JOURNAL OF SCIENCE OF THE HIROSHIMA UNIVERSITY, SER. A, VOL. 21, No. 3, JANUARY, 1958

On the Paths of an Analytic Two Dimensional Autonomous System in a Neighborhood of an Isolated Critical Point

By

Hisayoshi SHINTANI

(Received Sept. 25, 1957)

§1. Introduction

1.1 In a neighborhood of a critical point which is supposed to be the origin, the analytic two dimensional autonomous system is written as follows:

$$(1) \qquad \frac{dx}{dt}=ax+by+f^*(x, y), \qquad \frac{dy}{dt}=cx+dy+g^*(x, y),$$

where $f^*(x, y)$ and $g^*(x, y)$ are $o(r)$ as $r=\sqrt{x^2+y^2}\to 0$. In this note we consider such a system as above, for which the origin is an isolated critical point and $a^2+b^2+c^2+d^2\neq 0$.

When both roots of the equation

$$(2) \qquad \lambda^2-(a+d)\lambda+ad-bc=0$$

do not vanish, the origin is a simple critical point and the behavior of paths of (1) in a neighborhood of the origin is well known.

When only one of the roots of the equation (2) vanishes, the behavior of paths was investigated by I. Bendixson [1][1], N. A. Gubar [2] and K. A. Keil [3].

When both roots of the equation (2) vanish, the behavior of paths was studied by N. A. Gubar [2], K. A. Keil [3], A. F. Andreev [4] and S. Barocio [5]. K. A. Keil studied the continuously differentiable system by the isocline method, but, in his work, the distinction problem was not completely determined. In Mathematical Reviews 16 (1955) p. 360, we learned that N. A. Gubar studied the behavior of paths of (1) making use of simple critical points of neighboring systems, and also, in the same journal 17 (1956) pp. 364–365, we learned that A. F. Andreev obtained eight phase portraits by means of the method of Frommer [6]. But, to his regret, the writer has not yet been able to refer to the originals of these works. S. Barocio studied the behavior of paths by means of the orthogonal system to the original one and Poincaré's index formula, and he obtained also eight phase portraits.

In this note, the writer studies also the case where both roots of the equation (2) vanish. Making use of exceptional directions and Keil's result,

1) Numbers in the brackets refer to the references listed at the end of this note.

the writer shows that the paths of (1) have essentially only seven phase portraits in a neighborhood of the origin. The contact parabolas and conditions for each portrait are also obtained explicitly in terms of the coefficients of the right members of (1).

Our results are placed on the bases of Lonn's and Keil's theorems which do not assume analyticity, consequently, they are also valid when analyticity is not assumed but sufficient smoothness is assumed.

1.2 Given a continuous autonomous system

$$\frac{dx}{dt}=P(x, y), \quad \frac{dy}{dt}=Q(x, y), \tag{2}$$

which satisfies the condition of uniqueness of solutions. Let the origin be an isolated critical point and assume that $P(x, y)$ and $Q(x, y)$ are written in the form as follows:

$$P(x, y)=P_m(x, y)+o(r^m),$$
$$Q(x, y)=Q_n(x, y)+o(r^n),$$

as $r\to 0$, where $P_m(x, y)$ and $Q_n(x, y)$ are homogeneous polynomials of degree m and n respectively. Put

$$F(x, y)=xQ_n(x, y)-yP_n(x, y),$$
$$G(x, y)=xP_n(x, y)+yQ_n(x, y) \qquad \text{when} \quad n=m;$$
$$F(x, y)=xQ_n(x, y), \qquad G(x, y)=yQ_n(x, y) \qquad \text{when} \quad n<m;$$
$$F(x, y)=-yP_m(x, y), \qquad G(x, y)=xP_m(x, y) \qquad \text{when} \quad n>m.$$

