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On Convolutions in the Theory of Distributions

By

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The object of this paper is to carry forward the study concerning convolutions due to K. Yoshinaga and H. Ogata [*4*]. The notations used there will be adopted here without explicit references. It was shown there that if E is an admissible barrelled space, then a distribution T belongs to the c-dual E^* (the set of distributions composable with any distribution of E) only if $T*\varphi\in\check{E}'$ for any $\varphi\in(\mathcal{D})$. If this is also sufficient for T to belong to E^*, E is termed c-regular. It was verified ([*4*], Theorem *5*) that usual spaces of distributions discussed in [*3*] are c-regular. In order to give assurances for this situation we show (Theorem *2*) that any permitted barrelled space E with $E*(\mathcal{D})\subset E$ is c-regular if and only if $\check{E}'\subset E^*$. For certain spaces of distributions in [*3*] it is known that the relation $E^*=\check{E}'$ holds, e.g. $(\mathcal{D})^*=(\mathcal{D}')$, $(\mathcal{E})^*=(\mathcal{E}')$, $(\mathcal{S})^*=(\mathcal{S}')$, $(\mathcal{D}_+)^*=(\mathcal{D}'_+)$. We seek for the conditions imposed on E from which these situations arise (Theorem *3*, Theorem *4*). Finally we show that if E is any permitted reflexive space such that $\check{E}'=E^*$ and $E\bar{*}\check{E}'\subset\check{E}'$, then $E\bar{*}E^{**}\subset E$, E^{**} is an algebra over which E^* is a module, and $T\in E^{**}$ if and only if $T*\varphi\in E$ for any $\varphi\in(\mathcal{D})$. The hypotheses made here are certainly satisfied for $(\mathcal{S})$. In §*1* we give some remarks on permitted spaces. We show (Theorem *1*) that any sequentially continuous application of a permitted barrelled space into any locally convex space is continuous.

1. By an *admissible space* we mean any locally convex space E of distributions with properties:

(i) $(\mathcal{D})\subset E\subset(\mathcal{D}')$, *where the injections* $(\mathcal{D})\to E$ *and* $E\to(\mathcal{D}')$ *are continuous.*

(ii) $(\mathcal{D})$ *is dense in* E *in the topology of* E.

If E is admissible, any element $\xi\in E'$ is identified with a distribution T by the relation $\xi(\varphi)=\langle T,\varphi\rangle$, $\varphi\in(\mathcal{D})$. We write also $\xi(S)=\langle T,S\rangle$ for any $S\in E$. Unless otherwise stated, E' is assumed to be a locally convex space with the strong topology. It is then clear that (i) holds true for E', but it is not necessary that (ii) does. $E'^{\cdot}$, named *admissible part of* E', stands for the closed subspace of E' generated by $(\mathcal{D})$. It is noted that E' is admissible if E is semi-reflexive and therefore $E'=E'^{\cdot}$. E is termed *permitted* provided besides (i) and (ii), the following condition (iii) is satisfied ([4]):

(iii) *For any* $T \in E$, $(\alpha_k \cdot T) * \rho_k$ *and* $\alpha_k \cdot (T * \rho_k)$ *converge to* T *as* $k \to \infty$, *where* $\{\alpha_k\}$ *is a sequence of multiplicators and* $\{\rho_k\}$ *is a sequence of regularizations.*

For a permitted space E we have

LEMMA *1. Let E be a permitted space with any one of the properties:*

(1) E is semi-complete.

(2) E is semi-reflexive.

(3) E is barrelled.

Then $E'^{\cdot}$ *is also permitted.*

PROOF. We note that under the hypotheses any $\sigma(E, E)$-bounded subset of E' is also bounded in E'. Now if we put $\psi_k(x) = (\alpha_k \cdot x) * \rho_k$ for any $x \in E$, then ${}^t\psi_k(x') = \alpha_k(x' * \rho_k) \to x'$ in the topology $\sigma(E', E)$ as $k \to \infty$, and therefore $\{{}^t\psi_k(x')\}$ is bounded for any $x' \in E'$, which implies that the set $A = \{\bigcup_k \psi_k(B)\} \cup B$ is bounded for any bounded subset B of E. We define a semi-norm $\|x'\|_B$, $x' \in E'$, by

$$\|x'\|_B = \underset{x \in B}{\text{l.u.b.}} |\langle x', x \rangle|.$$

By definition, for any given positive number ε, there exists $\varphi \in (\mathcal{D})$ such that $\|x' - \varphi\|_A \leqslant \varepsilon$ for any fixed $x' \in E'^{\cdot}$. Then

$$\|{}^t\psi_k(x') - x'\|_B \leqslant \|{}^t\psi_k(x') - {}^t\psi_k(\varphi)\|_B + \|{}^t\psi_k(\varphi) - \varphi\|_B + \|\varphi - x'\|_B \leqslant 2\varepsilon + \|{}^t\psi(\varphi) - \varphi\|_B.$$

