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Moving Orthonormal System along a Closed Path of an Autonomous System

By

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1 Introduction

As was used Poincaré [1][1] and later by some writers [2], a moving orthonormal system along a closed path[2] representing a periodic solution of an autonomous system is very convenient for study of orbital stability and perturbation of a periodic solution of an autonomous system, if such a moving orthonormal system exists. However, so long as the writer knows, the existence of such a moving orthonormal system seems to have not yet been proved for a general autonomous system, consequently the utilization of such a moving orthonormal system also seems to have been insufficient.

In this note, for a continuous autonomous system, the existence of such a moving orthonormal system having the same smoothness as that of the given autonomous system is established. But the method of constructing such a moving orthonormal system in the proof is not convenient to practical construction. So the convenient method available in the most cases for practical construction is added. Then three applications of such a moving orthonormal system—applications to the variational equation, the stability problem and the perturbation problem—are shown.

2 Existence of a moving orthonormal system

Given an autonomous system

$$\frac{d\boldsymbol{x}}{dt}=\boldsymbol{X}(\boldsymbol{x})^{3)}, \tag{2.1}$$

where $\boldsymbol{X}(\boldsymbol{x})$ is continuous in a domain G of a phase n-space R^n. Assume that (2.1) has a closed path

$$C:\boldsymbol{x}=\boldsymbol{\varphi}(t)$$

lying in G, and let the positive period of $\boldsymbol{\varphi}(t)$ be ω[4].

From definition of a path, it is assumed that

1) Numbers in the crotchets refer to the references listed at the end of this note.

2) This means that the moving system is periodic with the same period as that of the closed path and one of unit vectors of the system is tangent to the closed path.

3) Letters in Gothic type denote the vectors.

4) We call ω also the period of the closed path C.

(2.2) $$\bar{X}\{\varphi(t)\}\neq 0$$

for any t. Consequently, if we put

$$\bar{X}=X/\|X\|^{1)},$$

$\bar{X}\{\varphi(t)\}$ becomes continuous and does not vanish in the interval $I=[0,\omega]$. Therefore, by Borel's covering theorem, I is covered by a finite number of intervals I_m $(m=0,1,2,\cdots)$ such that at least one of $\bar{X}^1\{\varphi(t)\}$, $\bar{X}^2\{\varphi(t)\},\cdots$ $\bar{X}^n\{\varphi(t)\}$[2] never vanishes in I_m. Then, as is well known, in each I_m, we can construct continuous $(n-1)$ vectors $e_{2m}(t), e_{3m}(t),\cdots, e_{nm}(t)$ so that these adjoined with $\bar{X}\{\varphi(t)\}$ make an orthonormal system[3]. Let $U_0(t)$ be the matrix of which the column vectors are $\bar{X}\{\varphi(t)\}$ and $e_{20}(t),\cdots,e_{n0}(t)$ constructed in the interval I_0 covering $t=0$. Then, evidently, $U_0(t)$ becomes an orthogonal matrix. At any point $t=t_1$ (>0) of I_0 contained in the consecutive interval I_1, we consider the orthogonal matrix

$$M_1=U_1^{-1}(t_1)\,U_0(t_1),$$

where $U_1(t)$ is an orthogonal matrix constructed in I_1, as $U_0(t)$ is in I_0. Then, if we define $U(t)$ so that

$$U(t)=U_0(t) \quad \text{for} \quad 0\leqq t\leqq t_1$$

and

$$U(t)=U_1(t)M_1 \quad \text{for} \quad t_1\leqq t\in I_1,$$

$U(t)$ becomes a continuous orthogonal matrix defined in $I_0\cup I_1$. Taking $U(t)$ instead of $U_0(t)$, we repeat the above process, then we obtain a continuous orthogonal matrix defined in $I_0\cup I_1\cup I_2$ where I_2 is a consecutive interval of $I_0\cup I_1$, and so on. Thus, ultimately, by a finite number of processes, we obtain a continuous orthogonal matrix $U(t)$ defined in I. Now, since the first column of each $U_m(t)$ is $\bar{X}\{\varphi(t)\}$, the first column of each M_m is $\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}$,

1) $\|X\|$ denotes the Euclidean norm of X.

