JOURNAL OF SCIENCE OF THE HIROSHIMA UNIVERSITY, SER. A, VOL. 21, NO. 3, JANUARY, 1958

# *Periodic Solution of Van der Pol's Equation with Damping Coefficient $\lambda=0$ (0.2) 1.0*

By

Minoru URABE

(Received Sept. 14, 1957)

## 1 Introduction

On a periodic solution of Van der Pol's equation

$$\frac{d^2x}{dt^2}-\lambda(1-x^2)\frac{dx}{dt}+x=0 \quad (\lambda>0), \tag{1}$$

its existence and uniqueness for any value of $\lambda$ is well known [1][1] and it is also well known [1] that the unique periodic solution is orbitally stable. For $\lambda=0.1$, 1.0 and 10, Van der Pol sought for the orbits of the periodic solutions by the graphical method [2]. From the behavior of these orbits, the properties of the periodic solutions of (1) are suggested somewhat, but, of course, the further and more minute properties of the periodic solutions are not known from these results. When $\lambda$ is sufficiently small [3] or large [4], the properties of the periodic solutions are known considerably in detail, but, when $\lambda$ lies between these two limits, hardly any property of the periodic solutions seems to be known yet.

In this report, in order to contribute to this respect, making use of the method of the previous paper [5], the author calculated the periodic solutions of (1) for $\lambda=0$ (0.2) 1.0, correct to three decimal places. From this result, the various properties of the periodic solutions for $\lambda=0$ (0.2) 1.0,—the shapes of the orbit and the oscillation, the amplitude and the period of the oscillation, the characteristic exponent $h$ of the orbit, etc.—are found. The remarkable results newly ascertained compared to those of Van der Pol are as follows:

With increasing $\lambda$ till 1.0,

1° the amplitude varies—probably increases—very slowly;

2° the period increases normally;

3° the stability increases very rapidly, and the sum of the characteristic exponent and the damping coefficient $\lambda$ varies very slowly.

## 2 Methods of calculation

Writing (1) in the simultaneous form

$$\begin{cases} \dfrac{dx}{dt}=y \ (=X) \\ \dfrac{dy}{dt}=-x+\lambda(1-x^2)y \ (=Y), \end{cases} \tag{2}$$

1) Numbers in the crotchets refer to the references listed at the end of the report.

we apply the method of the previous paper [5].

1° $\lambda=0.2$

When $\lambda=0$, (2) becomes

$$(3)\qquad \begin{cases} \dfrac{dx}{dt}=y, \\ \dfrac{dy}{dt}=-x, \end{cases}$$

which evidently admits of a continuum of periodic solutions

$$(4)\qquad x=u\cos t,\quad y=-u\sin t,$$

where $u$ is an arbitrary constant. Consequently, from the appendix of [5], we see that

$$\omega(u)=2\pi,\quad h(t)=0$$

and

$$\begin{aligned} I(2\pi)&=\int_0^{2\pi}(1-x^2)\,y^2\,dt \\ &=\int_0^{2\pi}(1-u^2\cos^2 t)\,u^2\sin^2 t\,dt \\ &=u^2\left(1-\frac{u^2}{4}\right)\pi. \end{aligned}$$

Then, by 3.2 of [5] or [3], for sufficiently small $|\lambda|$, the periodic solution of (2) is given approximately by

$$(5)\qquad x=2\cos t,\quad y=-2\sin t$$

and, for this periodic solution, it holds approximately that

$$\begin{aligned} h(\pi)-h(-\pi)&=\lambda\int_{-\pi}^{\pi}(1-x^2)dt \\ &=\lambda\int_{-\pi}^{\pi}(1-4\cos^2 t)\,dt \\ &=-2\pi\lambda, \end{aligned}$$

consequently it holds that

$$e^{h(\pi)}-e^{h(-\pi)}\doteqdot-2\pi\lambda.$$

Thus, for $\lambda=0.2$, we start our calculation from the approximate periodic solution given by (5) and

$$(6)\qquad e^{h(\pi)}-e^{h(-\pi)}\doteqdot-0.4\pi=-1.257.$$

For numerical integration of (2), the starting values are computed by Taylor series

$$(7)\qquad \begin{cases} x=x_0+\dot{x}_0 t+\dfrac{\ddot{x}_0}{2}t^2+\dfrac{\dddot{x}_0}{6}t^3+\dfrac{x_0^{(4)}}{24}t^4+\cdots, \\ y=y_0+\dot{y}_0 t+\dfrac{\ddot{y}_0}{2}t^2+\dfrac{\dddot{y}_0}{6}t^3+\dfrac{y_0^{(4)}}{24}t^4+\cdots, \end{cases}$$

where dots denote differentiation with respect to $t$ and the suffix zero means the value at $t=0$. The coefficients of (7) are computed by successive differentiation of (2) as follows:

$$(8)\quad \begin{cases} \dot{x}_0=y_0,\ \dot{y}_0=-x_0+\lambda y_0-\lambda y_0 x_0^2; \\ \ddot{x}_0=\dot{y}_0,\ \ddot{y}_0=-\dot{x}_0+\lambda\dot{y}_0-\lambda\{\dot{y}_0x_0^2+y_0(x^2)^{\cdot}_0\}; \\ \dddot{x}_0=\ddot{y}_0, \\ \qquad \dddot{y}_0=-\ddot{x}_0+\lambda\ddot{y}_0-\lambda\{\ddot{y}_0x_0^2+2\dot{y}_0(x^2)^{\cdot}_0+y_0(x^2)^{\cdot\cdot}_0\}; \\ x_0^{(4)}=\dddot{y}_0, \\ \qquad y_0^{(4)}=-\dddot{x}_0+\lambda\dddot{y}_0-\lambda\{\dddot{y}_0x_0^2+3\ddot{y}_0(x^2)^{\cdot}_0+3\dot{y}_0(x^2)^{\cdot\cdot}_0+y_0(x^2)^{\cdot\cdot\cdot}_0\}. \end{cases}$$

For the succeeding integration, the following five-point formulas are used:

$$(9)\quad \begin{cases} \text{for extrapolation,} \\ \qquad y_{r+1}=y_r+\dfrac{k}{720}\{251y'_{r-4}-1274y'_{r-3}+2616y'_{r-2}-2774y'_{r-1}+1901y'_r\}; \\ \text{for control,} \\ \qquad y_{r+1}=y_{r-1}+\dfrac{k}{90}\{-y'_{r-2}+34y'_{r-1}+114y'_r+34y'_{r+1}-y'_{r+2}\}, \end{cases}$$

where $k$ is an interval breadth and dashes denote differentiation with respect to the independent variable. The former of (9) is a deformation of Adam's formula:

$$y_{r+1}=y_r+k\left(y'_r+\frac{1}{2}\nabla y'_r+\frac{5}{12}\nabla^2y'_r+\frac{3}{8}\nabla^3y'_r+\frac{251}{720}\nabla^4y'_r\right)$$

and the latter of (9) is that of the central difference formula:

$$y_{r+1}=y_{r-1}+k\left(2y'_r+\frac{1}{3}\nabla^2y'_{r+1}-\frac{1}{90}\nabla^4y'_{r+2}\right).$$

From the starting values computed correct to three decimal places by (7) for $t=0,\ \pm0.1,\ \pm0.2$, putting $k=0.1$ and $-0.1$, we compute $x$ and $y$ correct to three decimal places successively by (9) for

$$t=0.3,\ 0.4,\cdots,\ 3.2,\ 3.3,\ 3.4$$

and for

$$t=-0.3,\ -0.4,\cdots,\ -3.1,\ -3.2,\ -3.3.$$

From this computation, we see that

$$x(-\omega_0/2)=x(-3.1)=-1.998,\quad y(-\omega_0/2)=y(-3.1)=0.099,$$
$$X(-3.1)=0.099,\quad Y(-3.1)=1.939$$

and

$$x(3.2)=-1.997,\quad y(3.2)=0.100.$$

From this, by (2.2.2) of [5], $\delta(\tilde{\omega}/2)$ is computed as follows:

$$\delta\left(\frac{\tilde{\omega}}{2}\right)=-\frac{(-1.997+1.998)\times0.099+(0.100-0.099)\times1.939}{0.099^2+1.939^2}=-0.001,$$

consequently $\tilde{\omega}/2$ becomes 3.199. Then, by Newton's backward interpolation formula, we see that

$$x(3.199)=-1.997, \quad y(3.199)=0.098.$$

For this value, we compute again $\delta(\tilde{\omega}/2)$ by (2.2.2) of [5] and then we find $\delta(\tilde{\omega}/2)=0.001$, consequently $\tilde{\omega}/2$ becomes 3.200. Thus we may assume that

$$\tilde{\omega}/2=3.200. \tag{10}$$

Then, by (5) of [5], from (6), we see that

$$c=\frac{(-1.997+1.998)\times 1.939-(0.100-0.099)\times 0.099}{2\times(-1.257)}\fallingdotseq -0.0007$$

consequently the modified initial values become

$$\begin{cases} x(0)=2.000-0.0007=1.9993\fallingdotseq 1.999, \\ y(0)=0.000. \end{cases}$$

Starting again from this set of initial values, we repeat the above process. The result shows that the modified initial values become

$$x(0)=1.9998=2.000, \quad y(0)=0.000.$$

Thus, taking a mean value of 1.9993 and 1.9998—or taking a value accompanied with the smaller correction, we may suppose that the desired periodic solution is a solution corresponding to the initial values

$$x(0)=2.000, \quad y(0)=0.000.$$

The period of this solution becomes

$$3.200+3.100=6.300$$

because of (10).