If we transform the variables x and y to the polar coordinates r and ϕ so that $x=r\cos\phi$, $y=r\sin\phi$, then (3) becomes

$$r\frac{d\phi}{dr}=\frac{F(\phi)+f(r, \phi)}{G(\phi)+g(r, \phi)},$$

where

$$F(\phi)=F(\cos\phi, \sin\phi), \quad G(\phi)=G(\cos\phi, \sin\phi),$$

and f, g are continuous functions such that

$$\lim_{r\to 0} f(r, \phi)=\lim_{r\to 0} g(r, \phi)=0.$$

The direction determined by the argument $\phi=\phi_0$ is called the exceptional direction of (3) at the origin if and only if $F(\phi_0)=0$. After Frommer [6] we call the exceptional direction $\phi=\phi_0$ the regular one if $G(\phi_0)=G_0\neq 0$ and the singular one otherwise. The regular exceptional directions are classified into those of three types, namely, if we write $F(\phi)$ as

$$F(\phi)=C(\phi-\phi_0)^k+o(|\phi-\phi_0|^k),$$

those of the 1st type such that k is odd and $CG_0>0$, those of the 2nd type such that k is odd and $CG_0<0$, and those of the 3rd type such that k is even [7, 8].

1.3 In this note, we use two theorems.

Lonn's theorem. *Let $\phi=\phi_0$ be a regular exceptional direction of the 2nd type of* (3) *such that $F'(\phi_0)G(\phi_0)<0$. If, for any $\varepsilon>0$, there exists a positive number δ such that*

$$|f(r,\phi_1)-f(r,\phi_2)|\leqq\varepsilon|\phi_1-\phi_2|,$$
$$|g(r,\phi_1)-g(r,\phi_2)|\leqq\varepsilon|\phi_1-\phi_2|$$

for $r<\delta$ when $|\phi_i-\phi_0|$ $(i=1,2)$ are sufficiently small, then one and only one path of (3) *tends to the origin in the directieon $\phi=\phi_0$.*

The other of the theorems is

Keil's theorem. *Given a continuously differentiable system*

$$\frac{dx}{dt}=x+f(x,y),\quad \frac{dy}{dt}=g(x,y), \tag{4}$$

where $f(x,y)$ and $g(x,y)$ are $o(r)$ as $r\to 0$ and the origin is an isolated critical point. Just one path of (4) *tends to the origin in the direction of positive x-axis and also just one in the direction of negative x-axis. We call the curve made of these two paths L. Put $g(x(y),y)=G(y)$, where $x(y)$ is a function satisfying $x+f(x,y)=0$. Then, if $G(y)<0$ for sufficiently small $y>0$, one and only one path tends to the origin in the direction of positive y-axis and any other path lying above L does not tend to the origin. If $G(y)>0$ for sufficiently small $y>0$, all paths lying above L tend to the origin in the direction of positive y-axis.*

For $y<0$, inverting inequality signs, we have the same results.

§2. Behavior of paths

2.1 In the system (1), we assume that $ad-bc=0$ and $a+d=0$. Then, by a suitable linear transformation, the system (1) is reduced to that of the form

$$\frac{dx}{dt}=y+\tilde{f}(x,y),\quad \frac{dy}{dt}=\tilde{g}(x,y),$$

or

$$\frac{dy}{dx}=\frac{\tilde{g}(x,y)}{y+\tilde{f}(x,y)},$$

where $\tilde{f}(x,y)$ and $\tilde{g}(x,y)$ satisfy the same conditions as $f^*(x,y)$ and $g^*(x,y)$. Here we may assume that $\tilde{f}(x,y)=yf(x,y)$, for, by the transformation of y to $y-y(x)$ where $y(x)$ is a function satisfying $y+\tilde{f}(x,y)=0$, we can transform the above system to that of the desired form. Thus if we put $\tilde{g}(x,y)/(1+f(x,y))=g(x,y)$, we see that the system (1) is ultimately reduced to the equation of the form

$$\frac{dy}{dx}=\frac{g(x,y)}{y}. \tag{5}$$

Here $g(x,0)$ cannot vanish identically, for, if not, the origin would not be an isolated critical point.

