathfrak{F}$ with the normal hyperplane of C_0 at $y=\psi(0)$ is an m-dimensional manifold, we may assume that that manifold V^m is represented as

(3.1) $$c=c(u),$$

where $u=(u^1, u^2, \cdots, u^m)$ is an m-dimensional vector. For (3.1), without loss of generality, we may suppose that $c(0)=0$. We assume that $c(u) \in C_u^2$ for sufficiently small $\|c\|$. Then the rank of the matrix whose elements are

(3.2) $$\frac{\partial c^i}{\partial u^\alpha} \qquad (i=1, 2, \cdots, n;\ \alpha=1, 2, \cdots, m)$$

cannot be identically smaller than m, because, otherwise, the manifold V^m would become a manifold of lower dimension than m. Therefore we may assume naturally that the rank of the matrix whose elements are (3.2) is m for $u=0$, for this means that the point $c=0$ is an ordinary point of the manifold.

Then, if we put

(3.3) $$\varphi[t, c(u)]=\tilde{\varphi}(t, u),$$

for this function $\tilde{\varphi}(t, u)$, we can prove

Lemma 1. *The function* $\tilde{\varphi}(t, u)$ *is twice continuously differentiable with respect to* u *and is periodic in* t *with period* ω_0 *and the rank of the matrix whose elements are*

(3.4) $$\frac{\partial \tilde{\varphi}^i}{\partial u^\alpha} \qquad (i=1, 2, \cdots, n;\ \alpha=1, 2, \cdots, m)$$

is m *for any* t.

1) A crtical point $x=x_0$ of (2.11) is called simple when
$$\frac{\partial(F_0^1, F_0^2, \cdots, F_0^n)}{\partial(x^1, x^2, \cdots, x^n)} \neq 0$$
for $x=x_0$.

Proof. Since $\varphi(t, c) \in C_c^2$ and $c(u) \in C_u^2$, it is evident that $\tilde{\varphi}(t, u) \in C_u^2$. Also the path represented by

$$x=\tilde{\varphi}(t, u)$$

passes through a point of the manifold V^m, therefore it is also evident that the function $\tilde{\varphi}(t, u)$ is periodic in t with period ω_0.

If the rank of the matrix whose elements are (3.4) is smaller than m, then there exist numbers λ^α's $(\alpha=1, 2, \cdots, m)$ not all vanishing such that

$$\sum_{\alpha=1}^{m} \lambda^\alpha \frac{\partial \tilde{\varphi}^i}{\partial u^\alpha}=0 \qquad (i=1, 2, \cdots, n),$$

which are written from (3.3) as follows:

$$\sum_{j=1}^{n} \frac{\partial \varphi^i}{\partial c^j}\left(\sum_{\alpha=1}^{m} \frac{\partial c^j}{\partial u^\alpha} \lambda^\alpha\right)=0 \qquad (i=1, 2, \cdots, n).$$

This implies

$$\sum_{\alpha=1}^{m} \frac{\partial c^j}{\partial u^\alpha} \lambda^\alpha=0 \qquad (j=1, 2, \cdots, n), \tag{3.5}$$

because

$$\frac{\partial(\varphi^1, \varphi^2, \cdots, \varphi^n)}{\partial(c^1, c^2, \cdots, c^n)} \neq 0$$

as is remarked on (2.4). (3.5) implies that the rank of the matrix whose elements are (3.2) is smaller than m. This contradicts the assumption on the parameter u.

Thus the lemma has been proved.

Now, since $x=\tilde{\varphi}(t, u)$ is a solution of (1.5), the n-dimensional vectors

$$\begin{pmatrix} \dfrac{\partial \tilde{\varphi}^1}{\partial u^\alpha} \\ \dfrac{\partial \tilde{\varphi}^2}{\partial u^\alpha} \\ \vdots \\ \dfrac{\partial \tilde{\varphi}^n}{\partial u^\alpha} \end{pmatrix} \qquad (\alpha=1, 2, \cdots, m) \tag{3.6}$$

becomes the solution of the equation of first variation of (1.5), consequently it is evident that $\dfrac{\partial \tilde{\varphi}}{\partial u^\alpha} \in C_t^1$. Then the above lemma says that the continuously differentiable n-dimensional vectors $\dfrac{\partial \tilde{\varphi}}{\partial u^\alpha}$ $(\alpha=1, 2, \cdots, m)$ constitute a periodic m-ple system of linearly independent vectors. Then, by Gram-Schmidt's process, we can construct a periodic orthonormal m-ple system of n-dimensional continuously differentiable unit vectors $P_\alpha(t)$'s $(\alpha=1, 2, \cdots, m)$ so that

$$P_\alpha(t)=\sum_{\beta=1}^{m} k_\alpha^\beta(t)\frac{\partial\tilde{\varphi}}{\partial u^\beta}\bigg|_{u=0} \qquad (\alpha=1, 2, \cdots, m) \tag{3.7}$$

may hold. For this m-ple system $\{P_\alpha(t)\}$ $(\alpha=1, 2, \cdots, m)$, we can prove

Lemma 2. *For a given continously differentiable periodic orthonormal m-ple system* $\{P_\alpha(t)\}$ $(\alpha=1, 2, \cdots, m)$, *there exists a continuously differentiable periodic orthonormal n-ple system* $\{P_i(t)\}$ $(i=1, 2, \cdots, n)$ *which contains the given m-ple system* $\{P_\alpha(t)\}$ $(\alpha=1, 2, \cdots, m)$.

This lemma is a generalization of Theorem 1 of [4] or Theorem 1 of [5]. The proof is carried out quite analogously as these theorems.

Proof. Evidently there exist local continuous orthonormal n-ple systems $\{P_i^{(k)}(t)\}$ $(k=0, 1, 2, \cdots)$ each of which contains the given m-ple system $\{P_\alpha(t)\}$. Let $U_k(t)$ be the matrix whose column vectors are $P_i^{(k)}(t)$'s $(i=1, 2, \cdots, n)$. Then, by the definition of $\{P_i^{(k)}(t)\}$, the matrices $U_k(t)$ $(k=0, 1, 2, \cdots)$ are orthogonal, and the interval $[0, \omega_0]$ is covered by a finite number of the intervals on each of which $U_k(t)$ is defined. Let $I_0<I_1<\cdots<I_N$[1] be such intervals covering $[0, \omega_0]$.

For any $t_1\in I_0\frown I_1$, construct an orthogonal matrix

$$M_1=U_1^{-1}(t_1)\,U_0(t_1) \tag{3.8}$$

and define $U(t)$ so that

$$U(t)=\begin{cases} U_0(t) & \text{for } 0\leqq t\leqq t_1, \\ U_1(t)M_1 & \text{for } t_1\leqq t\in I_1. \end{cases}$$

Then, from (3.8), M_1 is of the form

$$M_1=\begin{pmatrix} E_m & 0 \\ 0 & \tilde{M}_1 \end{pmatrix}\,^{2)}$$

and $U(t)$ becomes a continuous orthogonal matrix defined on $I_0\smile I_1$ such that its first m column vectors are always $P_\alpha(t)$'s $(\alpha=1, 2, \cdots, m)$. Taking this $U(t)$ instead of $U_0(t)$, we repeat the above process, then there is is obtained a continuous orthogonal matrix defined on $I_0\smile I_1\smile I_2$ such that its first m column vectors are always $P_\alpha(t)$'s $(\alpha=1, 2, \cdots, m)$, and so on. Thus, ultimately, after the N-th step, there is obtained a continuous orthogonal matrix $U(t)$ defined on $[0, \omega_0]$ such that its first m column vectors are always $P_\alpha(t)$'s $(\alpha=1, 2, \cdots, m)$.