The values of $h(t)$ are computed by the integrated Bessel's interpolation formula as follows:

$$\int_0^k y\,dx=\frac{k}{1440}(11y_{-2}-93y_{-1}+802y_0+802y_1-92y_2+11y_3)$$

where $y_r=y(rk)$. $I(3.2)$ and $I(-3.1)$ are computed by applying Simpson's rule to (10) of [5]. Thus the characteristic exponent $h$ of the orbit becomes

$$\begin{aligned} h&=\frac{1}{\omega}h(\omega) \\ &=\frac{1}{\omega}\int_0^\omega\left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}\right)dt \\ &=\{h(\tilde{\omega}/2)\}-h(-\omega_0/2)\}/\omega \\ &=\{h(3.200)-h(-3.100\}/6.300 \\ &=-1.263/6.300 \\ &=-0.200. \end{aligned}$$

The value $h(\omega)=-1.263$ is near the value given by (6).

2° $\lambda=0.4$

By the results of 1°, we see that

$$h(3.2)=-0.662, \quad h(-3.1)=0.601;$$
$$e^{h(3.2)}=\ 0.516, \quad e^{h(-3.1)}=1.824;$$
$$I(3.2)=\ 0.005, \quad I(-3.1)=-0.004.$$

Consequently, if we take 0.2 as $\lambda_0$ and assume that $\varepsilon=0.2$, then, for $\lambda=\lambda_0+\varepsilon=0.4$, by (11) of [5], the correction of initial values is computed as follows:

$$c=-\frac{0.516\times0.005-1.824\times(-0.004)}{2\times(0.516-1.824)}\times0.2=0.001.$$

Thus we have the approximate initial values

$$\begin{cases} x(0)=2.000+0.001=2.001, \\ y(0)=0.000. \end{cases}$$

From this set of initial values, putting $-\omega_0/2=-3.2$, we repeat the process of 1°. Here we use the approximate values

$$h(3.15)=-0.63, \qquad h(-3.2)=0.66;$$
$$e^{h(3.15)}=0.533, \qquad e^{h(-3.2)}=1.935,$$

which are readily obtained from the results of 1°. The obtained initial values become

$$x(0)=2.000, \qquad y(0)=0.000;$$
$$x(0)=2.001, \qquad y(0)=0.000.$$

Thus, comparing the corrections of the initial values as in 1°, we may suppose that the desired periodic solution is a solution corresponding to the initial values

$$x(0)=2.000, \qquad y(0)=0.000.$$

The period of this solution becomes

$$3.147+3.200=6.347.$$

As in 1°, the characteristic exponent $h$ is found.

3° $\lambda=0.6$

Repeating the process of 2°, we find successively the initial values of $x$ as follows:

2.007,
2.000,
2.007,

If we take the mean value 2.004 of the above values as $x(0)$ and correct successively the values of $x(0)$ again as in 1°, then we have successively the initial values as follows:

$$2.004,$$
$$2.005,$$
$$2.003,$$
$$2.004.$$

Thus, taking a middle value, we may suppose that the desired periodic solution is a solution corresponding to the initial value

$$x(0)=2.004, \qquad y(0)=0.000.$$

The period is found to be

$$3.200+3.222=6.422.$$

As in 1°, the characteristic exponent $h$ is found.

4° $\lambda=0.8$

From the results obtained till now, it is conjectured that the initial value of $x$ to which corresponds a periodic solution of (2) varies very slowly with $\lambda$. Thus we may start our calculation from the initial values

$$x(0)=2.004, \qquad y(0)=0.000.$$

When $\lambda=0.8$, since the values of $x$ and $y$ vary considerably with $t$, for numerical integration of (2), the starting values are computed correct to four decimal places by (7) for $t=0, \pm 0.05, \pm 0.10$ and the succeeding values likewise correct to four decimal places by (9) with interval breadth 0.05.

If we correct successively the initial values of $x$ as in 1°, then we have successively the initial values rounded to the third decimal places as follows:

$$2.004,$$
$$2.007,$$
$$2.005,$$
$$2.007.$$

If, as in 3°, we take the mean value 2.006 of 2.005 and 2.007, and if we seek for correction of this value again as in 1°, then we have zero correction, namely we see that the initial values

$$x(0)=2.006, \qquad y(0)=0.000$$

are the desired values to which corresponds a periodic solution of (2).

The period is found analogously as in 1°. The characteristic exponent $h$ is found by summing up the integral obtained by Simpson's rule and the integral obtained by the integrated Newton's backward interpolation formula.

5° $\lambda=1.0$

From the results of 3° and 4°, we may start our calculation from the initial values

$$x(0)=2.008, \quad y(0)=0.000.$$

As in 4°, correcting successively the initial values of $x$, we have:

2.008,
2.009,
2.008.

Since the correction of 2.008 is 0.0014 and that of 2.009 is $-0.0012$, we take 2.009 as the initial value of $x$.

The period and the characteristic exponent are found analogously as in 4°.

## 3 Results

Numerical values of $x$ and $y$ of a closed orbit corresponding to a periodic solution of (2) are shown in Tables 1–5 with some additive materials.

The shapes of closed orbits and of oscillations are shown respectively in Figs. 1 and 2.

Fig. 3 shows how the amplitude, the period and the characteristic exponent vary with $\lambda$.