The exceptional direction of (5) is $y=0$ alone, consequently, in order to know the behavior of paths, we need only to know the behavior of paths in a narrow sector containing positive x-axis and that in a sector containing negative one.

2.2 In a neighborhood of the origin, let

$$g(x, y)=\sum_{k,l} b_{kl}x^k y^l \quad (b_{kl}\neq 0)$$

and let $g(x, 0)=b_{\alpha 0}x^{\alpha}+o(x^{\alpha})$ as $x\to 0$. Further, when $g_y(x, 0)$ does not vanish identically, let $g_y(x, 0)=b_{\beta 1}x^{\beta}+o(x^{\beta})$ as $x\to 0$. When $g_y(x, 0)$ vanishes identically, we assume that $g_y(x, 0)$ has also the same form but with $\beta=\infty$.

Since the behavior of paths in a sector containing negative x-axis readily follows by substitution $x=-\xi$ from that of paths in a sector containing positive x-axis, let us, in the sequel, consider only the paths in a sector containing positive x-axis. By the transformation of variable $z=x^{1/2}$, (5) becomes

$$(E_0) \qquad \frac{dy}{dz}=\frac{2z\, g(z^2, y)}{y} \qquad (z\geqq 0).$$

The exceptional direction of this equation is $y=0$ alone and evidently paths of (5), which tend to the origin in a sector containing positive x-axis, correspond to paths of (E_0), which tend to the origin in the direction of positive z-axis.

2.3 Case 1° $\alpha<2\beta+1$.

First, for a positive integer m, let us consider the solutions of (E_0) such that $y=o(z^m)$ as $z\to 0$. For such a solution, if we put $y=u_m z^m$, then $u_m\to 0$ as $z\to 0$.

If $m<\alpha$, substitution of $y=u_m z^m$ into (E_0) entails

$$(E_m) \qquad \begin{aligned} \frac{du_m}{dz} &= \frac{\{2z\, g(z^2, u_m z^m)-mz^{2m-1}u_m^2\}z^{-2m+1}}{z^{2m}u_m z^{-2m+1}} \\ &= \frac{\sum 2b_{kl}z^{2k+2+(l-2)m}u_m^l-mu_m^2}{u_m z} \\ &= \frac{-mu_m^2+z^2\cdot o(1)}{u_m z}, \end{aligned}$$

because $2k+2+(l-2)m\geqq 2$ from $\alpha<2\beta+1$. For this (E_m), from definition in 1.2 follows

$$F(z, u_m)=-(m+1)z\, u_m^2, \quad G(z, u_m)=u_m(z^2-mu_m^2).$$

Therefore the exceptional directions of (E_m) are $z=0$ and $u_m=0$. Evidently $u_m=0$ is a singular exceptional direction. If we put $z=\rho\cos\phi$, $u_m=\rho\sin\phi$, then from the definition follows

$$F(\phi)=-(m+1)\cos\phi\cdot\sin^2\phi, \qquad G(\phi)=\sin\phi\cdot(\cos^2\phi-m\sin^2\phi).$$

Consequently it follows that

$$F'\left(\frac{\pi}{2}\right)G\left(\frac{\pi}{2}\right)=F'\left(-\frac{\pi}{2}\right)G\left(-\frac{\pi}{2}\right)=-m(m+1)<0,$$

therefore $z=0$ becomes a regular exceptional direction of the 2nd type. Hence, by Lonn's theorem, a unique path tends to the origin in the direction of u_m-axis. From the form of (E_m) evidently this unique path must be $z=0$.