Put

$$U^{-1}(\omega_0)\,U(0)=A, \tag{3.9}$$

then A is a proper orthogonal matrix, because $U(t)$ is a continous orthogonal matrix. Moreover, from (3.9), it is evident that A is of the form

1) $I_k<I_l$ means that I_l contains t greater than any t belonging to I_k.

2) E_m denotes a unit matrix of order m.

$$A=\begin{pmatrix} E_m & 0 \\ 0 & \tilde{A} \end{pmatrix}, \tag{3.10}$$

because

$$P_\alpha(\omega_0)=P_\alpha(0) \qquad (\alpha=1, 2, \cdots, m).$$

Consequently $\tilde{A}$ becomes also a proper orthogonal matrix. Then, as is well known, there exists an orthogonal matrix $\tilde{T}$ such that

$$\tilde{T}^*\tilde{A}\tilde{T}=\sum_r \oplus \begin{pmatrix} \cos\alpha_r & -\sin\alpha_r \\ \sin\alpha_r & \cos\alpha_r \end{pmatrix} \oplus E, \tag{3.11}$$

where $\oplus$ denotes the direct sum of matrices. Now, for matrices under the sign $\sum_r \oplus$ in the right-hand side of (3.11), it holds that

$$\begin{pmatrix} \cos\alpha_r & -\sin\alpha_r \\ \sin\alpha_r & \cos\alpha_r \end{pmatrix}=\exp\begin{pmatrix} 0 & -\alpha_r \\ \alpha_r & 0 \end{pmatrix},$$

consequently, if we put

$$(0)\oplus\sum_r \oplus \begin{pmatrix} 0 & -\alpha_r \\ \alpha_r & 0 \end{pmatrix} \oplus (0)=\omega_0 B_0 \tag{3.12}$$

and

$$\begin{pmatrix} E_m & 0 \\ 0 & \tilde{T} \end{pmatrix}=T, \tag{3.13}$$

then from (3.11) follows

$$T^*AT=\exp(\omega_0 B_0).$$

Therefore, if we put

$$TB_0T^*=B, \tag{3.14}$$

it holds that

$$A=\exp(\omega_0 B), \tag{3.15}$$

from which (3.9) is written as follows:

$$U(0)=U(\omega_0)e^{\omega_0 B}. \tag{3.16}$$

Then, let us consider the matrix

$$V(t)=U(t)e^{tB}.$$

From (3.13) and (3.14), e^{tB} is of the form

$$\begin{pmatrix} E_m & 0 \\ 0 & \times \end{pmatrix},$$

consequently the first m column vectors of $V(t)$ are also $P_\alpha(t)$'s ($\alpha=1, 2, \cdots, m$). Also, from (3.12) and (3.14), B is skew-symmetric, consequently e^{tB} is orthogonal. In addition, from (3.16),

$$V(\omega_0)=V(0).$$

Thus, if we define $V(t)$ outside the interval $[0, \omega_0]$ by

$$V(t+\omega_0)=V(t),$$

then $V(t)$ becomes a continuous periodic orthogonal matrix whose first m column vectors are always $P_\alpha(t)$'s $(\alpha=1, 2, \cdots, m)$.

Let $\tilde{P}_\nu(t)$'s $(\nu=m+1, \cdots, n)$ be the last $(n-m)$ column vectors of $V(t)$, then these are evidently continuous periodic vectors. In order to make these continuously differentiable periodic vectors, for any positive number ε, taking a sufficiently small positive number δ[1] so that

$$\|\tilde{P}_\nu(t')-\tilde{P}_\nu(t'')\|<\varepsilon \qquad (\nu=m+1, \cdots, n)$$

for $|t'-t''|\leqq\delta$, let us consider the vectors

$$\bar{P}_\nu(t)=\frac{1}{2\delta}\int_{t-\delta}^{t+\delta}\tilde{P}_\nu(s)ds \qquad (\nu=m+1, \cdots, n). \tag{3.17}$$

Then these are evidently periodic in t because of periodicity of $\tilde{P}_\nu(t)$'s $(\nu=m+1, \cdots, n)$. Moreover, by Schwartz's inequality, it holds that

$$\begin{aligned}\|\bar{P}_\nu(t)-\tilde{P}_\nu(t)\|&=\frac{1}{2\delta}\left\|\int_{t-\delta}^{t+\delta}[\tilde{P}_\nu(s)-\tilde{P}_\nu(t)]ds\right\|\\&\leqq\frac{\sqrt{2\delta}}{2\delta}\left[\int_{t-\delta}^{t+\delta}\|\tilde{P}_\nu(s)-\tilde{P}_\nu(t)\|^2ds\right]^{1/2}\\&\leqq\varepsilon.\end{aligned}$$

Therefore, when $|\varepsilon|$ is sufficiently small, $\bar{P}_\nu(t)$'s $(\nu=m+1, \cdots, n)$ are linearly independent relatively and to $P_\alpha(t)$'s $(\alpha=1, 2, \cdots, m)$, because $\tilde{P}_\nu(t)$'s $(\nu=m+1, \cdots, n)$ and $P_\alpha(t)$'s $(\alpha=1, 2, \cdots, m)$ are relatively orthonormal. In addition, $\bar{P}_\nu(t)$'s $(\nu=m+1, \cdots, n)$ are evidently continuously differentiable as follows:

$$\frac{d\bar{P}_\nu(t)}{dt}=\frac{1}{2\delta}[\tilde{P}_\nu(t+\delta)-\tilde{P}_\nu(t-\delta)] \qquad (\nu=m+1, \cdots, n).$$

Then, if we again apply Gram-Schmidt's process to this system $\{P_\alpha(t), \bar{P}_\nu(t)\}$ $(\alpha=1, 2, \cdots, m;\ \nu=m+1, \cdots, n)$, we can obtain a continuously differentiable periodic n-ple system of orthonormal vectors $P_i(t)$'s $(i=1, 2, \cdots, n)$ which contains initial m vectors $P_\alpha(t)$'s $(\alpha=1, 2, \cdots, m)$. Thus the lemma has been proved.