It follows since the injection $(\mathcal{D}) \to E'$ is continuous and $\|{}^t\psi_k(x) - \varphi\|_B \to 0$ in $(\mathcal{D})$ as $k \to \infty$ that

$$\overline{\lim_{k \to \infty}} \|{}^t\psi_k(x') - x'\|_B \leqslant 2\varepsilon,$$

ε being arbitrary so that ${}^t\psi_k(x') \to x'$ in E' as $k \to \infty$.

In a similar way we can show that $(\alpha_k \cdot x') * \rho_k \to x'$ in E' as $k \to \infty$. Thus the proof is complete.

Let E be a permitted space. If E is barrelled or a semi-complete space of type (**DF**), then any linear application $\mathcal{L}$ of E into any locally convex space is continuous if and only if $\mathcal{L}$ is sequentially continuous. This follows from the theorem *1* below since for any permitted space the endomorphisms ψ_k of the theorem are given by $\psi_k(x) = (\alpha_k \cdot x) * \rho_k$. In fact, ψ_k as an application of E into $(\mathcal{D})$ and the injection $(\mathcal{D}) \to (E)$ are continuous. By the aid of the theorem of Banach-Steinhaus applied to the applications $\psi_k(k) = (\alpha_k \cdot x) * \rho_k$, we see that if we let E with topology $\mathfrak{T}$ be a permitted barrelled space and if we assume that E with another topology $\mathfrak{T}'$ is permitted, then $\mathfrak{T}$ is stronger than $\mathfrak{T}'$, so that $\mathfrak{T}$ is the unique topology with which E is a permitted barrelled space.

THEOREM 1. *Let E be a locally convex space and let $\{\psi_k\}$ be a sequence of the endomorphisms of E with properties:*

(1) $\psi_k(x) \to x$ in E for any $x \in E$,

(2) $\psi_k = \alpha_k \circ \beta_k$ for any positive integer k, where β_k is a continuous linear application of E into a bornological space P_k and α_k a continuous linear application of P_k into E.

If E is a barrelled or a semi-complete space of type (DF), *then*

(i) *any sequentially continuous linear application ϕ of E into any locally convex space F is continuous,*

(ii) *the strong dual E' of E is complete,*

(iii) ([*2*], p. *109*) *if $\psi_k(x) \to x$ holds in Mackey's sense for any $x \in E$, then E is bornological.*

PROOF. *ad* (i). The continuity of ϕ_k defined by $\phi_k(x) = \phi(\psi_k(x))$, $x \in E$, $k = 1,2,3,\cdots$, follows from the fact that $\phi \circ \alpha_k$ is continuous since it is bounded linear from P_k into F. Now, by the hypotheses, E is a barrelled space or a semi-complete space of type (***DF***), which allows us to apply the theorem of Banach-Steinhaus to conclude that $\phi(x) = \lim_{k\to\infty} \phi_k(x)$ is continuous.

ad (ii). It suffices to show the statement for the case where E is barrelled, because the dual of a space of type (DF) is a space of type (F) and *a fortiori* complete. Let $\hat{E}'$ be the completion of E'. Then any $x' \in \hat{E}'$ as a functional is continuous on any bounded subset of E. This implies that x' is sequentially continuous, and therefore, by (i), implies that x' is an element of E'.

ad (iii). Let F be any locally convex space. It is enough to prove that every linear application ϕ which transforms any bounded subset of E into a bounded subset of F is continuous. Owing to the hypothesis that $\psi_k(x) \to x$ holds in Mackey's sense for any $x \in E$, it follows that $\phi\{\psi_k(x)\} \to \phi(x)$ in F. As proved in (i), $\phi\{\psi_k(x)\}$ is continuous, and so does $\phi(x)$. The proof is complete.