When $m=\alpha$, substitution of $y=u_m z^m$ into (E_0) likewise entails

$$(E_\alpha)\qquad \frac{du_\alpha}{dz}=\frac{2b_{\alpha 0}z^2-\alpha u_\alpha^2+z^2\cdot \boldsymbol{o}(1)}{u_\alpha z}.$$

Consequently, from the definition follows

$$\begin{array}{l} F(z, u_\alpha)=z\ \{2b_{\alpha 0}z^2-(\alpha+1)u_\alpha^2\} \\ G(z, u_\alpha)=u_\alpha\{(2b_{\alpha 0}+1)z^2-\alpha u_\alpha^2\} \end{array},$$

Therefore, as before, $z=0$ becomes a regular exceptional direction of the 2nd type and the path $z=0$ only tends to the origin in the direction of u_α-axis.

If $b_{\alpha 0}<0$, there is no exceptional direction other than $z=0$.

If $b_{\alpha 0}>0$, besides $z=0$, there are two exceptional directions $u_\alpha=\rho_1 z$ and $u_\alpha=\rho_2 z$, where $\rho_1=\sqrt{2b_{\alpha 0}/(\alpha+1)}$ and $\rho_2=-\rho_1$. For these directions it holds that

$$\begin{aligned} F'(\phi_1)G(\phi_1)&=-(\alpha+1)\cos\phi_1\cdot\sin\phi_1\cdot(\sin\phi_1-\rho_2\cos\phi_1)\times \\ &\quad\times(\rho_1\sin\phi_1+\cos\phi_1)=-(\alpha+1)\rho_1(\rho_1-\rho_2)(1+\rho_1^2)\cos^4\phi_1, \\ F'(\phi_2)G(\phi_2)&=-(\alpha+1)\rho_2(\rho_2-\rho_1)\ (1+\rho_2^2)\cos^4\phi_2, \end{aligned}$$

where $\tan\phi_i=\rho_i$ $(i=1,2)$. Therefore, both of these two directions are regular exceptional directions of the 2nd type, consequently, as before, one and only one path tends to the origin in each direction.

By the preceding discussion, we have seen that, when $m<\alpha$, any path of (E_m) other than $z=0$ cannot tend to the origin in any direction other than z-axis, in other words, if any path of (E_m) other than $z=0$, may tend to the origin, it must do so in the direction of z-axis. Such a path corresponds to a path of (E_{m+1}) tending to the origin. Thus, taking $m=0,1,2,\cdots,$ $\alpha-1$ successively, we see that the distribution of paths of (E_0) tending to the origin in the direction of z-axis is determined by that of (E_α) tending to the origin. Now, when $b_{\alpha 0}<0$, by the preceding results, besides $z=0$, there is no path of (E_α) tending to the origin, consequently, in this case, any path of (E_0) does not tend to the origin in the direction of z-axis. When $b_{\alpha 0}>0$, it is seen analogously that, along[1] each of the parabolas $y=\rho_1 z^{\alpha+1}$ and $y=\rho_2 z^{\alpha+1}$, one and only one path of (E_0) tends to the origin and any path of (E_0) other than these two cannot tend to the origin.

2.4 Case 2° $\alpha=2\beta+1$

In this case, β cannot be ∞, for, if not, α becomes ∞ and this contradicts the isolatedness of the critical point. As in the case 1°, we consider the solutions of (E_0) such that $y=\boldsymbol{o}(z^m)$ as $z\to 0$.

When $m<\alpha$, the results are the same as when $m<\alpha$ in the case 1°.

---

1) This means that $y/z\to\rho_i$ as $z\to 0$ along the path.

When $m=\alpha$, substitution of $y=u_m z^m$ into (E_0) entails

$$(E_\alpha)\qquad \frac{du_\alpha}{dz}=\frac{2b_{\alpha 0}z^2+2b_{\beta 1}zu_\alpha-\alpha u_\alpha^2+z^2\cdot o(1)}{u_\alpha z},$$

consequently, from the definition follows

$$F(z, u_\alpha)=z\{2b_{\alpha 0}z^2+2b_{\beta 1}zu_\alpha-(\alpha+1)u_\alpha^2\},$$
$$G(z, u_\alpha)=u_\alpha\{(2b_{\alpha 0}+1)z^2+2b_{\beta 1}zu_\alpha-\alpha u_\alpha^2\}.$$

Therefore, as before, $z=0$ becomes a regular exceptional direction of the 2nd type and the path $z=0$ only tends to the origin in the direction of u_α-axis.