2) X^i denotes the i-th component of X.

3) For instance, let $\bar{X}^1\neq 0$. Then, if we put

$$\tilde{e}_2=\begin{pmatrix}-\frac{\bar{X}^2}{\bar{X}^1}\\1\\0\\0\\\vdots\\0\\0\end{pmatrix},\quad \tilde{e}_3=\begin{pmatrix}-\frac{\bar{X}^3}{\bar{X}^1}\\0\\1\\0\\\vdots\\0\\0\end{pmatrix},\cdots,\ \tilde{e}_n=\begin{pmatrix}-\frac{\bar{X}^n}{\bar{X}^1}\\0\\0\\0\\\vdots\\0\\1\end{pmatrix}$$

and determine $a_{32}(t)$; $a_{42}(t)$, $a_{43}(t)$; $\cdots$; $a_{n2}(t),\cdots, a_{nn-1}(t)$ and $\sigma_2(t),\cdots,\sigma_n(t)$ so that

$$\begin{cases} e_{2m}(t)=\sigma_2(t)\tilde{e}_2, \\ e_{3m}(t)=\sigma_3(t)\{a_{32}(t)e_{2m}(t)+\tilde{e}_3\}, \\ \cdots\cdots\cdots\cdots \\ e_{nm}(t)=\sigma_n(t)\{a_{n2}(t)e_{2m}(t)+\cdots+a_{nn-1}(t)e_{n-1m}(t)+\tilde{e}_n\} \end{cases}$$

may be mutually orthonormal, then these $e_{2m}(t)$, $e_{3m}(t),\cdots, e_{nm}(t)$ become desired vectors.

consequently the first column of $U(t)$ is always $\bar{X}\{\varphi(t)\}$.

Put

$$U^{-1}(\omega)U(0)=A, \tag{2.3}$$

then A is evidently orthogonal and moreover is proper ($\det A=1$), for $\det U(t)=\det U(0)$ because of continuity of $U(t)$. Further the first columns of $U(0)$ and $U(\omega)$ are both $\bar{X}\{\varphi(0)\}=\bar{X}\{\varphi(\omega)\}$, consequently A is of the form

$$A=\begin{pmatrix} 1 & 0 \\ 0 & A_1 \end{pmatrix}, \tag{2.4}$$

where A_1 is a proper orthogonal matrix of the order $n-1$. Then, as is well known, there exists an orthogonal matrix T_1 such that

$$T_1^{-1}A_1T_1=\sum_r \oplus \begin{pmatrix} \cos\alpha_r & -\sin\alpha_r \\ \sin\alpha_r & \cos\alpha_r \end{pmatrix} \oplus \begin{pmatrix} 1 & & 0 \\ & 1 \ddots & \\ 0 & & 1 \end{pmatrix}, \tag{2.5}$$

where α_r's are not integral multiples of 2π and the symbol $\oplus$ denotes the direct sum of matrices. Now, for matrices under the sign $\sum\oplus$ in the right-hand side of (2.5), it holds that

$$\begin{pmatrix} \cos\alpha_r & -\sin\alpha_r \\ \sin\alpha_r & \cos\alpha_r \end{pmatrix}=\exp\begin{pmatrix} 0 & -\alpha_r \\ \alpha_r & 0 \end{pmatrix},$$

consequently, if we put

$$(0)\oplus\sum_r\oplus\begin{pmatrix} 0 & -\alpha_r \\ \alpha_r & 0 \end{pmatrix}\oplus\begin{pmatrix} 0 & & 0 \\ & 0 \ddots & \\ 0 & & 0 \end{pmatrix}=\omega B_0 \tag{2.6}$$

and

$$\begin{pmatrix} 1 & 0 \\ 0 & T_1 \end{pmatrix}=T, \tag{2.7}$$

then, from (2.4) and (2.5), we have

$$T^{-1}AT=e^{\omega B_0}.$$

Therefore, if we put

$$TB_0T^{-1}=B, \tag{2.8}$$

we obtain

$$A=e^{\omega B},$$

from which (2.3) is written as follows:

$$U(0)=U(\omega)e^{\omega B}. \tag{2.9}$$

Then, let us consider the matrix

$$\bar{V}(t)=U(t)e^{tB}. \tag{2.10}$$

From (2.6), (2.7) and (2.8), the first row and the first column of B are zero vectors, consequently the first column of e^{tB} is $\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}$. Therefore the first column of $\bar{V}(t)$ is always $\bar{\boldsymbol{X}}\{\boldsymbol{\varphi}(t)\}$. On the other hand, from (2.6) follows

$$B_0^* = -B_0{}^{1)},$$

consequently it holds that

$$\begin{aligned}(e^{tB})^* e^{tB} &= (Te^{tB_0}T^{-1})^*(Te^{tB_0}T^{-1})\\ &= T^{*-1}e^{tB_0{}^*}T^* Te^{tB_0}T^{-1}\\ &= E \qquad\qquad \text{(unit matrix)},\end{aligned}$$

from which follows orthogonality of $\bar{V}(t)$. In addition, from (2.9) follows

$$\bar{V}(\omega) = U(0) = \bar{V}(0).$$

Thus, making extension of $\bar{V}(t)$ outside I by the relation

$$\bar{V}(t+\omega) = \bar{V}(t),$$

we see that there exists a continuous orthogonal periodic matrix $\bar{V}(t)$ along C[2], namely that *there exists a continuous moving orthonormal system along C of which unit vectors are the column vectors of $\bar{V}(t)$.*

Next, let us consider the case where $\boldsymbol{X}(\boldsymbol{x}) \in C^N[G]$. Let $\bar{\xi}_\nu(t)$ $(\nu=2,3,\cdots,n)$ be the ν-th column vector of $\bar{V}(t)$. Since $\bar{\xi}_\nu(t)$'s are continuous and periodic, they are uniformly continuous, therefore, for any given $\varepsilon>0$, there exists a positive number δ such that

$$\|\bar{\xi}_\nu(s)-\bar{\xi}_\nu(t)\|<\varepsilon \tag{2.11}$$

when $|s-t|<\delta$. Put

$$\tilde{\xi}_\nu(t) = \frac{1}{(2\delta)^N}\int_{t-\delta}^{t+\delta}\int_{s_{N-1}-\delta}^{s_{N-1}+\delta}\cdots\int_{s_1-\delta}^{s_1+\delta}\bar{\xi}_\nu(s_0)\,ds_0\,ds_1\cdots ds_{N-1}{}^{3)} \tag{2.12}$$

and let $\tilde{V}(t)$ be the matrix of which the column vectors are $\bar{\boldsymbol{X}}\{\boldsymbol{\varphi}(t)\}$ and $\tilde{\xi}_\nu(t)$ $(\nu=2,3,\cdots,n)$. Then, by Schwarz's inequality, from (2.11) follows

$$\|\tilde{\xi}_\nu(t)-\bar{\xi}_\nu(t)\| = \left\|\frac{1}{(2\delta)^N}\int_{t-\delta}^{t+\delta}\int_{s_{N-1}-\delta}^{s_{N-1}+\delta}\cdots\int_{s_1-\delta}^{s_1-\delta}[\bar{\xi}_\nu(s_0)-\bar{\xi}_\nu(t)]\,ds_0\,ds_1\cdots ds_{N-1}\right\|$$

1) The symbol * means transposed.
2) The term "along" means that the first column vector is tangent to the path C.
3) When $N=1$, we suppose that $s_1=t$.

$$\leq \frac{1}{(2\delta)^N}\left[2\delta\cdot\int_{t-\delta}^{t+\delta}\left\|\int_{s_{N-1}-\delta}^{s_{N-1}+\delta}\cdots\int_{s_1-\delta}^{s_1+\delta}[\bar{\xi}_\nu(s_0)-\bar{\xi}_\nu(t)]\,ds_0\cdots ds_{N-2}\right\|^2 ds_{N-1}\right]^{\frac{1}{2}}$$

$$\leq \varepsilon,$$

consequently, when ε is sufficiently small,

$$\det \tilde{V}(t)\neq 0 \quad \text{in} \quad -\infty<t<\infty \tag{2.13}$$

because $\det \bar{V}(t)=1$ or -1.