It follows immediately from this theorem that any permitted reflexive space is complete. We also note that if E is a permitted barrelled space then a distribution T belongs to E' if and only if $\lim_{k\to\infty} \langle S, (\alpha_k T) * \rho_k \rangle$ exists and is finite for any $S \in E$.

2. It was shown by K. Yoshinaga and H. Ogata [*4*] that, for any admissible barrelled space E, a distribution T belongs to E^* only if

$$(\alpha) \qquad \check{T} * \varphi \in E' \quad \text{for any} \quad \varphi \in (\mathcal{D}).$$

E is called c-regular if (α) is sufficient for T to belong to E^*. It is known that usual spaces of distributions discussed in [*3*] are c-regular. In order to give assurances for this situation we give

THEOREM 2. *Let E be an admissable barrelled space. If $\check{E}' \subset E^*$ holds, then E is c-regular and $E^* = \check{E}'^{**}$. If E is a permitted barrelled space with $E*(\mathcal{D}) \subset E$, then E is c-regular if and only if $\check{E}' \subset E^*$.*

PROOF. Let T be any distribution satisfying (α). Then the condition $\check{E}' \subset E^*$ implies that $T*\varphi \in \check{E}' \subset E^*$ for any $\varphi \in (\mathcal{D})$ and therefore, by a theorem of C. Chevalley ([1], p. 119), $T \in E^*$, whence E is c-regular. To complete the proof of the first statement it is sufficient to show that $E^* \subset \check{E}'^{**}$ because $E^* \supset \check{E}'^{**}$ follows from $E \subset \check{E}'^{*}$. For any $T \in E^*$ we see that $T*\varphi \in \check{E}' \subset \check{E}'^{**}$ for any $\varphi \in (\mathcal{D})$, and in turn by a theorem of C. Chevalley cited just above we have $T \in E'^{**}$ that is, $E^* \subset \check{E}'^{**}$, as desired. Now turn to the proof of the second statement. Assume that E is c-regular. Any $T \in \check{E}'$ uniquely determines a continuous linear form ξ on E such that $\xi(\varphi) = \langle \check{T}, \varphi \rangle$ for and $\varphi \in (\mathcal{D})$. Since the linear endomorphism $S \in E \rightarrow S*\varphi \in E$ for any fixed $\varphi \in (\mathcal{D})$ is continuous ([4], Theorem 2) so that a linear form $\eta(S) = \xi(S*\varphi)$ defines an element of the dual of E. Since $\eta(\psi) = \xi(\psi*\varphi)$ $= \langle \check{T}, \psi*\varphi \rangle = \langle \check{T}*\check{\varphi}, \psi \rangle$ for and $\psi \in (\mathcal{D})$, there corresponds to η a distribution $\check{T}*\check{\varphi} \in E'$. Then that E is c-regular implies $T \in E^*$, where $\check{E}' \subset E^*$, completing the proof.

Next we show

THEOREM 3. *Let E be an admissible space of type* (F) *and such that $\check{E}' \subset E^*$ and $D \cdot E \subset E$ for any differential operators $D = D_{x_i}$, $i = 1, \cdots, n$, then $E^* = \check{E}'$ holds.*

PROOF. We have only to show that $E^* \subset \check{E}'$ since $E^* \supset \check{E}'$ holds from our hypotheses. Suppose $T \in E^*$. The linear application $\varphi \rightarrow \check{T}*\varphi$ is continuous from $(\mathcal{D})$ into E' ([4], Theorem 2), so that the bilinear form $\langle T*\varphi, S \rangle$, $S \in E$, is separately continuous on $(\mathcal{D}) \times E$, whence continuous on $(\mathcal{D}_{\bar{\Omega}}) \times E$, since $(\mathcal{D}_{\bar{\Omega}})$ and E are spaces of type (F), where Ω is any relatively compact neighbourhood of the origin of R^n. This means that there exists a positive integer k, a neighbourhood U of zero in $(\mathcal{D}_{\bar{\Omega}}^k)$, and a neighbourhood V of zero in E such that

$$(1) \qquad |\langle T*(U \frown (\mathcal{D}_{\bar{\Omega}})), V \rangle| \leqq 1.$$

As usually done, choose a positive integer m, function $u \in (\mathcal{D}_{\Omega}^k)$ and $\eta \in (\mathcal{D}_{\bar{\Omega}})$ such that $\delta = \Delta^m u + \eta$. We can now take a sequence $\varphi_j \in (\mathcal{D}_{\bar{\Omega}})$ such that $\varphi_j \rightarrow u$ in $(\mathcal{D}_{\bar{\Omega}})$. Put for any $S \in E$

$$(2) \qquad \xi_j(S) = \chi_j(S) + \langle \check{T}*\eta, S \rangle,$$

where $\chi_j(S) = \langle \check{T}*\varphi_j, \Delta^m S \rangle$.