**Table 1** $\lambda=0.2$

| $t$ | $x$ | $y$ |
|---|---|---|
| 0.0 | 2.000 | 0.000 |
| 0.1 | 1.990 | −0.194 |
| 0.2 | 1.962 | −0.375 |
| 0.3 | 1.916 | −0.544 |
| 0.4 | 1.854 | −0.701 |
| 0.5 | 1.777 | −0.847 |
| 0.6 | 1.685 | −0.984 |
| 0.7 | 1.580 | −1.112 |
| 0.8 | 1.465 | −1.233 |
| 0.9 | 1.334 | −1.348 |
| 1.0 | 1.194 | −1.458 |
| 1.1 | 1.043 | −1.562 |
| 1.2 | 0.882 | −1.661 |
| 1.3 | 0.711 | −1.753 |
| 1.4 | 0.531 | −1.837 |
| 1.5 | 0.344 | −1.911 |
| 1.6 | 0.150 | −1.972 |
| 1.7 | −0.050 | −2.017 |
| 1.8 | −0.253 | −2.041 |
| 1.9 | −0.457 | −2.041 |
| 2.0 | −0.660 | −2.013 |
| 2.1 | −0.859 | −1.954 |
| 2.2 | −1.050 | −1.862 |
| 2.3 | −1.230 | −1.737 |
| 2.4 | −1.396 | −1.581 |
| 2.5 | −1.545 | −1.399 |
| 2.6 | −1.675 | −1.196 |
| 2.7 | −1.784 | −0.979 |
| 2.8 | −1.871 | −0.755 |
| 2.9 | −1.935 | −0.531 |
| 3.0 | −1.977 | −0.311 |
| 3.1 | −1.997 | −0.100 |
| 3.2 | −1.997 | 0.100 |
| 3.3 | −1.978 | 0.287 |
| 3.4 | −1.940 | 0.462 |

| $t$ | $x$ | $y$ |
|---|---|---|
| 0.0 | 2.000 | 0.000 |
| −0.1 | 1.990 | 0.206 |
| −0.2 | 1.959 | 0.421 |
| −0.3 | 1.906 | 0.643 |
| −0.4 | 1.830 | 0.868 |
| −0.5 | 1.732 | 1.089 |
| −0.6 | 1.612 | 1.299 |
| −0.7 | 1.472 | 1.492 |
| −0.8 | 1.314 | 1.661 |
| −0.9 | 1.141 | 1.802 |
| −1.0 | 0.955 | 1.911 |
| −1.1 | 0.760 | 1.987 |
| −1.2 | 0.559 | 2.030 |
| −1.3 | 0.355 | 2.044 |
| −1.4 | 0.151 | 2.031 |
| −1.5 | −0.051 | 1.996 |
| −1.6 | −0.248 | 1.942 |
| −1.7 | −0.439 | 1.874 |
| −1.8 | −0.622 | 1.795 |
| −1.9 | −0.797 | 1.707 |
| −2.0 | −0.963 | 1.612 |
| −2.1 | −1.119 | 1.511 |
| −2.2 | −1.265 | 1.404 |
| −2.3 | −1.400 | 1.292 |
| −2.4 | −1.523 | 1.174 |
| −2.5 | −1.634 | 1.049 |
| −2.6 | −1.732 | 0.917 |
| −2.7 | −1.817 | 0.776 |
| −2.8 | −1.887 | 0.625 |
| −2.9 | −1.941 | 0.462 |
| −3.0 | −1.979 | 0.287 |
| −3.1 | −1.998 | 0.099 |
| −3.2 | −1.998 | −0.101 |
| −3.3 | −1.978 | −0.310 |