This lemma says that there exists a continuously differentiable moving orthonormal system along C_0 such that its first m normal unit vectors are $\sum_{i=1}^{n}P_\alpha^i\xi_i$ $(\alpha=1, 2, \cdots, m)$. Then, if we denote $x=(x^i)$ with respect to this moving orthonormal system by $\hat{x}=(\hat{x}^i)$, from

$$\sum_{i,j=1}^{n}\hat{x}^iP_i^j\xi_j=\sum_{j=1}^{n}x^j\xi_j,$$

1) This is really possible because $\tilde{P}_\nu(t)$'s $(\nu=m+1, \cdots, n)$ are uniformly continuous in $-\infty<t<\infty$ on account of their periodicity.

it is valid that

$$P(t)\hat{x}=x, \tag{3.18}$$

where $P(t)$ is a matrix whose elements are $P^i_j(t)$ $(i, j=1, 2, \cdots, n)$. Then, from (1.5), for any path lying near C_0, $\hat{x}=\hat{x}(t)$ becomes a solution of the equation

$$\frac{d\hat{x}}{dt}=\hat{X}(\hat{x}, t) \tag{3.19}$$

of the same form as (1.5), where

$$\begin{aligned}\hat{X}(\hat{x}, t)&=P^{-1}(t)X[P(t)\hat{x}, t]-P^{-1}(t)\frac{dP(t)}{dt}\hat{x}\\&=\left(P^{-1}AP-P^{-1}\frac{dP}{dt}\right)\hat{x}+o(\|\hat{x}\|).\end{aligned}$$

Then, since $x=\tilde{\varphi}(t, u)$ is a periodic solution of (1.5), from (3.18),

$$\hat{x}=\hat{\varphi}(t, u)=P^{-1}(t)\tilde{\varphi}(t, u) \tag{3.20}$$

becomes a periodic solution of (3.19). Since, from the above relation follows

$$\sum_{j=1}^{n} P^i_j(t)\frac{\partial\hat{\varphi}^j}{\partial u^\alpha}=\frac{\partial\tilde{\varphi}^i}{\partial u^\alpha} \qquad (i=1, 2, \cdots, n;\ \alpha=1, 2, \cdots, m),$$

from (3.7), we have

$$\sum_{j=1}^{n} P^i_j(t)\frac{\partial\hat{\varphi}^j}{\partial u^\alpha}\bigg|_{u=0}=\sum_{\beta=1}^{m} K^\beta_\alpha(t)P^i_\beta(t), \tag{3.21}$$

where $K^\beta_\alpha(t)$ $(\alpha, \beta=1, 2, \cdots, m)$ are the elements of the inverse matrix of the matrix whose elements are $k^\beta_\alpha(t)$ $(\alpha, \beta=1, 2, \cdots, m)$[1]. (3.21) can be written as follows:

$$\sum_{j=1}^{n} P^i_j(t)\left[\frac{\partial\hat{\varphi}^j}{\partial u^\alpha}\bigg|_{u=0}-\sum_{\beta=1}^{m} K^\beta_\alpha(t)\delta^j_\beta\right]=0 \qquad (i=1, 2, \cdots, n;\ \alpha=1, 2, \cdots, m).$$

Then, since $\det P(t)\neq 0$, it follows that

$$\frac{\partial\hat{\varphi}^j}{\partial u^\alpha}\bigg|_{u=0}=\sum_{\beta=1}^{m} K^\beta_\alpha(t)\delta^j_\beta \qquad (j=1, 2, \cdots, n;\ \alpha=1, 2, \cdots, m),$$

which says that

$$\frac{\partial\hat{\varphi}^\nu}{\partial u^\alpha}\bigg|_{u=0}=0 \qquad (\nu=m+1, \cdots, n;\ \alpha=1, 2, \cdots, m),$$

$$\frac{\partial\hat{\varphi}^\beta}{\partial u^\alpha}\bigg|_{u=0}=K^\beta_\alpha(t) \qquad (\alpha, \beta=1, 2, \cdots, m).$$

From the second of the above formulas, it follows that

$$\frac{\partial(\hat{\varphi}^1, \hat{\varphi}^2, \cdots, \hat{\varphi}^m)}{\partial(u^1, u^2, \cdots, u^m)}$$

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1) The determinant of the matrix whose elements are $k^\beta_\alpha(t)$ $(\alpha, \beta=1, 2, \cdots, m)$ does not vanish as a matter of course because (3.7) is an expression of the Gram-Schmidt's process.

never vanishes for any t when $\|u\|$ is sufficiently small.

Then, the relations

$$\hat{x}^{\alpha}=\hat{\varphi}^{\alpha}(t, u) \qquad (\alpha=1, 2, \cdots, m) \tag{3.22}$$

can be solved with respect to $(u^1, u^2, \cdots, u^m)$ for any t as follows:

$$u^{\alpha}=u^{\alpha}(t, \hat{x}^1, \hat{x}^2, \cdots, \hat{x}^m) \qquad (\alpha=1, 2, \cdots, m) \tag{3.23}$$

where $u^{\alpha}(t, \hat{x}^1, \hat{x}^2, \cdots, \hat{x}^m)$'s are periodic in t with period ω_0. Then, substituting (3.23) into the others of (3.20), i. e.

$$\hat{x}^{\nu}=\hat{\varphi}^{\nu}(t, u) \qquad (\nu=m+1, \cdots, n),$$

we obtain

$$\hat{x}^{\nu}=f^{\nu}(t, \hat{x}^1, \hat{x}^2, \cdots, \hat{x}^m) \qquad (\nu=m+1, \cdots, n), \tag{3.24}$$

where $f^{\nu}(t, \hat{x}^1, \hat{x}^2, \cdots, \hat{x}^m)$'s are periodic in t with period ω_0 and twice continuously differentiable with respect to $\hat{x}^1, \hat{x}^2, \cdots, \hat{x}^m$. Of course,

$$f^{\nu}(t, 0, 0, \cdots, 0)=0 \qquad (\nu=m+1, \cdots, n),$$

for $u^{\alpha}=0$ $(\alpha=1, 2, \cdots, m)$ when $\hat{x}^{\alpha}=0$ $(\alpha=1, 2, \cdots, m)$. Thus the equation (3.24) represents an $(m+1)$-dimensional closed integral strip S^{m+1} passing through C_0 as is seen from its formation.

Then, if we transform the variable $\hat{x}$ to z by

$$\begin{cases} z^{\alpha}=\hat{x}^{\alpha} \qquad (\alpha=1, 2, \cdots, m), \\ z^{\nu}=\hat{x}^{\nu}-f^{\nu}(t, \hat{x}^1, \hat{x}^2, \cdots, \hat{x}^m) \qquad (\nu=m+1, \cdots, n), \end{cases} \tag{3.25}$$

the equation (3.19) is transformed to the equation

$$\frac{dz}{dt}=Z(z, t) \tag{3.26}$$

of the same form[1] as (3.19), namely as (1.5), and $z^{\nu}=0$ $(\nu=m+1, \cdots, n)$ satisfy (3.26) identically. Hence $Z(z, t)$ can be written in the form

$$Z^{\nu}(z, t)=\sum_{\mu=m+1}^{n} Z^{\nu}_{\mu}(z, t)z^{\mu} \qquad (\nu=m+1, \cdots, n),$$

where $Z^{\nu}_{\mu}(z, t)$'s $(\nu, \mu=m+1, \cdots, n)$ are periodic in t and continuous in z and t. Thus the equation (3.26) becomes the equation of the form

$$\begin{cases} \dfrac{dz^{\alpha}}{dt}=Z^{\alpha}(z, t) \qquad (\alpha=1, 2, \cdots, m), \\ \dfrac{dz^{\nu}}{dt}=\displaystyle\sum_{\mu=m+1}^{n} Z^{\nu}_{\mu}(z, t)z^{\mu} \qquad (\nu=m+1, \cdots, n). \end{cases} \tag{3.27}$$