It follows from (1) that $\chi_j(S)$ simply converges to a limit, which we denote by $\chi(S)$, as $j \rightarrow \infty$. Indeed, $\alpha \Delta^m S \in V$ for a suitably chosen $\alpha > 0$

depending on S, and for sufficiently large j and l we have $\varphi_j-\varphi_l \in \varepsilon U$, ε being a given positive number, we have

$$|\langle \check{T}*\varphi_j, \Delta^m S\rangle - \langle \check{T}*\varphi_l, \Delta^m S\rangle| = |\langle \check{T}*(\varphi_j-\varphi_l), \Delta^m S\rangle| < \varepsilon/\alpha.$$

While the linear endomorphism $S\in E \to \Delta^m S \in E$ is continuous ([4], Theorem 2), and therefore $\chi_j(S)$ is continuous and in turn, by the theorem of Banach-Steinhaus, $\chi(S)$ is continuous since E is barrelled. Let

$$(3) \qquad \xi(S) = \chi(S) + \langle \check{T}*\eta, S\rangle,$$

then $\xi(S)$ is a continuous linear form, to which corresponds $\check{T}$. Indeed

$$\begin{aligned}\xi(\varphi) &= \lim_{j\to\infty} \langle \check{T}*\varphi_j, \Delta^m\varphi\rangle + \langle \check{T}*\eta, \varphi\rangle \\ &= \lim_{j\to\infty} \langle \check{T}, \check{\varphi}_j * \Delta^m\varphi + \check{\eta}*\varphi\rangle \\ &= \langle \check{T}, \check{u}*\Delta^m\varphi + \check{\eta}*\varphi\rangle \\ &= \langle \check{T}, (\Delta^m u+\eta)^\vee * \varphi\rangle \\ &= \langle \check{T}, \varphi\rangle.\end{aligned}$$

This shows that $\check{T}\in E'$, and therefore $T\in \check{E}'$ completing the proof.

Applying the theorem to the spaces $(\mathcal{E})$ and $(\mathcal{S})$, we obtain the well known results $(\mathcal{E})^* = (\mathcal{E}')$ and $(\mathcal{S})^* = (\mathcal{S}')$. Any space E of distributions considered in Theorem 3 is contained in $(\mathcal{E})$. In fact, this is an immediate consequence of the following

THEOREM 4. *Let E be any barrelled space of distributions such that the injection $E\to(\mathcal{D}')$ is continuous and $\Delta E\subset E$. Assume that there exists for any $T\in E$ a sequence of positive integers λ_k depending on each T such that $\{\lambda_k \Delta^k T\}$ is bounded in E.*
Then

(i) $E\subset(\mathcal{E})$

(ii) *S belongs to E^* if and only if $T(\check{S}*\varphi)\in L^1$ for any $T\in E$ and $\varphi\in(\mathcal{D})$.*

PROOF. Suppose $T\in E$. For any $S\in E^*$, we have

$$(1) \qquad (T*\varphi)\cdot(\check{S}*\psi)\in L^1 \quad \text{for any} \quad \varphi, \psi\in(\mathcal{D}).$$

Consider the bilinear application $B(T,\varphi) = (T*\varphi)(\check{S}*\psi)$, which is hypocontinuous on $E\times(\mathcal{D})$ ([4], Theorem 2). Let A be the bounded subset $\{\lambda_k \Delta^k T\}$. There exists then a neighbourhood U of zero of $(\mathcal{D}_{\bar{\Omega}}^k)$ such that

$$(2) \qquad B(A, U\frown(\mathcal{D}_{\bar{\Omega}}))\subset \textit{the unit sphere of } L^1.$$

As in the proof of the preceding theorem 3, we choose $u\in(\mathcal{D}_\Omega{}^k)$, $\eta\in(\mathcal{D}_\Omega)$ and $\varphi_j\in(\mathcal{D}_\Omega)$ such that $\delta = \Delta^m u+\eta$ and $\varphi_j\to u$ in $(\mathcal{D}_{\bar{\Omega}}^k)$. It follows from (2) that $(\Delta^m T*\varphi_l)(\check{S}*\psi)\to(\Delta^m T*u)(\check{S}*\psi)$ in L^1 as $j\to\infty$. This yields that $T(S*\varphi) = (T*\Delta^m u)(\check{S}*\psi) + (T*\eta)(\check{S}*\psi)\in L^1$. Since we may take S and ψ so that $S*\psi$ is 1 on any given compact subset of R^n, and therefore T is a function, whence so does any $\Delta^k T$ since $\Delta^k T \in E$, $k=1,2,\cdots$. Hence $T \in (\mathcal{E})$. Thus (i) is proved.