amplitude $=2.000$
period $=6.300$
characteristic exponent$=-1.263/6.300=-0.200$

**Table 2** $\lambda=0.4$

| $t$ | $x$ | $y$ | $t$ | $x$ | $y$ |
|---|---|---|---|---|---|
| 0.0 | 2.000 | 0.000 | 0.0 | 2.000 | 0.000 |
| 0.1 | 1.990 | −0.188 | −0.1 | 1.990 | 0.212 |
| 0.2 | 1.963 | −0.355 | −0.2 | 1.957 | 0.448 |
| 0.3 | 1.920 | −0.501 | −0.3 | 1.900 | 0.703 |
| 0.4 | 1.863 | −0.632 | −0.4 | 1.816 | 0.972 |
| 0.5 | 1.794 | −0.750 | −0.5 | 1.706 | 1.240 |
| 0.6 | 1.713 | −0.859 | −0.6 | 1.568 | 1.497 |
| 0.7 | 1.622 | −0.961 | −0.7 | 1.408 | 1.723 |
| 0.8 | 1.521 | −1.059 | −0.8 | 1.225 | 1.910 |
| 0.9 | 1.410 | −1.155 | −0.9 | 1.028 | 2.043 |
| 1.0 | 1.290 | −1.250 | −1.0 | 0.818 | 2.125 |
| 1.1 | 1.160 | −1.347 | −1.1 | 0.605 | 2.152 |
| 1.2 | 1.021 | −1.445 | −1.2 | 0.389 | 2.140 |
| 1.3 | 0.871 | −1.546 | −1.3 | 0.178 | 2.088 |
| 1.4 | 0.712 | −1.649 | −1.4 | −0.028 | 2.017 |
| 1.5 | 0.541 | −1.753 | −1.5 | −0.225 | 1.924 |
| 1.6 | 0.361 | −1.855 | −1.6 | −0.413 | 1.828 |
| 1.7 | 0.170 | −1.953 | −1.7 | −0.590 | 1.722 |
| 1.8 | −0.029 | −2·038 | −1.8 | −0.758 | 1.621 |
| 1.9 | −0.237 | −2.107 | −1.9 | −0.914 | 1.516 |
| 2.0 | −0.449 | −2.147 | −2.0 | −1.061 | 1.418 |
| 2.1 | −0.665 | −2.151 | −2.1 | −1.198 | 1.318 |
| 2.2 | −0.878 | −2.107 | −2.2 | −1.325 | 1.224 |
| 2.3 | −1.085 | −2.013 | −2.3 | −1.443 | 1.127 |
| 2.4 | −1.279 | −1.862 | −2.4 | −1.551 | 1.033 |
| 2.5 | −1.456 | −1.665 | −2.5 | −1.649 | 0.932 |
| 2.6 | −1.611 | −1.426 | −2.6 | −1.737 | 0.831 |
| 2.7 | −1.741 | −1.166 | −2.7 | −1.815 | 0.718 |
| 2.8 | −1.844 | −0.893 | −2·8 | −1.880 | 0.599 |
| 2.9 | −1.920 | −0.630 | −2.9 | −1.934 | 0.462 |
| 3.0 | −1.970 | −0.377 | −3.0 | −1.972 | 0.313 |
| 3.1 | −1.996 | −0.150 | −3.1 | −1.996 | 0.139 |
| 3.2 | −2.001 | 0.057 | −3.2 | −1.999 | −0.053 |
| 3.3 | −1.986 | 0.237 | −3.3 | −1.984 | −0.274 |
| 3.4 | −1.954 | 0.399 | −3.4 | −1.944 | −0.513 |

amplitude =2.000
period =6.347
characteristic exponent=−2.500/6.347=−0.394

**Table 3** $\lambda=0.6$

| $t$ | $x$ | $y$ | $t$ | $x$ | $y$ |
|---|---|---|---|---|---|
| 0.0 | 2.004 | 0.000 | 0.0 | 2.004 | 0.000 |
| 0.1 | 1.995 | −0.183 | −0.1 | 1.993 | 0.219 |
| 0.2 | 1.968 | −0.336 | −0.2 | 1.959 | 0.478 |
| 0.3 | 1.929 | −0.463 | −0.3 | 1.896 | 0.771 |
| 0.4 | 1.876 | −0.572 | −0.4 | 1.804 | 1.093 |
| 0.5 | 1.815 | −0.667 | −0.5 | 1.677 | 1.419 |
| 0.6 | 1.743 | −0.753 | −0.6 | 1.521 | 1.728 |
| 0.7 | 1.665 | −0.832 | −0.7 | 1.333 | 1.985 |
| 0.8 | 1.577 | −0.910 | −0.8 | 1.126 | 2.174 |
| 0.9 | 1.483 | −0.986 | −0.9 | 0.901 | 2.278 |
| 1.0 | 1.380 | −1.065 | −1.0 | 0.673 | 2.306 |
| 1.1 | 1.270 | −1.147 | −1.1 | 0.442 | 2.267 |
| 1.2 | 1.150 | −1.235 | −1.2 | 0.221 | 2.182 |
| 1.3 | 1.023 | −1.330 | −1.3 | 0.007 | 2.068 |
| 1.4 | 0.884 | −1.433 | −1.4 | −0.192 | 1.940 |
| 1.5 | 0.736 | −1.545 | −1.5 | −0.381 | 1.808 |
| 1.6 | 0.575 | −1.665 | −1.6 | −0.554 | 1.680 |
| 1.7 | 0.403 | −1.793 | −1.7 | −0.717 | 1.558 |
| 1.8 | 0.216 | −1.924 | −1.8 | −0.866 | 1.446 |
| 1.9 | 0.018 | −2.054 | −1.9 | −1.006 | 1.341 |
| 2.0 | −0.194 | −2.170 | −2.0 | −1.135 | 1.246 |
| 2.1 | −0.415 | −2.260 | −2.1 | −1.255 | 1.157 |
| 2.2 | −0.644 | −2.304 | −2.2 | −1.367 | 1.074 |
| 2.3 | −0.874 | −2.287 | −2.3 | −1.470 | 0.995 |
| 2.4 | −1.099 | −2.190 | −2.4 | −1.566 | 0.918 |
| 2.5 | −1.309 | −2.013 | −2.5 | −1.654 | 0.841 |
| 2.6 | −1.499 | −1.761 | −2.6 | −1.734 | 0.761 |
| 2.7 | −1.660 | −1.459 | −2.7 | −1.806 | 0.676 |
| 2.8 | −1.790 | −1.131 | −2.8 | −1.869 | 0.582 |
| 2.9 | −1.886 | −0.810 | −2.9 | −1.922 | 0.475 |
| 3.0 | −1.953 | −0.510 | −3.0 | −1.963 | 0.349 |
| 3.1 | −1.989 | −0.249 | −3.1 | −1.991 | 0.200 |
| 3.2 | −2.004 | −0.024 | −3.2 | −2.002 | 0.019 |
| 3.3 | −1.995 | 0.162 | −3.3 | −1.994 | −0.195 |
| 3.4 | −1.973 | 0.319 | −3.4 | −1.962 | −0.450 |