From (2.12) and periodicity of $\bar{\xi}_\nu(s)$ follows

$$\tilde{\xi}_\nu(t+\omega)=\frac{1}{(2\delta)^N}\int_{t+\omega-\delta}^{t+\omega+\delta}\int_{s_{N-1}-\delta}^{s_{N-1}+\delta}\cdots\int_{s_1-\delta}^{s_1+\delta}\bar{\xi}_\nu(s_0)ds_0ds_1\cdots ds_{N-1}$$

$$=\frac{1}{(2\delta)^N}\int_{t-\delta}^{t+\delta}\int_{s_{N-1}+\omega-\delta}^{t_{N-1}+\omega+\delta}\cdots\int_{s_1-\delta}^{s_1+\delta}\bar{\xi}_\nu(s_0)\,ds_0\,ds_1\cdots ds_{N-1}$$

$$=\frac{1}{(2\delta)^N}\int_{t-\delta}^{t+\delta}\int_{s_{N-1}-\delta}^{s_{N-1}+\delta}\cdots\int_{s_1-\delta}^{s_1+\delta}\bar{\xi}_\nu(s_0+\omega)\,ds_0\,ds_1\cdots ds_{N-1}$$

$$=\tilde{\xi}_\nu(t),$$

consequently it holds that

$$\tilde{V}(t+\omega)=\tilde{V}(t).$$

Further, from (2.12) follows

$$\frac{d\tilde{\xi}_\nu(t)}{dt}=\frac{1}{(2\delta)^N}\left[\int_t^{t+2\delta}\int_{s_{N-2}-\delta}^{s_{N-2}+\delta}\cdots\int_{s_1-\delta}^{s_1+\delta}\bar{\xi}_\nu(s_0)\,ds_0\cdots ds_{N-2}\right.$$

$$\left.-\int_{t-2\delta}^{t}\int_{s_{N-2}-\delta}^{s_{N-2}+\delta}\cdots\int_{s_1-\delta}^{s_1+\delta}\bar{\xi}_\nu(s_0)\,ds_0\cdots ds_{N-2}\right],$$

. . . .

$$\frac{d^N\tilde{\xi}_\nu(t)}{dt^N}=\frac{1}{(2\delta)^N}[\bar{\xi}_\nu(t+N\delta)-\binom{N}{1}\bar{\xi}_\nu\{t+(N-2)\delta\}+\cdots+(-1)^N\bar{\xi}_\nu(t-N\delta)],$$

namely $\tilde{\xi}_\nu(t)\in C^N$.

Then, by the well-known method, we can construct an orthogonal matrix $V(t)\in C^N$ along C with period ω of which the column vectors are

$$\bar{X}\{\varphi(t)\},\ \xi_2(t),\cdots,\ \xi_n(t),$$

namely by choosing coefficients $a_{21}(t)$; $a_{31}(t)$, $a_{32}(t)$;$\cdots$; $a_{n1}(t),\cdots,a_{nn-1}(t)$ and factors $\sigma_2(t),\cdots,\sigma_n(t)$ so that

$$\xi_2(t)=\sigma_2(t)\{a_{21}(t)\bar{X}+\tilde{\xi}_2\},$$

$$\xi_3(t)=\sigma_3(t)\{a_{31}(t)\overline{X}+a_{32}(t)\xi_2+\tilde{\xi}_3\},$$

$$\cdots\cdots\cdots\cdots$$

$$\xi_n(t)=\sigma_n(t)\{a_{n1}(t)\overline{X}+a_{n2}(t)\xi_2+\cdots+a_{nn-1}(t)\xi_{n-1}+\tilde{\xi}_n\}$$

may be orthogonal to $\overline{X}\{\varphi(t)\}$ and be mutually orthonormal. Thus it is seen that *there exists a moving orthonormal system along C belonging to* C^N *of which the unit vectors are the column vectors of* $V(t)$.