The "*only if*" part of (ii) is already proved above. We shall now show the "*if*" part. The application $h \to \tau_h \psi$ is continuous from R^n into $(\mathcal{D})$, and therefore $h \to (\tau_h T)(S * \psi)$ is continuous from R^n into L^1 ([*4*], Theorem *2*), and in turn (*1*) holds, as desired.

COROLLARY. *There is no admissible Banach space E with $\Delta E \subset E$.*

PROOF. Suppose the contrary. The dual space E' is a Banach space such that the injection $E' \to (\mathcal{D}')$ is continuous. We show that $\Delta E' \subset E'$. Since the application $S \in E \to \Delta S \in E$ is continuous ([*4*], Theorem *2*), it follows that the linear form $\xi_T(S) = \langle T, \Delta S \rangle$, $S \in E$, for any $T \in E'$ is continuous. It is easy to verify that the distribution identified with ξ is ΔT, and therefore $\Delta T \in E'$, as desired. Then both E and E' satisfy the hypothesis of the preceding theorem *4*, so that $E, E' \subset (\mathcal{E})$. While the injection $E \to (\mathcal{D}')$ is continuous and so does the injection $E \to (\mathcal{E})$ ([*4*], Theorem *2*), which implies $(\mathcal{E}') \subset E'$ and in turn $(\mathcal{E}') \subset (\mathcal{E})$, a contradiction. The proof is complete.

A locally convex space E is termed *a space of type* (LF) provided that it is algebraically a union of an increasing sequence of space E_j of type (F) such that injections $E_j \to E_{j+1}$ are continuous, and furthermore that the topology of E is the inductive limit of those of E_j. The sequence $\{E_j\}$ is called *a sequence of definition of E*. We note that any bounded subset of E is contained in some E_j and bounded there if E is semi-complete. For we may suppose that B is a absolutely convex closed subset of E. The normed space E_B of elements of E absorbed by B with norm $\|x\| = \text{g.l.b.} \{\lambda : x \in \lambda B, \lambda > 0\}$ is a Banach space since E is semi-complete. Consider the injection $E_B \to E$. Then owing to a theorem of Grothendieck ([*2*], Théorème A, p. *16*) we see that $E_B \subset$ some E_j and B is bounded in E_j, as desired.

THEOREM *5*. *Let E be any semi-complete permitted space of type* (LF) *with (E_i) as a sequence of definition of E and assume that*

(*1*) $\check{E}' \subset E^*$

(*2*) *there exists for each E_i an E_l such that $DE_i \subset E_l$ for any derivation D of arbitrary order. Then $E^* = \check{E}'$ holds true.*

PROOF. Suppose $T \in E^*$. To prove $\check{T} \in E'$, it is sufficient to show that $\lim_{k \to \infty} \langle \check{T}, (\alpha_k \cdot S) * \rho_k \rangle$ exists since $\langle \check{T}, (\alpha_k \cdot S) * \rho_k \rangle$ is continuous. Let l_i be the least positive integer l satisfying (*2*). The bilinear from $B(\varphi, S) = \langle \check{T} * \varphi, S \rangle$ is continuous on $(\mathcal{D}_{\bar{\Omega}}) \times E_{l_i}$ since the application $\varphi \in (\mathcal{D}) \to \check{T} * \varphi \in E'$ is continuous ([*4*], Theorem *3*). As shown in the proof of Theorem *3* we can now choose $u^{(i)} \in (\mathcal{D}_\Omega{}^{k_i})$, $\eta^{(i)} \in (\mathcal{D}_\Omega)$, $\varphi_j^{(i)} \in (\mathcal{D}_\Omega)$ in such a way that

(3) $$\delta=\Delta^{m_i}u^{(i)}+\eta^{(i)},$$

(4) $$\varphi_j^{(i)}\to u^{(i)} \text{ in } (\mathcal{D}_{\bar{\Omega}}^{k_i}) \text{ as } j\to\infty,$$

(5) $$\chi_j^{(i)}=\langle \check{T}*\varphi_j^{(i)}, \Delta^{m_i}S\rangle, \ S\in E_i,$$

converges uniformly to a limit denoted by $\chi^{(i)}(S)$ on any bounded subset of E_i as $j\to\infty$.