amplitude =2.004
period =6.422
characteristic exponent= −3.937/6.422= −0.613

**Table 4** $\lambda=0.8$

| $t$ | $x$ | $y$ |
|---|---|---|
| 0.0 | 6.006 | 0.000 |
| 0.1 | 1.997 | −0.178 |
| 0.2 | 1.972 | −0.318 |
| 0.3 | 1.934 | −0.429 |
| 0.4 | 1.887 | −0.519 |
| 0.5 | 1.831 | −0.595 |
| 0.6 | 1.768 | −0.663 |
| 0.7 | 1.698 | −0.725 |
| 0.8 | 1.623 | −0.785 |
| 0.9 | 1.541 | −0.845 |
| 1.0 | 1.454 | −0.908 |
| 1.1 | 1.360 | −0.975 |
| 1.2 | 1.259 | −1.048 |
| 1.3 | 1.150 | −1.129 |
| 1.4 | 1.032 | −1.220 |
| 1.5 | 0.905 | −1.323 |
| 1.6 | 0.767 | −1.440 |
| 1.7 | 0.617 | −1.572 |
| 1.8 | 0.452 | −1.719 |
| 1.9 | 0.272 | −1.880 |
| 2.0 | 0.076 | −2.050 |
| 2.1 | −0.137 | −2.216 |
| 2.2 | −0.366 | −2.362 |
| 2.3 | −0.608 | −2.460 |
| 2.4 | −0.856 | −2.478 |
| 2.5 | −1.100 | −2.388 |
| 2.6 | −1.330 | −2.178 |
| 2.7 | −1.532 | −1.864 |
| 2.4 | −1.700 | −1.486 |
| 2.9 | −1.829 | −1.091 |
| 3.0 | −1.919 | −0.722 |
| 3.1 | −1.975 | −0.402 |
| 3.2 | −2.002 | −0.140 |
| 3.3 | −2.005 | 0.068 |
| 3.4 | −1.990 | 0.231 |

| $t$ | $x$ | $y$ |
|---|---|---|
| 0.0 | 2.006 | 0.000 |
| −0.1 | 1.995 | 0.226 |
| −0.2 | 1.959 | 0.509 |
| −0.3 | 1.891 | 0.847 |
| −0.4 | 1.788 | 1.229 |
| −0.5 | 1.645 | 1.624 |
| −0.6 | 1.464 | 1.985 |
| −0.7 | 1.251 | 2.266 |
| −0.8 | 1.015 | 2.433 |
| −0.9 | 0.768 | 2.483 |
| −1.0 | 0.522 | 2.433 |
| −1.1 | 0.284 | 2.315 |
| −1.2 | 0.060 | 2.159 |
| −1.3 | −0.148 | 1.989 |
| −1.4 | −0.338 | 1.822 |
| −1.5 | −0.512 | 1.666 |
| −1.6 | −0.672 | 1.524 |
| −1.7 | −0.818 | 1.397 |
| −1.8 | −0.952 | 1.286 |
| −1.9 | −1.075 | 1.187 |
| −2.0 | −1.189 | 1.100 |
| −2.1 | −1.295 | 1.021 |
| −2.2 | −1.394 | 0.951 |
| −2.3 | −1.486 | 0.885 |
| −2.4 | −1.571 | 0.824 |
| −2.5 | −1.650 | 0.764 |
| −2.6 | −1.724 | 0.704 |
| −2.7 | −1.791 | 0.640 |
| −2.8 | −1.852 | 0.570 |
| −2.9 | −1.905 | 0.490 |
| −3.0 | −1.949 | 0.394 |
| −3.1 | −1.983 | 0.274 |
| −3.2 | −2.003 | 0.122 |

amplitude $=2.006$
period $=6.531$
characteristic exponent $=-5.432/6.531=-0.832$