Then, in S^{m+1}, the equation (3.27) becomes

$$\begin{cases} \dfrac{dz^{\alpha}}{dt}=Z^{\alpha}(z^1, \cdots, z^m, 0, \cdots, 0, t) \quad (\alpha=1, 2, \cdots, m), \\ \dfrac{dz^{\nu}}{dt}=0 \qquad (\nu=m+1, \cdots, n). \end{cases} \tag{3.28}$$

But, in S^{m+1}, all paths lying near C_0 are closed and they are expressed by

1) The condition that $Z(z, t)$ is continuously differentiable with respect to z is included.

$$z^\alpha=\hat{\varphi}^\alpha(t, u) \qquad (\alpha=1, 2, \cdots, m)$$

as is seen from (3.22). Therefore, in like manner as in Theorem 1, by the periodic transformation

$$z^\alpha=\hat{\varphi}^\alpha(t, z') \qquad (\alpha=1, 2, \cdots, m), \tag{3.29}$$

the first of (3.28) is transformed to the equation

$$\frac{dz'^\alpha}{dt}=0 \qquad (\alpha=1, 2, \cdots, m),$$

namely the equation of the same form as (3.28) but with

$$Z^\alpha(z^1, \cdots, z^m, 0, \cdots, 0, t)=0 \qquad (\alpha=1, 2, \cdots, m).$$

When these hold, $Z^\alpha(z, t)$ $(\alpha=1, 2, \cdots, m)$ are of the same form as $Z^\nu(z, t)$ $(\nu=m+1, \cdots, n)$. Thus we have

Theorem 3. *For the twice continuously differentiable m-parameter*[1] *partially oscillatory system* (1.1), *the relative equation of any path C lying near C_0 in the normal hyperplane of C_0 is transformed to the continuously differentiable equation*[2]

$$\frac{dx^i}{dt}=\sum_{\nu=m+1}^{n} X^i_\nu(x, t)x^\nu \qquad (i=1, 2, \cdots, n) \tag{3.30}$$

by the periodic transformation, when a moving orthonormal system along C_0 is suitably chosen. Here $X^i_\nu(x, t)$'s $(i=1, 2, \cdots, n;\ \nu=m+1, \cdots, n)$ are periodic in t and continuous in x and t.

As is seen from the form of (3.30), for the present system, there exists an $(m+1)$-dimensional closed integral strip S^{m+1}, which is represented by

$$x^\nu=0 \qquad (\nu=m+1, \cdots, n)$$

for the equation (3.30). In addition, for the equation (3.30), the paths in S^{m+1} are represented by the fixed points in the normal hyperplane of C_0.

4. Perturbation problem for the partially oscillatory system

In this paragraph, we consider the case where the three times continuously differentiable m-parameter[3] partially oscillatory system (1.1) is perturbed as

$$\frac{dy}{dt}=Y(y)+\varepsilon H(y;\varepsilon) \tag{4.1}$$

where $\varepsilon H(y;\varepsilon)\in C^2_{y;\varepsilon}$. In this case, as is seen from the proof of Theorem 3, the equation (3.30) is perturbed as

$$\frac{dx}{dt}=\sum_{\nu=m+1}^{n} X_\nu(x, t)x^\nu+\varepsilon K(x, t;\varepsilon) \tag{4.2}$$

and it becomes that $\sum_{\nu=m+1}^{n} X_\nu(x, t)x^\nu\in C^2_x$ and $\varepsilon K(x, t;\varepsilon)\in C^2_{x;\varepsilon}$.

---

1) This means that $Y(y)\in C^2_y$ and $c(u)$ of (3.1) belongs to C^2_u.

2) This means that the function of the right-hand side of the equation belongs to C^1_x.

3) This means that $Y(y)\in C^3_y$ and $c(u)$ of (3.1) belongs to C^3_u.

Let $x=\varphi(t, c; \varepsilon)$ be the solution of (4.2) such that

$$\varphi(0, c; \varepsilon)=c. \tag{4.3}$$

Then the function $\varphi(t, c; \varepsilon)$ is twice continuously differentiable with respect to c and ε, and, from the form of (4.2), it is seen that

$$\begin{cases} \varphi^{\alpha}(t, c^1, \cdots, c^m, 0, \cdots, 0; 0)=c^{\alpha} & (\alpha=1, 2, \cdots, m), \\ \varphi^{\nu}(t, c^1, \cdots, c^m, 0, \cdots, 0; 0)=0 & (\nu=m+1, \cdots, n). \end{cases}$$

Therefore $\varphi(t, c; \varepsilon)$ can be expressed as follows:

$$\begin{cases} \varphi^{\alpha}(t, c; \varepsilon)=c^{\alpha}+\sum_{\mu=m+1}^{n} G^{\alpha}_{\mu}c^{\mu}+\varepsilon r^{\alpha}+o(|c^{m+1}|+\cdots+|c^n|+|\varepsilon|) \\ \qquad\qquad (\alpha=1, 2, \cdots, m), \\ \varphi^{\nu}(t, c; \varepsilon)=\sum_{\mu=m+1}^{n} G^{\nu}_{\mu}c^{\mu}+\varepsilon r^{\nu}+o(|c^{m+1}|+\cdots+|c^n|+|\varepsilon|) \\ \qquad\qquad (\nu=m+1, \cdots, n) \end{cases} \tag{4.4}$$

as $c^{\mu}, \varepsilon\to 0$ $(\mu=m+1, \cdots, n)$. Here

$$G^{i}_{\mu}=G^{i}_{\mu}(t, c^1, \cdots, c^m), \quad r^i=r^i(t, c^1, \cdots, c^m)$$
$$(i=1, 2, \cdots, n;\ \mu=m+1, \cdots, n)$$

and, from (4.3),

$$\begin{cases} G^{\alpha}_{\mu}(0, c^1, \cdots, c^m)=r^i(0, c^1, \cdots, c^m)=0, \\ G^{\nu}_{\mu}(0, c^1, \cdots, c^m)=\delta^{\nu}_{\mu} \end{cases} \tag{4.5}$$
$$(\alpha=1, 2, \cdots, m;\ \nu, \mu=m+1, \cdots, n;\ i=1, 2, \cdots, n).$$

In the sequel, for brevity, let us denote $G^{i}_{\mu}(t, c^1, \cdots, c^m)$ and $r^i(t, c^1, \cdots, c^m)$ by $G^{i}_{\mu}(t)$ and $r^i(t)$ respectively.