Let $\{\overline{X}\{\varphi(t)\}, \zeta_2(t),\cdots,\zeta_n(t)\}$ be any moving orthonormal system along C, then the matrix $Z(t)$ of which the column vectors are $\overline{X}, \zeta_2,\cdots,\zeta_n$ is evidently expressed as follows:

$$Z(t)=Z_0(t)M(t), \tag{2.14}$$

where $Z_0(t)$ is a matrix of which the column vectors are unit vectors of any moving orthonormal system along C—for example, the system determined by the above $V(t)$, and $M(t)$ is an arbitrary orthogonal periodic matrix of the form

$$M(t)=(1)\oplus M_1(t). \tag{2.15}$$

Thus summarizing the results, we have

Theorem 1. *For a continuous autonomous system* (2.1), *there exists always a moving orthonormal system along its closed path C having the same smoothness as that of the given autonomous system.*

If $Z_0(t)$ *is a matrix of which the column vectors are unit vectors of a moving orthonormal system along C, then any other moving orthonormal system along C is given by the column vectors of* $Z(t)$ *given by* (2.14). *Here* $M(t)$ *is an arbitrary orthogonal matrix of the form* (2.15) *having the same smoothness as that of* $Z(t)$ *and* $Z_0(t)$.

From (2.14), it is readily seen that, *when any orthonormal system in the normal hyperplane of C is given at any point P of C, we can always construct a moving orthonormal system along C so that its orthonormal system in the normal hyperplane at P may coincide with the given orthonormal system at P.* For, this is always possible by choosing a suitable constant matrix $M(t)$ of the form (2.15).

3 Convenient method to construct a moving orthonormal system

In the preceding paragraph, in order to prove the existence of a moving orthonormal system along a closed path, we have shown a method to construct such a moving orthonormal system. But, as is seen, that method is not suited to practical construction of a moving orthonormal system along a closed path. Therefore, in this paragraph, we seek for a convenient method available in the most cases for practical construction.

First, let us consider the case where $n \geqq 3$.

We choose a unit vector $\boldsymbol{e}_1$ so that $\overline{\boldsymbol{X}}\{\boldsymbol{\varphi}(t)\}$ never coincides with $-\boldsymbol{e}_1$. This is possible in the most cases. For, the point $\overline{\boldsymbol{X}}\{\boldsymbol{\varphi}(t)\}$ describes a curve on the unit hypersphere F in the n-dimensional Euclidean space, consequently the set of points $\overline{\boldsymbol{X}}\{\boldsymbol{\varphi}(t)\}$ $(-\infty < t < \infty)$ does not cover the hypersphere F in the most cases.

Starting from this $\boldsymbol{e}_1$, we construct an arbitrary constant orthonormal system $\{\boldsymbol{e}_i\}$ $(i=1,2,\cdots,n)$ and put

$$\boldsymbol{e}_i^* \overline{\boldsymbol{X}}\{\boldsymbol{\varphi}(t)\} = \cos\theta_i \qquad (i=1,2,\cdots,n). \tag{3.1}$$

Then, from $\|\overline{\boldsymbol{X}}\|=1$ follows

$$\cos^2\theta_1 + \cos^2\theta_2 + \cdots + \cos^2\theta_n = 1 \tag{3.2}$$

and, from the manner of choosing $\boldsymbol{e}_1$ follows

$$\cos\theta_1 + 1 \neq 0. \tag{3.3}$$

For a moment, assuming that

$$\cos\theta_1 \neq 1, \tag{3.4}$$

we rotate the system $\{\boldsymbol{e}_i\}$ about the $(n-2)$-dimensional subspace S perpendicular both to $\boldsymbol{e}_1$ and $\overline{\boldsymbol{X}}$ until $\boldsymbol{e}_1$ coincides with $\overline{\boldsymbol{X}}$. Let $\boldsymbol{\xi}_\nu$ $(\nu=2,3,\cdots,n)$ be the final position of $\boldsymbol{e}_\nu$. If we write $\boldsymbol{e}_\nu$ $(\nu=2,3,\cdots,n)$ as

$$\boldsymbol{e}_\nu = \bar{\boldsymbol{e}}_\nu + \lambda_\nu \boldsymbol{e}_1 + \mu_\nu \overline{\boldsymbol{X}} \tag{3.5}$$

where $\bar{\boldsymbol{e}}_\nu$ is a component vector of $\boldsymbol{e}_\nu$ in S, then $\boldsymbol{\xi}_\nu$ is written in the form

$$\boldsymbol{\xi}_\nu = \bar{\boldsymbol{e}}_\nu + \lambda'_\nu \bar{\boldsymbol{e}}_1 + \mu'_\nu \overline{\boldsymbol{X}}. \tag{3.6}$$