If $D=b_{\beta 1}^2+4(\beta+1)b_{\alpha 0}<0$, there is no exceptional direction other than $z=0$.

If $D\geqq 0$, besides $z=0$, there are two exceptional directions $u_\alpha=\rho_3 z$ and $u_\alpha=\rho_4 z$, where ρ_3 and ρ_4 ($|\rho_3|\geqq|\rho_4|$) are roots of the equation

$$(\beta+1)\rho^2-b_{\beta 1}\rho-b_{\alpha 0}=0.$$

For these directions, it holds that

$$F'(\phi_3)G(\phi_3)=-(\alpha+1)\cos\phi_3\sin\phi_3\cdot(\cos\phi_3+\rho_3\sin\phi_3)\times$$
$$\times(\sin\phi_3-\rho_4\cos\phi_3)=-(\alpha+1)\rho_3(\rho_3-\rho_4)(1+\rho_3^2)\cos^4\phi_3,$$
$$F'(\phi_4)G(\phi_4)=-(\alpha+1)\rho_4(\rho_4-\rho_3)(1+\rho_4^2)\cos^4\phi_4,$$

where $\tan\phi_j=\rho_j$ ($j=3,4$).

Consequently, when $b_{\alpha 0}>0$, both these directions are regular exceptional directions of the 2nd type, because $\rho_3\rho_4<0$. Therefore, as before in each direction one and only one path tends to the origin.

When $b_{\alpha 0}<0$, if $D>0$, the directions $u_\alpha=\rho_3 z$ and $u_\alpha=\rho_4 z$ become regular exceptional directions of the 2nd type and of the 1st type respectively, because $\rho_3\rho_4>0$, and if $D=0$, these two directions coincide with each other and the coincident direction, for instance, $u_\alpha=\rho_3 z$ becomes a regular exceptional direction of the 3rd type. In the former case, the behavior of the

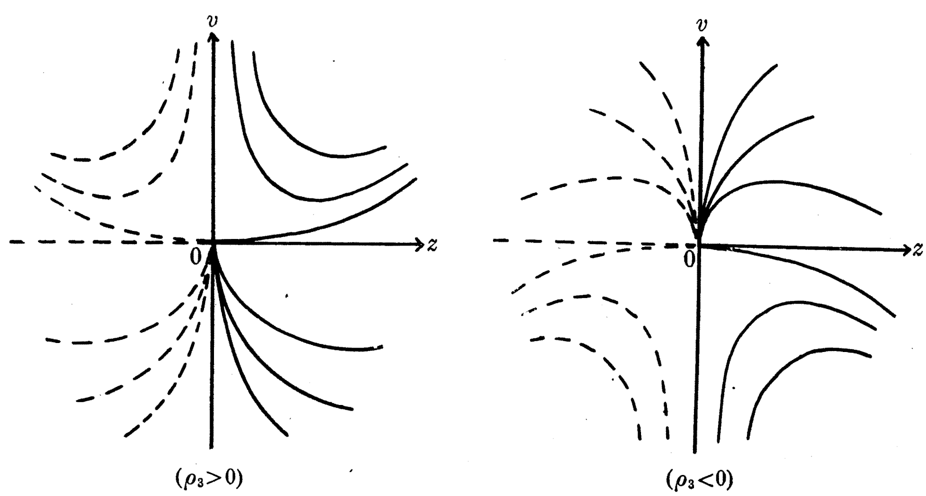


Fig. 1.

paths is easily seen as before, for, in a narrow sector containing an exceptional direction of the 1st type, all paths tend to the origin in that exceptional direction. In the latter case, substitution of $u_\alpha=(\rho_3+v)z$ into (E_α) entails

$$\frac{dv}{dz}=\frac{-(\alpha+1)v^2+z^2\cdot \boldsymbol{O}(1)}{\rho_3 z+vz}.$$

Consequently, from Keil's theorem, it is seen that, in the (z, v)-plane, the origin is a saddle-node, therefore that, in the (z, u_α)-plane, infinitely many paths tend to the origin in the direction $u_\alpha=\rho_3 z$. As is easily seen, this is a limiting case as $\rho_4\to\rho_3$ in the former case.