Put

$$\xi_j^{(i)}(S)=\chi_j^{(i)}(S)+\langle \check{T}*\eta^{(i)}, S\rangle$$

and

$$\xi^{(i)}(S)=\chi^{(i)}(S)+\langle \check{T}*\eta^{(i)}, S\rangle.$$

Then $\xi_j^{(i)}(S)\to\xi^{(i)}(S)$ uniformly on any bounded subset of E_i as $j\to\infty$.

Now let S be any fixed element of E. Take i so large that the set $(\alpha_k\cdot S)*\rho_k$ is bounded in E_i. Then $\xi_j^{(i)}(\alpha_k\cdot S*\rho_k)\to\xi^{(i)}(\alpha_k\cdot S*\rho_k)$ uniformly as $j\to\infty$. This implies that $\xi^{(i)}(\alpha_k\cdot S*\rho_k)\to\lim\limits_{j\to\infty}\lim\limits_{k\to\infty}\xi_j^{(i)}(\alpha_k S*\rho_k)=\xi^{(i)}(S)$, as $k\to\infty$. While simple calculation shows us that $\xi^{(i)}((\alpha_k\cdot S)*\rho_k)=\langle \check{T}, (\alpha_k\cdot S)*\rho_k\rangle$, which proves the theorem.

LEMMA 2. *Let E be an admissible barrelled space, then the relation $E^{**}\supset E'^{\cdot\prime}$ holds.*

PROOF. Let T be any element of E^*. Then

$$\mathcal{L}(S)=S*\varphi\cdot\check{T}*\psi\in L^1, \ S\in E,$$

for any $\varphi, \psi\in(\mathcal{D})$. $\mathcal{L}$ is continuous from E into L^1 ([4], Theorem 2). Let us denote by $\mathcal{L}_1$, the restriction of ${}^t\mathcal{L}$ on the subspace C_0^∞ of L^∞, where C_0^∞ is the set of continuous function on R^n vanishing at infinity. Then for any $f\in C_0^\infty$ we have

$$\begin{aligned}\langle \mathcal{L}_1(f), S\rangle&=\langle f, \mathcal{L}(S)\rangle\\&=\int S*\varphi(x)\cdot\check{T}*\psi(x)\cdot f(x)\,dx.\end{aligned}$$

From this we have $\mathcal{L}_1(\chi)=\{(\check{T}*\psi)\cdot\chi\}*\check{\varphi}$, for any $\chi\in(\mathcal{D})$. Let $\{\xi_j\}$ be a sequence of elements of $(\mathcal{D})$ such that $\xi_j\to f$ in C_0^∞ as $j\to\infty$. Then $\{\mathcal{L}_1(\xi_j)\}$ converges to $\mathcal{L}_1(f)$ and $\mathcal{L}_1(\xi_j)\in(\mathcal{D})$, whence $\mathcal{L}_1(f)$ belongs to $E'^{\cdot}$, and $\mathcal{L}_1(f)=\{(T*f)\cdot\chi\}*\check{\varphi}$. Thus $\mathcal{L}_1$ is continuous from C_0^∞ into $E'^{\cdot}$. Then for any $U\in E'^{\cdot\prime}$ we see that ${}^t\mathcal{L}_1(U)$ is a bounded measure on R^n.

$$\begin{aligned}\langle {}^t\mathcal{L}_1(U), \chi\rangle&=\langle U, \mathcal{L}_1(\chi)\rangle\\&=\langle U, \{(\check{T}*\psi)\cdot\chi\}*\check{\varphi}\rangle\\&=\int (U*\varphi)(x)\cdot(\check{T}*\psi)(x)\chi(x)\,dx.\end{aligned}$$

Therefore

$${}^t\mathcal{L}_1(U)=U*\varphi\cdot\check{T}*\psi\in L^1,$$

that is, $U\in E^{**}$, completing the proof.