Then, from orthogonality of $\bar{\boldsymbol{e}}_\nu$, it holds that

$$\begin{cases} \boldsymbol{e}_1^* (\boldsymbol{e}_\nu - \lambda_\nu \boldsymbol{e}_1 - \mu_\nu \overline{\boldsymbol{X}}) = 0, \\ \overline{\boldsymbol{X}}^* (\boldsymbol{e}_\nu - \lambda_\nu \boldsymbol{e}_1 - \mu_\nu \overline{\boldsymbol{X}}) = 0, \end{cases}$$

from which follows

$$\begin{cases} \lambda_\nu + \mu_\nu \cos\theta_1 = 0, \\ \lambda_\nu \cos\theta_1 + \mu_\nu = \cos\theta_\nu. \end{cases}$$

From (3.3) and (3.4), these equations are solved as follows:

$$\lambda_\nu = -\frac{\cos\theta_1 \cos\theta_\nu}{\sin^2\theta_1}, \quad \mu_\nu = \frac{\cos\theta_\nu}{\sin^2\theta_1}. \tag{3.7}$$

Since the transformation of $\lambda_\nu \boldsymbol{e}_1 + \mu_\nu \overline{\boldsymbol{X}}$ to $\lambda'_\nu \boldsymbol{e}_1 + \mu'_\nu \overline{\boldsymbol{X}}$ is a rotation in a plane by an angle θ_1, it is readily seen that

$$(3.8)\qquad \lambda_\nu' = -\mu_\nu,\quad \mu_\nu' = \lambda_\nu + 2\mu_\nu \cos\theta_1.$$

Substituting (3.8) into (3.6), from (3.5) and (3.7), we have

$$(3.9)\qquad \begin{aligned}\boldsymbol{\xi}_\nu &= \boldsymbol{e}_\nu - (\lambda_\nu \boldsymbol{e}_1 + \mu_\nu \overline{\boldsymbol{X}}) - \mu_\nu \boldsymbol{e}_1 + (\lambda_\nu + 2\mu_\nu \cos\theta_1)\overline{\boldsymbol{X}} \\ &= \boldsymbol{e}_\nu - \frac{\cos\theta_\nu}{1+\cos\theta_1}(\boldsymbol{e}_1 + \overline{\boldsymbol{X}}).\end{aligned}$$

Now we remove the temporary assumption (3.4) and define $\boldsymbol{\xi}_\nu$ anew by (3.9). Then, from (3.1) and (3.3), it is evident that $\boldsymbol{\xi}_\nu(t)$'s have the same smoothness as that of $\boldsymbol{X}\{\boldsymbol{\varphi}(t)\}$, namely that of $\overline{\boldsymbol{X}}(\boldsymbol{x})$ and that they are periodic with period ω. Thus the orthonormal system $\{\overline{\boldsymbol{X}}\{\boldsymbol{\varphi}(t)\}, \boldsymbol{\xi}_2(t), \cdots, \boldsymbol{\xi}_n(t)\}$ provides a moving orthonormal system along C having the same smoothness as that of the given autonomous system (2.1).

When $n=2$, the vector $\boldsymbol{\xi}_2$ defined by (3.9) becomes

$$\begin{aligned}\boldsymbol{\xi}_2 &= \boldsymbol{e}_2 - \frac{\cos\theta_2}{1+\cos\theta_1}\{\boldsymbol{e}_1 + (\cos\theta_1\cdot\boldsymbol{e}_1 + \cos\theta_2\cdot\boldsymbol{e}_2)\} \\ &= -\cos\theta_2\cdot\boldsymbol{e}_1 + \cos\theta_1\cdot\boldsymbol{e}_2.\end{aligned}$$

As is readily seen, this means that

$$\xi_2^1 = \mp\overline{X}^2,\quad \xi_2^2 = \pm\overline{X}^1.$$

Then $\boldsymbol{\xi}_2$ defined by (3.9) adjoined with $\overline{\boldsymbol{X}}\{\boldsymbol{\varphi}(t)\}$ provides a desired orthonormal system also when $n=2$, even though (3.3) does not hold in this case.