**Table 5** $\lambda=1.0$

| $t$ | $x$ | $y$ | $t$ | $x$ | $y$ |
|---|---|---|---|---|---|
| 0.0 | 2.009 | 0.000 | 0.0 | 2.009 | 0.000 |
| 0.1 | 2.000 | −0.173 | −0.1 | 1.998 | 0.234 |
| 0.2 | 1.976 | −0.301 | −0.2 | 1.960 | 0.544 |
| 0.3 | 1.941 | −0.398 | −0.3 | 1.886 | 0.934 |
| 0.4 | 1.897 | −0.472 | −0.4 | 1.771 | 1.388 |
| 0.5 | 1.847 | −0.534 | −0.5 | 1.608 | 1.860 |
| 0.6 | 1.791 | −0.586 | −0.6 | 1.401 | 2.273 |
| 0.7 | 1.730 | −0.634 | −0.7 | 1.158 | 2.557 |
| 0.8 | 1.664 | −0.680 | −0.8 | 0.895 | 2.676 |
| 0.9 | 1.594 | −0.727 | −0.9 | 0.628 | 2.641 |
| 1.0 | 1.519 | −0.776 | −1.0 | 0.370 | 2.499 |
| 1.1 | 1.438 | −0.829 | −1.1 | 0.130 | 2.300 |
| 1.2 | 1.353 | −0.887 | −1.2 | −0.089 | 2.083 |
| 1.3 | 1.261 | −0.953 | −1.3 | −0.287 | 1.874 |
| 1.4 | 1.162 | −1.028 | −1.4 | −0.465 | 1.684 |
| 1.5 | 1.055 | −1.114 | −1.5 | −0.624 | 1.518 |
| 1.6 | 0.939 | −1.214 | −1.6 | −0.769 | 1.374 |
| 1.7 | 0.812 | −1.332 | −1.7 | −0.900 | 1.251 |
| 1.8 | 0.672 | −1.468 | −1.8 | −1.020 | 1.145 |
| 1.9 | 0.517 | −1.628 | −1.9 | −1.130 | 1.055 |
| 2.0 | 0.346 | −1.810 | −2.0 | −1.231 | 0.977 |
| 2.1 | 0.155 | −2.014 | −2.1 | −1.325 | 0.909 |
| 2.2 | −0.058 | −2.230 | −2.2 | −1.413 | 0.849 |
| 2.3 | −0.291 | −2.438 | −2.3 | −1.495 | 0.795 |
| 2.4 | −0.544 | −2.604 | −2.4 | −1.572 | 0.746 |
| 2.5 | −0.809 | −2.678 | −2.5 | −1.644 | 0.699 |
| 2.6 | −1.075 | −2.612 | −2.6 | −1.712 | 0.654 |
| 2.7 | −1.326 | −2.380 | −2.7 | −1.775 | 0.609 |
| 2.8 | −1.546 | −2.002 | −2.8 | −1.834 | 0.560 |
| 2.9 | −1.723 | −1.541 | −2.9 | −1.887 | 0.506 |
| 3.0 | −1.854 | −1.074 | −3.0 | −1.935 | 0.440 |
| 3.1 | −1.940 | −0.661 | −3.1 | −1.975 | 0.358 |
| 3.2 | −1.988 | −0.325 | −3.2 | −2.005 | 0.249 |
| 3.3 | −2.008 | −0.068 | −3.3 | −2.023 | 0.103 |
| 3.4 | −2.004 | 0.123 | | | |

amplitude $=2.009$
period $=6.687$
characteristic exponent $=-7.158/6.687=-1.070$

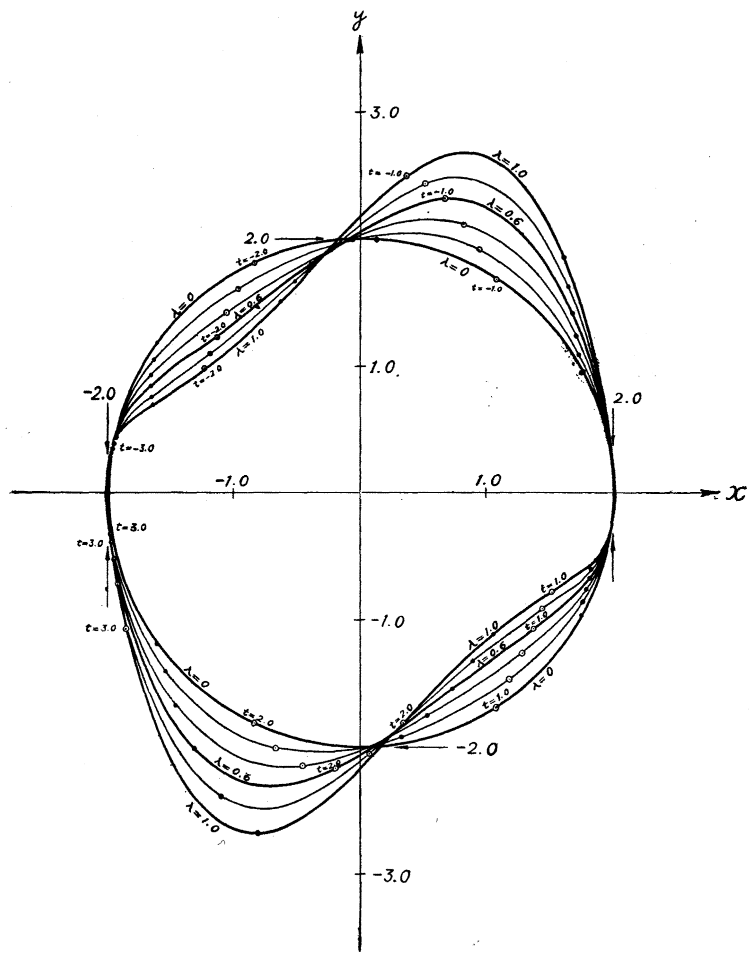


Fig. 1.