When (4.4) is substituted into (4.2), there are obtained

$$\begin{cases} \text{(i)} & \dfrac{dG^{\alpha}_{\mu}}{dt}=\sum_{\nu=m+1}^{n} X^{\alpha}_{\nu}(c^1, \cdots, c^m, 0, \cdots, 0, t)G^{\nu}_{\mu} \\ & \qquad (\alpha=1, 2, \cdots, m;\ \mu=m+1, \cdots, n), \\ \text{(ii)} & \dfrac{dr^{\alpha}}{dt}=\sum_{\nu=m+1}^{n} X^{\alpha}_{\nu}(c^1, \cdots, c^m, 0, \cdots, 0, t)r^{\nu} \\ & \qquad +K^{\alpha}(c^1, \cdots, c^m, 0, \cdots, 0, t; 0) \\ & \qquad (\alpha=1, 2, \cdots, m), \\ \text{(iii)} & \dfrac{dG^{\nu}_{\mu}}{dt}=\sum_{\kappa=m+1}^{n} X^{\nu}_{\kappa}(c^1, \cdots, c^m, 0, \cdots, 0, t)G^{\kappa}_{\mu} \\ & \qquad (\nu, \mu=m+1, \cdots, n), \\ \text{(iv)} & \dfrac{dr^{\nu}}{dt}=\sum_{\kappa=m+1}^{n} X^{\nu}_{\kappa}(c^1, \cdots, c^m, 0, \cdots, 0, t)r^{\kappa} \\ & \qquad +K^{\nu}(c^1, \cdots, c^m, 0, \cdots, 0, t; 0) \\ & \qquad (\nu=m+1, \cdots, n). \end{cases} \tag{4.6}$$

Therefore, from (4.5), the matrix G whose elements are G^{ν}_{μ} $(\nu, \mu=m+1, \cdots, n)$ becomes a fundamental matrix of the linear homogeneous equation

$$\frac{dg}{dt}=Xg \tag{4.7}$$

such that

$$G(0)=E, \tag{4.8}$$

where X is a matrix whose elements are $X^\nu_\kappa(c^1, \cdots, c^m, 0, \cdots, 0, t)$ $(\nu, \kappa=m+1, \cdots, n)$. Then, from (4.5), the equation (iv) of (4.6) is easily integrated as follows:

$$r=G(t)\int_0^t G^{-1}(s)K(s)\,ds, \tag{4.9}$$

where r and $K(s)$ are $(n-m)$-dimensional vectors whose components are r^ν and $K^\nu(c^1, \cdots, c^m, 0, \cdots, 0, t; 0)$ $(\nu=m+1, \cdots, n)$ respectively. Making use of these results, G^α_μ $(\alpha=1, 2, \cdots, m;\ \mu=m+1, \cdots, n)$ and r^α $(\alpha=1, 2, \cdots, m)$ are easily found from (4.5) by integrating (i) and (ii) of (4.6) as follows:

$$G^\alpha_\mu(t)=\int_0^t \sum_{\nu=m+1}^n X^\alpha_\nu(c^1, \cdots, c^m, 0, \cdots, 0, s)G^\nu_\mu(s)\,ds \qquad (\alpha=1, 2, \cdots, m;\ \mu=m+1, \cdots, n), \tag{4.10}$$

$$r^\alpha(t)=\int_0^t \Big[\sum_{\nu=m+1}^n X^\alpha_\nu(c^1, \cdots, c^m, 0, \cdots, 0, s)r^\nu(s) + K^\alpha(c^1, \cdots, c^m, 0, \cdots, 0, s; 0)\Big]\,ds \qquad (\alpha=1, 2, \cdots, m). \tag{4.11}$$

Since $\varphi(t, c; \varepsilon)\in C^2_{c;\,\varepsilon}$, it is evident that

$$G^i_\mu(t)=\frac{\partial\varphi^i(t, c; \varepsilon)}{\partial c^\mu}\bigg|_{\tilde{c}=\varepsilon=0}, \quad r^i(t)=\frac{\partial\varphi^i(t, c; \varepsilon)}{\partial \varepsilon}\bigg|_{\tilde{c}=\varepsilon=0} \qquad (i=1, 2, \cdots, n;\ \mu=m+1, \cdots, n)$$

belong to $C^1_{c^1, c^2, \ldots, c^m}$, where $\tilde{c}$ is an $(n-m)$-dimensional vector whose components are c^μ $(\mu=m+1, \cdots, n)$. For the functions $G^i_\mu(t)$, $r^i(t)$ $(i=1, 2, \cdots, n;\ \mu=m+1, \cdots, n)$, there holds

Lemma 3. *For any positive integer p, it holds that*

$$\begin{cases} \text{(i)} & G(p\omega_0)=G_0^p, \\ \text{(ii)} & r(p\omega_0)=(E+G_0+\cdots+G_0^{p-1})r(\omega_0), \\ \text{(iii)} & G'(p\omega_0)=G'(\omega_0)(E+G_0+\cdots+G_0^{p-1}), \\ \text{(iv)} & r'(p\omega_0)=pr'(\omega_0)+[G'(\omega_0)+G'(2\omega_0)+\cdots+G'((p-1)\omega_0)]r(\omega_0), \end{cases} \tag{4.12}$$

where

$G_0=G(\omega_0)$;

$G'(t)$ is a matrix whose elements are G^α_μ $(\alpha=1, 2, \cdots, m;\ \mu=m+1, \cdots, n)$;

$r'(t)$ is an m-dimensional vector whose components are r^α $(\alpha=1, 2, \cdots, m)$.

Proof. Since $G(t)$ is a fundamental matrix of the periodic linear homogeneous differential equation (4.7) such that (4.8) holds, as is well known, it holds that

$$G(t+\omega_0)=G(t)G(\omega_0), \tag{4.13}$$

from which immediately follows (i).

When we calculate $r(t+\omega_0)$ from (4.9) and (4.13), we obtain:

$$(4.14)\qquad \begin{aligned} r(t+\omega_0) &= G(t)G(\omega_0)\left[\int_0^{\omega_0} G^{-1}(s)K(s)\,ds + \int_{\omega_0}^{\omega_0+t} G^{-1}(s)K(s)\,ds\right] \\ &= G(t)r(\omega_0) + G(t)G(\omega_0)\int_0^t G^{-1}(\omega_0)G^{-1}(s)K(s)\,ds \\ &= G(t)r(\omega_0) + r(t), \end{aligned}$$

from which immediately follows (ii).

For $G'(t)$, from (4.10) and (4.13), we obtain:

$$\begin{aligned} G'(t+\omega_0) &= G'(\omega_0) + \int_{\omega_0}^{\omega_0+t} X'(s)G(s)\,ds \\ &= G'(\omega_0) + \int_0^t X'(s)G(s+\omega_0)\,ds \\ &= G'(\omega_0) + G'(t)G_0, \end{aligned}$$

where $X'(s)$ is the matrix whose elements are $X_\nu^\alpha(c^1, \cdots, c^m, 0, \cdots, 0, s)$ $(\alpha=1, 2, \cdots, m;\ \nu=m+1, \cdots, n)$. (iii) of (4.12) is immediately derived from the above formula.

For $r'(t)$, from (4.11) and (4.14), we obtain:

$$\begin{aligned} r'(t+\omega_0) &= r'(\omega_0) + \int_{\omega_0}^{\omega_0+t} [X'(s)r(s) + K'(s)]\,ds \\ &= r'(\omega_0) + \int_0^t [X'(s)(r(s) + G(s)r(\omega_0)) + K'(s)]\,ds \\ &= r'(\omega_0) + r'(t) + G'(t)r(\omega_0), \end{aligned}$$

where $K'(s)$ is a vector whose components are $K^\alpha(c^1, \cdots, c^m, 0, \cdots, 0, s; 0)$ $(\alpha=1, 2, \cdots, m)$. From the above formula, (iv) of (4.12) is derived readily by induction.

Thus the lemma has been proved.