Thus, as in the case 1°, we can see how the paths of (E_0) tending to the origin in the direction of z-axis behave in a neighborhood of the origin.

2.5 Case 3° $\alpha>2\beta+1$

In this case, as in the case 2°, β cannot be ∞. As in the case 1°, we consider the solutions of (E_0) such that $y=\boldsymbol{o}(z^m)$ as $z\to 0$.

When $m<2\beta+1$, the results are the same as when $m<\alpha$ in the case 1°.

When $m=2\beta+1$, substitution of $y=u_m z^m$ into (E_0) entails

$$\frac{du_{2\beta+1}}{dz}=\frac{2b_{\beta 1}zu_{2\beta+1}-(2\beta+1)u_{2\beta+1}^2+z^2\cdot \boldsymbol{o}(1)}{u_{2\beta+1}z}$$

consequently, from definition follows

$$F(z, u_{2\beta+1})=2zu_{2\beta+1}\{b_{\beta 1}z-(\beta+1)u_{2\beta+1}\},$$
$$G(z, u_{2\beta+1})=u_{2\beta+1}\{z^2+2b_{\beta 1}zu_{2\beta+1}-(2\beta+1)u_{2\beta+1}^2\}.$$

Therefore $u_{2\beta+1}=0$ becomes a singular exceptional direction, and $z=0$ and $u_{2\beta+1}=\rho_5 z$ $(\rho_5=b_{\beta 1}/(\beta+1))$ become both regular exceptional directions of the 2nd type. Hence, as before, in the direction $z=0$, the path $z=0$ only tends to the origin and, in the direction $u_{2\beta+1}=\rho_5 z$, one and only one path tends to the origin.

When $2(\alpha-\beta)-1>m>2\beta+1$, substitution of $y=u_m z^m$ into (E_0) entails

$$\frac{du_m}{dz}=\frac{\{2zg(z^2, u_m z^m)-mz^{2m-1}u_m^2\}z^{-(2\beta+m+1)}}{z^{2m}u_m z^{-(2\beta+m+1)}}$$
$$=\frac{\sum 2b_{kl}z^{2k-2\beta+(l-1)m}u_m^l-mz^{m-2\beta-2}u_m^2}{z^{m-2\beta-1}u_m}$$
$$=\frac{2b_{\beta 1}u_m+z^2\cdot \boldsymbol{O}(1)+u_m^2\cdot \boldsymbol{O}(1)}{z^{m-2\beta-1}u_m}$$

consequently, from the definition follows

$$F(z, u_m)=2b_{\beta 1}zu_m, \qquad G(z, u_m)=2b_{\beta 1}u_m^2.$$

Therefore $u_m=0$ becomes a singular exceptional direction and $z=0$ becomes a regular exceptional direction of the 2nd type. Hence, as before, in the direction $z=0$, the path $z=0$ only tends to the origin.