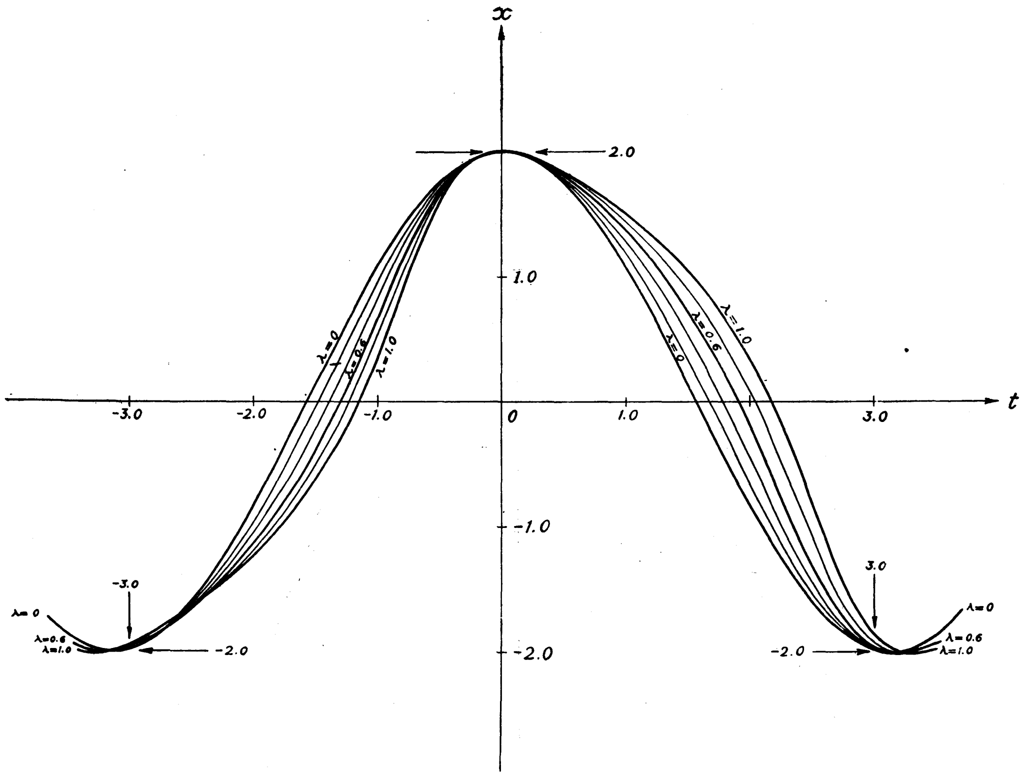
x
t
2.0
1.0
-1.0
-2.0
-3.0
-2.0
-1.0
0
1.0
3.0
λ=0
λ=0.6
λ=1.0
3.0
-3.0

Fig. 2.

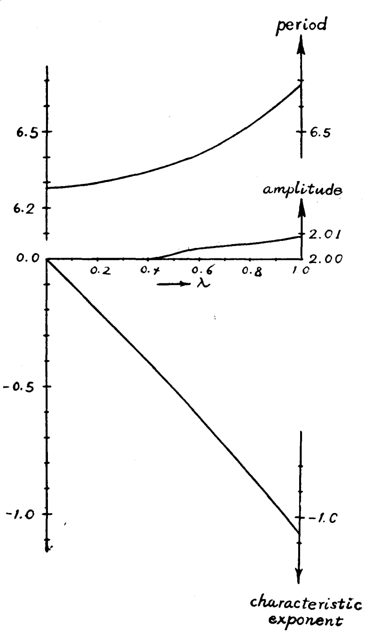
period
6.5
6.2
6.5
amplitude
2.01
2.00
0.0
0.2
0.4
0.6
0.8
1.0
λ
-0.5
-1.0
characteristic
exponent

Fig. 3.

## References

1. For instance, cf.
S. Lefschetz, *Lectures on differential equations*, Annals Math. Studies 14, Princeton (1946).
2. For instance, cf.
N. W. McLachlan, *Ordinary non-linear differential equations in engineering and physical sciences*, Oxford (1950).
3. For instance, cf.
M. Urabe, *Infinitesimal deformation of cycles*, J. Sci. Hiroshima Univ., Ser. A, **18** (1954), 37–53.
4. For instance, cf.
M. L. Cartwright, *Van der Pol's equation for relaxation oscillations*, Contributions to the theory of nonlinear oscillations II, Annals Math. Studies 29, Princeton (1952), pp. 3–18.
5. M. Urabe, *Numerical determination of periodic solution of nonlinear system*, J. Sci. Hiroshima Univ., Ser. A, **20** (1957), 125–148.

Department of Mathematics,
Faculty of Science,
Hiroshima University.