From this lemma we have immediately

COROLLARY. *Let E be a c-reflexive admissible barrelled space, then $E=E''$. Therefore if E' is also admissible, then E is reflexive.*

LEMMA *3*. *Let E be an admissible barrelled space such that $\check{E}'\subset E^*$. If E'' is a barrelled space, then $E^{**}=E''^*$, and $E^{**}=\{T: T*(\mathcal{D})\in E''\}$.*

PROOF. $\check{E}'\subset E^*$ implies that $\check{E}''\subset E^*$ and therefore $\check{E}''^*\supset E^{**}$. To prove the inverse implication, let T be any element of $\check{E}''^*$, then $T*\varphi\in E''$ for any $\varphi\in(\mathcal{D})$ and in turn $T*\varphi\in E^{**}$ since $E''\subset E^{**}$ holds by Lemma *2*. It now follows from a theorem of C. Chevalley ([*1*], p. *119*) that $T\in E^{**}$, as desired. $E''\subset E^{**}=\check{E}''^*$ implies that E'' is c-regular, and therefore $T\in E^{**}$ holds if and only if $T*\varphi\in E''$ holds for any $\varphi\in(\mathcal{D})$. This completes the proof.

From these lemmas we have immediately

THEOREM *6*. *Let E be an admissible barrelled space with admissible dual and such that $\check{E}'\subset E^*$. Then E is c-reflexive if and only if (1) E is reflexive and (2) $T*(\mathcal{D})\in E$ implies $T\in E$.*

PROOF. Let E be c-reflexive. Then $E''\subset E^{**}=E$ implies that E is reflexive. (*2*) follows from Lemma *3*. Suppose that (*1*) and (*2*) hold. For any $T\in E^{**}$ it follows from Lemma *3* that $T*(\mathcal{D})\in E$, and therefore by (*2*) $T\in E$, completing the proof.

THEOREM *7*. *Let E be any permitted barrelled space of type* (DF) *such that $\check{E}'\subset E^*$ and $DE\subset E$ for any derivation D, then $E^{**}=E''$.*

PROOF. $D\check{E}''\subset\check{E}''$ for any derivation D. In fact the application $S\in E\rightarrow DS\in E$ is continuous by Theorem *2* of [*4*], and therefore ${}^tD$ is a continuous endomorphism of E', whence $DE''\subset E''$ and in turn $D\cdot\check{E}''\subset\check{E}''$. Owing to Lemma *3* we have $E''\subset E^{**}=\check{E}''^*$. We can now apply Theorem *3* to conclude that $\check{E}''^*=E''$. Therefore $E^{**}=E''$, as desired.

LEMMA *4*. *Let F be an admissible barrelled space and G a permitted reflexive space with $\check{G}'\subset G^*$, such that, for a given distribution T, $T\in F^*$ and $T\bar{*}F\subset G$ hold true. Then $T\in\check{G}'^*$ and $T\bar{*}\check{G}'\subset\check{F}'$.*

PROOF. The linear application $\mathcal{L}(S)=T\bar{*}S$ is continuous from F into G ([*4*], Theorem *2*), and $\mathcal{L}(\tau_h\varphi)=\tau_h\mathcal{L}(\varphi)$ for any translation τ_h and any $\varphi\in(\mathcal{D})$, that is, $\mathcal{L}$ is a composition operator with range in G. Then the conjugate application ${}^t\mathcal{L}$ is also a composition operator of G' into F'. Hence ${}^t\mathcal{L}(W)=T_1*W$ for any $W\in G'$, where T_1 is the distribution associated with ${}^t\mathcal{L}$ ([*4*], Theorem *4*). It is easy to see that $T_1=\check{T}$, and therefore $T\in\check{G}'^*$ and $T\bar{*}\check{G}'\in\check{F}'$, as desired.

Finally we show

Lemma 5. *Let E, F, G be permitted reflexive spaces such that $\check{E}' \subset E^*$, $\check{F}' \subset F^*$, $\check{G}' \subset G^*$, $E \subset F^*$ and $E \overline{*} F \subset G$. Then*

(*1*) $E^{**} \overline{*} F^{**} \subset G^{**}$,

(*2*) $E \overline{*} F^{**} \subset G$ *if* $E^* = \check{E}'$ *holds.*

Proof. *ad* (*1*). Let $S \in E^{**}$ and $T \in F^{**}$. By Lemma *3*, $S * \varphi \in E$, $T * \psi \in F$ for any $\varphi, \psi \in (\mathscr{D})$. It follows since $E \overline{*} F \subset G$ that S, T are composable and $(S \overline{*} T) * \varphi * \psi = (S * \varphi) \overline{*} (T * \psi) \in G$, whence $(S \overline{*} T) * \varphi \in G^{**}$ by Lemma *3* and in turn $S \overline{*} T \in G^{**}$, completing the proof.