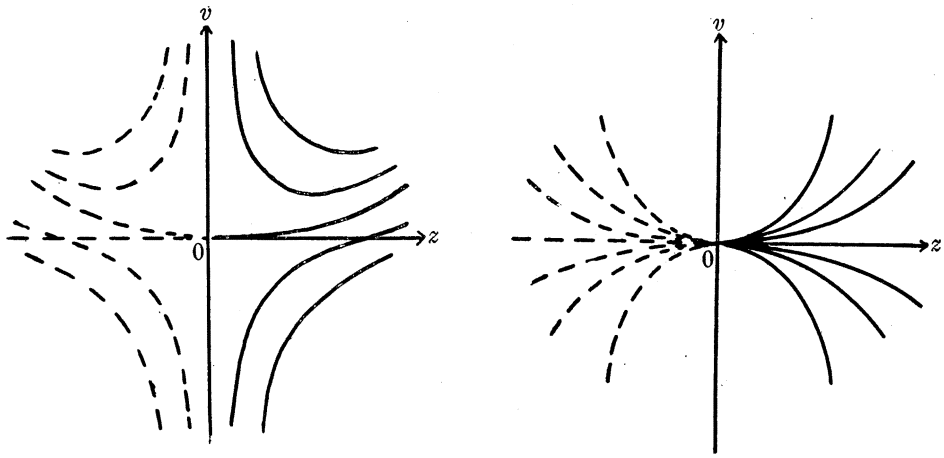
When $m=2(\alpha-\beta)-1$, as before, substitution of $y=u_m z^m$ into (E_0) entails

$$\frac{du_{2\alpha-2\beta-1}}{dz}=\frac{2b_{\beta 1}u_{2\alpha-2\beta-1}+2b_{\alpha 0}z+z^2\cdot \boldsymbol{O}(1)+zu_{2\alpha-2\beta-1}^2\cdot \boldsymbol{O}(1)}{u_{2\alpha-2\beta-1}z^{2\alpha-4\beta-2}},$$

which is transformed to

$$\frac{dv}{dz}=\frac{2b_{\beta 1}v+z^2\cdot\boldsymbol{O}(1)+zv^2\cdot\boldsymbol{O}(1)}{(\rho_6 z+v)z^{2\alpha-4\beta-2}}$$

by the transformation $v=u_{2\alpha-2\beta-1}-\rho_6 z$ ($\rho_6=-b_{\alpha 0}/b_{\beta 1}$). Then, by Keil's theorem, it is seen that, in the (z, v)-plane the origin is a saddle-point or a node according as $b_{\alpha 0}>0$ or <0.



$(b_{\alpha 0}>0)$ $(b_{\alpha 0}<0)$

Fig. 2.

This says that, in the $(z, u_{2\alpha-2\beta-1})$-plane, in the direction $u_{2\alpha-2\beta-1}=\rho_6 z$, just one or infinitely many paths tend to the origin according as $b_{\alpha 0}>0$ or <0.

Thus, as in the case 1°, we can see how the paths of (E_0) tending to the origin in the direction of z-axis behave in a neighborhood of the origin.

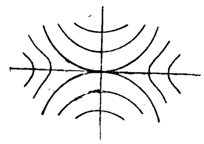
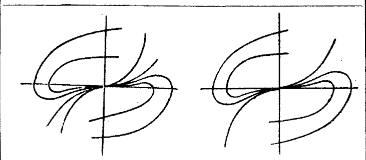
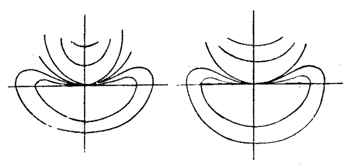
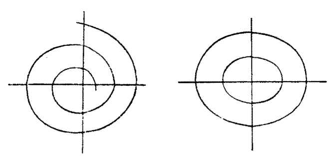
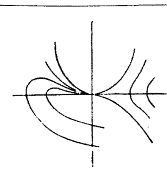
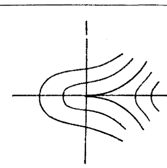
2.6 Results

The results obtained in the preceding sections are summarized in the tables as follows:

Table 1. Contact parabolas and number of paths tending to the origin in the direction of positive x-axis