The condition that the solution $x=\varphi(t, c; \varepsilon)$ may represent a periodic solution of (4.1) is that the following relation holds for a certain positive integer p:

$$(4.15)\qquad \varphi(p\omega_0, c; \varepsilon) = c.$$

When (4.4) is substituted into the left-hand side, this relation is written as follows:

$$(4.16)\qquad \begin{cases} G'(p\omega_0)\tilde{c} + \varepsilon r'(p\omega_0) + o(|\tilde{c}| + |\varepsilon|) = 0, \\ [G(p\omega_0) - E]\tilde{c} + \varepsilon r(p\omega_0) + o(|\tilde{c}| + |\varepsilon|) = 0, \end{cases}$$

where $|\tilde{c}| = \sum_{\mu=m+1}^{n} |c^\mu|$. Thus the problem to seek for a periodic solution of (4.1) becomes the problem to determine c satisfying (4.16) so that $c^\alpha \to c_0^\alpha$ $(\alpha=1, 2, \cdots, m)$, $\tilde{c} \to 0$ as $\varepsilon \to 0$.

At present let us assume that

(4.17) $$\det[G(p\omega_0)-E]\neq 0,$$
which, by (i) of (4.12), is equivalent to
$$\det[G^p(\omega_0)-E]\neq 0,$$
namely to that the p-th power of any characteristic roots of $G(\omega_0)$ is not equal to unity. Then, in the present case, it evidently holds that
(4.18) $$\det[G(\omega_0)-E]\neq 0.$$

When (4.17) is valid, from the latter of (4.16), it follows that
(4.19) $$\tilde{c}=-\varepsilon[G(p\omega_0)-E]^{-1}r(p\omega_0)+o(|\varepsilon|)\in C^2_{c^1,\dots,c^m;\varepsilon}.$$
Consequently substitution of this into the former of (4.16) entails
(4.20) $$-G'(p\omega_0)[G(p\omega_0)-E]^{-1}r(p\omega_0)+r'(p\omega_0)+0(\varepsilon)=0,$$
of which the left-hand side belongs to $C^1_{c^1,\dots,c^m;\varepsilon}$ because $\varphi(t,c;\varepsilon)\in C^2_{c;\varepsilon}$. Then, letting $\varepsilon\to 0$, $c^\alpha=c^\alpha_0$ $(\alpha=1,2,\cdots,m)$ becomes the solution of the equation
(4.21) $$-G'(p\omega_0)[G(p\omega_0)-E]^{-1}r(p\omega_0)+r'(p\omega_0)=0.$$
However this is equivalent to the equation
(4.22) $$-G'(\omega_0)[G(\omega_0)-E]^{-1}r(\omega_0)+r'(\omega_0)=0.$$
For, by Lemma 3, the left-hand side of (4.21) is calculated as follows:
$$\begin{aligned}
&-G'(\omega_0)(E+G_0+\cdots+G_0^{p-1})(G_0^p-E)^{-1}(E+G_0+\cdots+G_0^{p-1})r(\omega_0)+pr'(\omega_0)+\\
&\qquad+G'(\omega_0)[E+(E+G_0)+\cdots+(E+G_0+\cdots+G_0^{p-2})]r(\omega_0)\\
&=-G'(\omega_0)[(E+G_0+\cdots+G_0^{p-1})(G_0-E)^{-1}\\
&\qquad-(E+(E+G_0)+\cdots+(E+G_0+\cdots+G_0^{p-2}))]r(\omega_0)+pr'(\omega_0)\\
&=-G'(\omega_0)[(E+G_0+\cdots+G_0^{p-1})\\
&\qquad-((G_0-E)+(G_0^2-E)+\cdots+(G_0^{p-1}-E))](G_0-E)^{-1}r(\omega_0)+pr'(\omega_0)\\
&=p[-G'(\omega_0)(G_0-E)^{-1}r(\omega_0)+r'(\omega_0)].
\end{aligned}$$

Thus we see that, when there exists no real solution $c^\alpha=c^\alpha_0$ $(\alpha=1,2,\cdots,m)$ of (4.22), there exists no c satisfying (4.16), namely there exists no periodic solution of (4.1).

When there exists a real solution $c^\alpha=c^\alpha_0$ $(\alpha=1,2,\cdots,m)$ of (4.22), let us assume that *the Jacobian J of the left-hand side of* (4.22) *with respect to c^α* $(\alpha=1,2,\cdots,m)$ does not vanish for $c^\alpha=c^\alpha_0$ $(\alpha=1,2,\cdots,m)$. Then, since the Jacobian of the left-hand side of (4.20) with respect to c^α $(\alpha=1,2,\cdots,m)$ becomes pJ for $\varepsilon=0$, by the theorem on implicit functions, there exists a unique solution $c^\alpha=c^\alpha_p(\varepsilon)\in C^1_\varepsilon$ $(\alpha=1,2,\cdots,m)$ of (4.20) such that $c^\alpha_p(0)=c^\alpha_0$ $(\alpha=1,2,\cdots,m)$. Substituting this solution $c^\alpha=c^\alpha_p(\varepsilon)$ $(\alpha=1,2,\cdots,m)$ into (4.19), there is obtained $\tilde{c}$ of the form $\tilde{c}=\varepsilon\tilde{c}_p(\varepsilon)$ where $\tilde{c}_p(\varepsilon)\in C^1_\varepsilon$.

Now, in the present case, since (4.18) is valid, there also exists a solution $\{c^\alpha=c^\alpha_1(\varepsilon)$ $(\alpha=1,2,\cdots,m)$, $\tilde{c}=\varepsilon\tilde{c}_1(\varepsilon)\}$ of (4.16) such that $c^\alpha_1(0)=c^\alpha_0$ $(\alpha=1,2,\cdots,m)$. Then, for such $c=(c^1,\cdots,c^m,\tilde{c})$, there holds
$$\varphi(\omega_0,c;\varepsilon)=c,$$

consequently, for any positive integer p, there holds

$$\varphi(p\omega_0, c; \varepsilon)=c.$$

This says that $\{c^\alpha=c_1^\alpha(\varepsilon)\ (\alpha=1, 2, \cdots, m),\ \tilde{c}=\varepsilon\tilde{c}_1(\varepsilon)\}$ also satisfy (4.16). But, as is stated above, the solution of (4.16) such that $c^\alpha(0)=c_0^\alpha\ (\alpha=1, 2, \cdots, m)$ is unique, therefore

$$c_p^\alpha(\varepsilon)=c_1^\alpha(\varepsilon) \quad (\alpha=1, 2, \cdots, m),\ \tilde{c}_p(\varepsilon)=\tilde{c}_1(\varepsilon),$$

namely, in the present case, there does not exist any subperiodic solution[1] for p such that (4.17) holds.

When (4.18) holds but (4.17) does not hold for a certain positive integer p (>1), there exists a solution of (4.16), but is not necessarily unique for such p, because $\tilde{c}$ is in general not determined uniquely as (4.19).

When (4.17) holds but $J=0$ for $c^\alpha=c_0^\alpha\ (\alpha=1, 2, \cdots, m)$, there may not exist any solution of (4.16).