ad (*2*). It follows from Lemma *4* that $E \overline{*} F \subset G$ implies $\check{G}' \overline{*} F \subset \check{E}'$. Then by (*1*) we have $\check{G}' \overline{*} F^{**} \subset \check{E}'^{**}$, and therefore $\check{G}' \overline{*} F^{**} \subset \check{E}'$ since $\check{E}'^{**} = E^{***} = E^* = \check{E}'$. Now by Lemma *4* we obtain $E \overline{*} F^{**} \subset G$, completing the proof.

Theorem 8. *Let E be a permitted reflexive space such that $\check{E}' \subset E^*$ and $E \overline{*} \check{E}' \subset \check{E}'$. Then under convolution as multiplication*

(*1*) *E forms an algebra and $\check{E}'$ is an E-module.*

(*2*) *E^{**} forms an algebra $\subset E^*$ and E^* is an E^{**}-module.*

(*3*) $E \overline{*} E^{**} \subset E$ *if* $\check{E}' = E^*$.

Proof. *ad* (*1*). By Lemma *4*, $E \overline{*} E \subset E$. Then for the proof of (*1*) it suffices to show that $S \overline{*} (T \overline{*} U) = (S \overline{*} T) \overline{*} U$ for any $S, T \in E$ and any $U \in E$ or $\in \check{E}'$. Put $\mathscr{L}(S) = (S \overline{*} T) \overline{*} U$ for fixed T and S. Clearly $\mathscr{L}$ is a composition operator. Let W be the distribution associated with $\mathscr{L}$ such that $\mathscr{L}(S) = S \overline{*} W$ ([*4*], Theorem *4*). Then $W * (\mathscr{D}) = ((\mathscr{D}) * T) \overline{*} U = (T \overline{*} U) * (\mathscr{D})$. This yields $W = T \overline{*} U$, which completes the proof.

ad (*2*). It follows from Lemma *5* that $E \overline{*} \check{E}' \subset \check{E}'$ implies $E^{**} \overline{*} \check{E}'^{**} \subset \check{E}'^{**}$, and therefore $E^{**} \overline{*} E^* \subset E^*$ since $E^* = \check{E}'^{**}$ holds by Theorem *2*. As E^* contains the Dirac measure δ, so we obtain $E^{**} \subset E^*$. Now, for the end of the proof of (*2*) it suffices to show that

$$(T_1 \overline{*} T_2) \overline{*} S = T_1 \overline{*} (T_2 \overline{*} S)$$

for any $T_1, T_2 \in E^{**}$ and any $S \in E^{**}$ or $\in E^*$. Let φ, ψ, χ be any elements of $(\mathscr{D})$. Then

$$\begin{aligned}\{(T_1 \overline{*} T_2) \overline{*} S\} * \varphi * \psi * \chi \\ &= \{(T_1 \overline{*} T_2) * \varphi * \psi\} \overline{*} (S * \chi) \\ &= \{(T_1 * \varphi) \overline{*} (T_2 * \psi)\} \overline{*} (S * \chi),\end{aligned}$$

where $T_1 * \varphi$, $T_2 * \psi \in E$ and $S * \chi \in E$ or $\in \check{E}$ according to the cases $S \in E^{**}$ or $S \in E^*$. We then obtain from (*1*)

$$\begin{aligned}\{(T_1*\varphi)\bar{*}(T_2*\psi)\bar{*}(S*\chi)\\ &=(T_1*\varphi)\bar{*}\{(T_2*\psi)\bar{*}(S*\chi)\}\\ &=\{T_1\bar{*}(T_2\bar{*}S)\}*\varphi*\psi*\chi.\end{aligned}$$

Hence $(T_1\bar{*}T_2)\bar{*}S=T_1\bar{*}(T_2\bar{*}S)$, as desired.

ad (*3*). As proved in (*1*), $E\bar{*}E\subset E$. Then it follows by Lemma *5* that $E\bar{*}E^{**}\subset E$. The proof is complete.

We note that the space $(\mathscr{S})$ satisfies the conditions of this theorem.

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