<table>
<tr><th>cases</th><th>$\alpha<2\beta+1$</th><th colspan="2">$\alpha=2\beta+1$</th><th>$\alpha>2\beta+1$</th></tr>
<tr><td>$b_{\alpha 0}>0$</td><td>$y=\rho_1 x^{\frac{\alpha+1}{2}}$ (one)
$y=-\rho_1 x^{\frac{\alpha+1}{2}}$ (one)</td><td colspan="2">$y=\rho_3 x^{\beta+1}$ (one)
$y=\rho_4 x^{\beta+1}$ (one)</td><td>$y=\rho_5 x^{\beta+1}$ (one)
$y=\rho_6 x^{\alpha-\beta}$ (one)</td></tr>
<tr><td rowspan="3">$b_{\alpha 0}<0$</td><td rowspan="3">none</td><td>$D>0$</td><td>$y=\rho_3 x^{\beta+1}$ (one)
$y=\rho_4 x^{\beta+1}$ (infinitely many)
($|\rho_3|>|\rho_4|$)</td><td rowspan="3">$y=\rho_5 x^{\beta+1}$ (one)
$y=\rho_6 x^{\alpha-\beta}$ (infinitely many)</td></tr>
<tr><td>$D=0$</td><td>$y=\rho_3 x^{\beta+1}$ (infinitely many)</td></tr>
<tr><td>$D<0$</td><td>none</td></tr>
<tr><td>Notes</td><td>$\rho_1=\sqrt{\frac{2b_{\alpha 0}}{\alpha+1}}$</td><td colspan="2">$(\beta+1)\rho_j^2-b_{\beta 1}\rho_j-b_{\alpha 0}=0$ ($j=3, 4$)
$D=b_{\beta 1}^2+4(\beta+1)b_{\alpha 0}$</td><td>$\rho_5=b_{\beta 1}/(\beta+1)$
$\rho_6=-b_{\alpha 0}/b_{\beta 1}$</td></tr>
</table>

Table 2. Types of distribution of paths.

types \ figures \ cases		$\alpha<2\beta+1$	$\alpha=2\beta+1$	$\alpha>2\beta+1$
degenerate saddle		$b_{\alpha 0}>0$ α: odd	$b_{\alpha 0}>0$	$b_{\alpha 0}>0$ α: odd
degenerate node			$b_{\alpha 0}<0$ $D\geqq 0$ β: even	$b_{\alpha 0}<0$ α: odd β: even
closed saddle-node			$b_{\alpha 0}<0$ $D\geqq 0$ β: odd	$b_{\alpha 0}<0$ α: odd β: odd
focus or center		$b_{\alpha 0}<0$ α: odd	$D<0$	
saddle-node				α: even
saddle-center		α: even		

In conclusion, the writer wishes to express his hearty gratitude to Prof. M. Urabe for his kind guidance and constant advice.

References

[1] I. Bendixson, *Sur les courbes définies par des équations différentielles.* Acta Math., **24** (1901), 1–88.

[2] N. A. Gubar, *A characteristic of composite singular points of a system of two differential equations by means of simple singular points of neighboring systems.* Dokl. Akad. Nauk SSSR, **95** (1954), 435–438.

[3] K.A. Keil, *Das qualitative Verhalten der Integralkurven einer gewöhnlichen Differentialgleichung erster Ordnung in der Umgebung eines singulären Punktes*. Jber. Deutsch. Math. Verein., **57** (1955), 111–132.

[4] A. F. Andreev, *Investigation of the behavior of integral curves of a system of two differential equations in the neighborhood of a singular point*. Vestnik Leningrad. Univ., **10** (1955), 43–65.

[5] S. Barocio, *On certain critical points of a differential system in the plane*. Contributions to the theory of nonlinear oscillations, III, Princeton (1956), 127–135.

[6] M. Frommer, *Die Integralkurven einer gewöhnlichen Differentialgleichung erster Ordnung in der Umgebung rationaler Unbestimmtheitsstellen*. Math. Ann., **99** (1928), 222–272.

[7] V. V. Nemytzkii and V. V. Stepanov, *Qualitative theory of differential equations* (Russian). Moscow (1947).

[8] G. Sansone e R. Conti, *Equazioni differenziali non lineari*. Roma (1956).

Department of Mathematics,
Faculty of Science,
Hiroshima University.