Thus, summarizing the above results, we have

Theorem 4. *When the three times continuously differentiable m-parameter partially oscillatory system* (1.1) *is perturbed as* (4.1), *consequently when the relative equation* (3.30) *is perturbed as* (4.2), *on the periodic solution of* (4.1), *we have the following conclusions:*

1° *when* (4.17) *holds and* (4.22) *has no real solution* c^α $(\alpha=1, 2, \cdots, m)$, *there exists no periodic solution of* (4.1);

2° *when* (4.17) *holds and* (4.22) *has a real solution* $c^\alpha=c_0^\alpha$ $(\alpha=1, 2, \cdots, m)$, *there exists a unique periodic solution of* (4.1) *provided* $J\neq 0$ *for* $c^\alpha=c_0^\alpha$ $(\alpha=1, 2, \cdots, m)$; *in this case, there exists no subperiodic solution for p for which* (4.17) *holds;*

3° *when* (4.17) *holds but* $J=0$ *for* $c^\alpha=c_0^\alpha$ $(\alpha=1, 2, \cdots, m)$, *there may exist no periodic solution of* (4.1) *even when* (4.22) *has a real solution* $c^\alpha=c_0^\alpha$ $(\alpha=1, 2, \cdots, m)$;

4° *when* (4.17) *does not hold but* (4.18) *holds, the same conclusions as* 1° *and* 3° *are valid and the conclusion of* 2° *is replaced by:*

provided $J\neq 0$ *for* $c^\alpha=c_0^\alpha$ $(\alpha=1, 2, \cdots, m)$, *there exists a periodic solution of* (4.1) *and there may also exist a subperiodic solution;*

5° *when* (4.18) *does not hold, there may or may not exist the periodic solution of* (4.1) *and it may not be unique when it may exist.*

5. Stability of the periodic solution of the perturbed system

In this paragraph, we study on the stability of the periodic solution of (4.1), whose existence is asserted by Theorem 4, namely on the stability of the periodic solution of (4.1) corresponding to

$$\{c^\alpha=c_1^\alpha(\varepsilon) \quad (\alpha=1, 2, \cdots, m),\ \tilde{c}=\varepsilon\tilde{c}_1(\varepsilon)\}.$$

---

1) The periodic solution corresponding to $(c_p^1, \cdots, c_p^m, \varepsilon\tilde{c}_p)$ different from $(c_1^1, \cdots, c_1^m, \varepsilon\tilde{c}_1)$ is called a *subperiodic solution*. Cf. **3.2** of [5].

For the subsequent discussion, a lemma on the eigenvalues of matrices will be proved.

Lemma 4. *When* $\det B(0)\neq 0$ *and* $|\varepsilon|$ *is sufficiently small, the eigenvalues of the continuous square matrix of the form*

$$\begin{pmatrix} \varepsilon A_1(\varepsilon) & A(\varepsilon) \\ \varepsilon B_1(\varepsilon) & B(\varepsilon) \end{pmatrix}$$

are the numbers of the forms

$$\varepsilon[\lambda+o(1)] \quad \text{and} \quad \mu+o(1)$$

as $\varepsilon\to 0$, *where* λ *and* μ *are respectively the eigenvalues of the matrices*

$$A_1(0)-A(0)B^{-1}(0)B_1(0) \quad \text{and} \quad B(0).$$

Proof. Since, by the assumption, there exists $B^{-1}(\varepsilon)$ for sufficiently small $|\varepsilon|$, put

$$T=E+Z=E+\begin{pmatrix} 0 & AB^{-1} \\ -\varepsilon B^{-1}B_1 & 0 \end{pmatrix},$$

then

$$\begin{aligned} T^{-1}&=(E+Z)^{-1} \\ &=E-Z+Z^2-Z^3+\cdots \\ &=\begin{pmatrix} E_1-\varepsilon AB^{-2}B_1 & -AB^{-1}+\varepsilon AB^{-2}B_1AB^{-1} \\ \varepsilon B^{-1}B_1 & E_2-\varepsilon B^{-1}B_1AB^{-1} \end{pmatrix}+O(\varepsilon^2)^{1)}. \end{aligned}$$

Therefore it follows that

$$T^{-1}\begin{pmatrix} \varepsilon A_1 & A \\ \varepsilon B_1 & B \end{pmatrix}T=\begin{pmatrix} \varepsilon(A_1-AB^{-1}B_1) & \varepsilon(A_1-AB^{-1}B_1)AB^{-1} \\ 0 & B+\varepsilon B_1AB^{-1} \end{pmatrix}+O(\varepsilon^2). \tag{5.1}$$

If we put

$$f(\rho,\varepsilon)=\det\begin{pmatrix} \varepsilon A_1-\rho E_1 & A \\ \varepsilon B_1 & B-\rho E_2 \end{pmatrix},$$

then

$$f(\rho,0)=\det(-\rho E_1)\cdot\det(B(0)-\rho E_2),$$

consequently the roots of

$$f(\rho,\varepsilon)=0,$$

namely the eigenvalues of the given matrix are the numbers of the forms $o(1)$ and $\mu+o(1)$ as $\varepsilon\to 0$.

If we write the eigenvalues of the forms $o(1)$ as $\varepsilon\rho$, then, from $f(\varepsilon\rho,\varepsilon)=0$, we have

$$\det\begin{pmatrix} \mathring{A}_1-\mathring{A}\mathring{B}^{-1}\mathring{B}_1-\rho E_1+o(1) & O(\varepsilon) \\ O(\varepsilon) & \mathring{B}+o(1) \end{pmatrix}=0, \tag{5.2}$$

because, from (5.1), $f(\varepsilon\rho,\varepsilon)$ can be written as follows:

1) E_1 and E_2 denote the unit matrices.

$$f(\varepsilon\rho,\varepsilon)=\det\begin{pmatrix}\varepsilon(\mathring{A}_1-\mathring{A}\mathring{B}^{-1}\mathring{B}_1-\rho E_1)+o(\varepsilon) & \varepsilon(\mathring{A}_1-\mathring{A}\mathring{B}^{-1}\mathring{B}_1)\mathring{A}\mathring{B}^{-1}+o(\varepsilon)\\ O(\varepsilon^2) & \mathring{B}+\varepsilon\mathring{B}_1\mathring{A}\mathring{B}^{-1}+o(1)-\varepsilon\rho E_2\end{pmatrix}.$$

Here $\mathring{A}_1$, $\mathring{A}$, $\cdots$ etc. denote respectively $A_1(0)$, $A(0)$, $\cdots$ etc. The equation (5.2) says that, as $\varepsilon\to 0$, ρ tends to a root of the equation

$$\det[(\mathring{A}_1-\mathring{A}\mathring{B}^{-1}\mathring{B}_1)-\rho E_1]=0,$$

namely one of λ's. This says that the eigenvalues of the form $o(1)$ of the given matrix are of the forms

$$\varepsilon[\lambda+o(1)].$$

Thus the lemma has been proved.

Now, the multipliers of the solutions of the equation of the first variation of (4.2) with respect to the periodic solution of (4.1) corresponding to $\{c^\alpha=c_1^\alpha(\varepsilon)\ (\alpha=1,2,\cdots,m),\ \tilde{c}=\varepsilon\tilde{c}_1(\varepsilon)\}$ are the characteristic roots of the matrix Φ whose elements are

$$\frac{\partial\varphi^i(\omega_0,c;\varepsilon)}{\partial c^j}\qquad(i,j=1,2,\cdots,n)$$

for $c^\alpha=c_1^\alpha(\varepsilon)$ $(\alpha=1,2,\cdots,m)$, $\tilde{c}=\varepsilon\tilde{c}_1(\varepsilon)$. But, from (4.4), Φ is written as follows:

$$\begin{pmatrix}E+\varepsilon\left(\sum_{\mu=m+1}^{n}\frac{\partial G'_\mu}{\partial c_0}c_1^\mu+\frac{\partial r'}{\partial c_0}\right)+o(\varepsilon) & G'(\omega_0)+o(1)\\ \varepsilon\left(\sum_{\mu=m+1}^{n}\frac{\partial G_\mu(\omega_0)}{\partial c_0}c_1^\mu+\frac{\partial r}{\partial c_0}\right)+o(\varepsilon) & G(\omega_0)+o(1)\end{pmatrix}$$

where $\dfrac{\partial G'_\mu}{\partial c_0}$, $\dfrac{\partial G_\mu(\omega_0)}{\partial c_0}$, $\dfrac{\partial r'}{\partial c_0}$ and $\dfrac{\partial r}{\partial c_0}$ denote the matrices whose elements are respectively $\dfrac{\partial G_\mu^\alpha(\omega_0)}{\partial c^\beta}$, $\dfrac{\partial G_\mu^\nu(\omega_0)}{\partial c^\beta}$, $\dfrac{\partial r^\alpha(\omega_0)}{\partial c^\beta}$ and $\dfrac{\partial r^\nu(\omega_0)}{\partial c^\beta}$ $(\alpha,\beta=1,2,\cdots,m;$ $\nu,\mu=m+1,\cdots,n)$ for $c^\alpha=c_1^\alpha(\varepsilon)$ $(\alpha=1,2,\cdots,m)$. Then, from (4.19), Φ can be written as

(5.3) $$\Phi=E+\Psi,$$

where

(5.4) $$\Psi=\begin{pmatrix}\varepsilon\left[-\sum_{\mu=m+1}^{n}\frac{\partial G'_\mu}{\partial c_0}(H^{-1}r(\omega_0))^\mu+\frac{\partial r'}{\partial c_0}+o(1)\right] & G'(\omega_0)+o(1)\\ \varepsilon\left[-\sum_{\mu=m+1}^{n}\frac{\partial H_\mu}{\partial c_0}(H^{-1}r(\omega_0))^\mu+\frac{\partial r}{\partial c_0}+o(1)\right] & H+o(1)\end{pmatrix}.$$

Here $H=G(\omega_0)-E$, $\partial H_\mu/\partial c_0=\partial G_\mu(\omega_0)/\partial c_0$ and $(H^{-1}(\omega_0))^\mu$ denotes the μ-th component of $H^{-1}r(\omega_0)$.

Let us consider the case where the eigenvalues of $G(\omega_0)=G_0$ are all smaller than 1 in absolute values. As is readily seen from (4.4), this means that the integral strip S^{m+1} of the unperturbed system (3.30) is completely stable[1]. In this case, the condition (4.18) is of course fulfilled, namely $\det H \neq 0$. Consequently, for sufficiently small $|\varepsilon|$, $\det(H+o(1)) \neq 0$ and $(H+o(1))^{-1}=H^{-1}+o(1)$. Then, by Lemma 4, for sufficiently small $|\varepsilon|$, the eigenvalues of Ψ are approximately equal to the eigenvalues of H and those of the matrix

$$\varepsilon\Big[-\sum_{\mu=m+1}^{n}\frac{\partial G'_\mu}{\partial c_0}(H^{-1}r(\omega_0))^\mu+\frac{\partial r'}{\partial c_0}-G'(\omega_0)H^{-1}\Big(-\sum_{\mu=m+1}^{n}\frac{\partial H_\mu}{\partial c_0}(H^{-1}r(\omega_0))^\mu+\frac{\partial r}{\partial c_0}\Big)\Big]$$

$$=\varepsilon\Big[-\sum_{\mu=m+1}^{n}\frac{\partial G'_\mu}{\partial c_0}(H^{-1}r(\omega_0))^\mu+G'(\omega_0)H^{-1}\sum_{\mu=m+1}^{n}\frac{\partial H_\mu}{\partial c_0}(H^{-1}r(\omega_0))^\mu-G'(\omega_0)H^{-1}\frac{\partial r}{\partial c_0}+\frac{\partial r'}{\partial c_0}\Big]$$

$$=\varepsilon\cdot\frac{\partial}{\partial c_0}[-G'(\omega_0)H^{-1}r(\omega_0)+r'(\omega_0)].$$

Here the matrix $\frac{\partial}{\partial c_0}[-G'(\omega_0)H^{-1}r(\omega_0)+r'(\omega_0)]$ is a Jacobian matrix of the left-hand side of (4.22) with respect to c^α $(\alpha=1, 2, \cdots, m)$. Therefore, if we denote this matrix for the value $c^\alpha=c_0^\alpha$ $(\alpha=1, 2, \cdots, m)$ by J_0, it follows that, for sufficiently small $|\varepsilon|$, the eigenvalues of Ψ are approximately equal to the eigenvalues of H and of εJ_0. Then, from (5.3), it is seen that the eigenvalues of Φ are approximately equal to the eigenvalues of G_0 and the eigenvalues of εJ_0 increased by 1. Thus, if the real parts of the eigenvalues of εJ_0 are all negative, the eigenvalues of Φ become all smaller than 1 in absolute values because of the assumption on the eigenvalues of G_0[2]. Thus, on the stability, we have

Theorem 5. *When the eigenvalues of G_0 are all smaller than 1 in absolute values and the real parts of the eigenvalues of εJ_0 are all negative, there exists a unique periodic solution of the perturbed system* (4.1) *and this periodic solution is stable with the integral strip S^{m+1} of the unperturbed system. Here J_0 is a Jacobian matrix of the left-hand side of* (4.22) *with respect to c^α $(\alpha=1, 2, \cdots, m)$ for the value $c^\alpha=c_0^\alpha$ $(\alpha=1, 2, \cdots, m)$.*

When $\det(G_0-E)\neq 0$ and $J\neq 0$ for $c^\alpha=c_0^\alpha$ $(\alpha=1, 2, \cdots, m)$, if some of the eigenvalues of G_0 are greater than 1 in absolute values, or if the real parts of some of the eigenvalues of εJ_0 are positive, there exists a unique periodic solution of the perturbed system (4.1), but it can not be stable, for, in this case, some of the eigenvalues of Φ become greater than 1 in absolute values.

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1) The stable periodic solution is called *completely stable* when the multipliers of solutions of the equation of the first variation with respect to that periodic solution are all smaller than 1 in absolute values.

2) Because, for any complex number $a+ib$, we have

$$|1+\varepsilon(a+ib)|=\sqrt{1+2\varepsilon a+\varepsilon^2(a^2-b^2)}=1+\varepsilon a+O(\varepsilon^2).$$

6. Remarks

Except for slight generality on the order of periodic solutions, Theorem 4 is the same as Theorem 13 of the preceding paper [5]. However the method of the present paper is different from, and far simpler than, that of the preceding paper. On account of this simplicity, we could find Theorem 5—the criteria of stability of the periodic solution of the perturbed system, which has not been obtained in the preceding paper.

But, for actual calculation of periodic solutions of the perturbed system, the method of the preceding paper will probably be preferable to the method of the present paper, for the reduction of the relative equation of the paths to the form (4.2) will not be so simple for actual calculation.

Thus it may be said that *the method of the preceding paper is suited to actual calculation and the method of the present paper is suited to theoretical research.*

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