Thus, *so long as, for* $n\geqq 3$, *we can choose* $\boldsymbol{e}_1$ *so that* (3.3) *may hold always, we can easily construct a moving orthonormal system by taking* $\boldsymbol{\xi}_\nu$*'s given by* (3.9) *as the normal unit vectors.*

4 Variational equation

In this paragraph, assuming that $\boldsymbol{X}(\boldsymbol{x})\in C^1$, let us consider the variational equation with respect to a moving orthonormal system and compare it to the initial variational equation

$$(4.1)\qquad \frac{d\boldsymbol{y}}{dt} = A(t)\boldsymbol{y}$$

where $A(t) = (\partial X^i/\partial x^j)_{\boldsymbol{x}=\boldsymbol{\varphi}(t)}$.

Let the unit vectors of a continuously differentiable moving orthonormal system along a closed path C be $\overline{\boldsymbol{X}} = \boldsymbol{X}/\|\boldsymbol{X}\|$, $\boldsymbol{\xi}_\nu(t)$ $(\nu=2,3,\cdots,n)$, then any point $\boldsymbol{x}=\boldsymbol{x}(\tau)$ of a path C' lying near C is represented by

$$(4.2)\qquad \boldsymbol{x} = \boldsymbol{x}(\tau) = \boldsymbol{\varphi}(t) + \sum_{\nu=2}^{n} \rho^\nu \boldsymbol{\xi}_\nu(t)$$

and this satisfies the equation

(4.3)
$$\frac{d\boldsymbol{x}}{d\tau}=\boldsymbol{X}(\boldsymbol{x}),$$

where τ is a time required to reach any normal hyperplace of C along C' from any fixed normal hyperplane of C. If we put

$$\boldsymbol{F}(\tau, \rho^\nu, t)=\boldsymbol{x}(\tau)-\boldsymbol{\varphi}(t)-\sum_{\nu=2}^{n}\rho^\nu\boldsymbol{\xi}_\nu(t),$$

then, from (4.3), it is evident that

$$\frac{\partial\boldsymbol{F}}{\partial\tau}=\boldsymbol{X}\{\boldsymbol{x}(\tau)\}, \qquad \frac{\partial\boldsymbol{F}}{\partial\rho^\nu}=-\boldsymbol{\xi}_\nu(t),$$

consequently the Jacobian of F^i with respect to τ and ρ^ν does not vanish so long as C' lies near C. Therefore, for sufficiently small $|\rho^\nu|$, we can solve (4.2) with respect to τ and ρ^ν, and we obtain the continuously differentiable functions $\tau=\tau(t)$ and $\rho^\nu=\rho^\nu(t)$. Then, substituting (4.2) into (4.3), we have

$$\frac{d\boldsymbol{\varphi}}{dt}+\sum_\nu\frac{d\rho^\nu}{dt}\boldsymbol{\xi}_\nu+\sum_\nu\rho^\nu\frac{d\boldsymbol{\xi}_\nu}{dt}=\boldsymbol{X}(\boldsymbol{\varphi}+\sum_\nu\rho^\nu\boldsymbol{\xi}_\nu)\frac{d\tau}{dt},$$

namely

(4.4)
$$\sum_\nu\frac{d\rho^\nu}{dt}\boldsymbol{\xi}_\nu+\sum_\nu\rho^\nu\frac{d\boldsymbol{\xi}_\nu}{dt}=\boldsymbol{X}'\frac{d\tau}{dt}-\boldsymbol{X}$$

where

$$\boldsymbol{X}'=\boldsymbol{X}(\boldsymbol{\varphi}+\sum_\nu\rho^\nu\boldsymbol{\xi}_\nu).$$

Multiplying $\boldsymbol{X}^*$ on both sides of (4.4), we have:

(4.5)
$$\frac{d\tau}{dt}=\frac{\|\boldsymbol{X}\|^2+\sum_\nu\rho^\nu\boldsymbol{X}^*\frac{d\boldsymbol{\xi}_\nu}{dt}}{\boldsymbol{X}^*\boldsymbol{X}'},$$

consequently, multiplying $\boldsymbol{\xi}_\nu^*$ on both sides of (4.4), we have:

(4.6)
$$\frac{d\rho^\nu}{dt}=\frac{\|\boldsymbol{X}\|^2+\sum_\mu\rho^\mu\boldsymbol{X}^*\frac{d\boldsymbol{\xi}_\mu}{dt}}{\boldsymbol{X}^*\boldsymbol{X}'}\cdot\boldsymbol{\xi}_\nu^*\boldsymbol{X}'-\sum_\mu\rho^\mu\boldsymbol{\xi}_\nu^*\frac{d\boldsymbol{\xi}_\mu}{dt}.$$

Now $\boldsymbol{X}'$ is written as

$$\boldsymbol{X}'=\boldsymbol{X}+\sum_\mu\rho^\mu A(t)\boldsymbol{\xi}_\mu+o(\|\boldsymbol{\rho}\|)$$

where $\boldsymbol{\rho}$ is an $(n-1)$-dimensional vector whose components are ρ^ν. Therefore the right-hand sides of (4.6) are written in the vector form as follows:

$$\boldsymbol{R}(\boldsymbol{\rho}, t)=\Xi(t)\boldsymbol{\rho}+o(\|\boldsymbol{\rho}\|)$$

where $\Xi(t)$ is an $(n-1)$-$(n-1)$ matrix of which $\nu\mu$-element $\Xi_\mu^\nu(t)$ is

$$\Xi_{\mu}^{\nu}(t)=\xi_{\nu}^{*}A(t)\xi_{\mu}-\xi_{\nu}^{*}\frac{d\xi_{\mu}}{dt}.$$

Thus, in vector notations, (4.6) is written as follows:

$$\frac{d\boldsymbol{\rho}}{dt}=\Xi(t)\boldsymbol{\rho}+o(\|\boldsymbol{\rho}\|). \tag{4.7}$$

If we neglect the terms of higher order, we obtain

$$\frac{d\boldsymbol{\rho}}{dt}=\Xi(t)\boldsymbol{\rho}. \tag{4.8}$$

This is an equation for normal variation of solutions with respect to a moving orthonormal system along C.

In the sequel, we seek for the bearing of this equation to the initial variational equation (4.1). Since any solution of (4.1) can be expressed as

$$\boldsymbol{y}=p(t)\boldsymbol{X}+\sum_{\mu=2}^{n}p^{\mu}(t)\xi_{\mu}, \tag{4.9}$$

substitution of this into (4.1) entails

$$\frac{dp}{dt}\boldsymbol{X}+p\frac{d\boldsymbol{X}}{dt}+\sum_{\mu}\frac{dp^{\mu}}{dt}\xi_{\mu}+\sum_{\mu}p^{\mu}\frac{d\xi_{\mu}}{dt}=A(t)(p\boldsymbol{X}+\sum_{\mu}p^{\mu}\xi_{\mu}),$$

namely

$$\sum_{\mu}\frac{dp^{\mu}}{dt}\xi_{\mu}+\frac{dp}{dt}\boldsymbol{X}=\sum_{\mu}p^{\mu}\left(A(t)\xi_{\mu}-\frac{d\xi_{\mu}}{dt}\right)$$

because $d\boldsymbol{X}/dt=A(t)\boldsymbol{X}$. Multiplying ξ_{ν}^{*} and $\boldsymbol{X}^{*}$ on both sides, we have respectively

$$\frac{d\boldsymbol{p}}{dt}=\Xi(t)\boldsymbol{p}, \tag{4.10}$$

and

$$\frac{dp}{dt}=\frac{1}{\|\boldsymbol{X}\|^{2}}\sum_{\mu}p^{\mu}\left(\boldsymbol{X}^{*}A(t)\xi_{\mu}-\boldsymbol{X}^{*}\frac{d\xi_{\mu}}{dt}\right), \tag{4.11}$$

where $\boldsymbol{p}$ denotes an $(n-1)$-dimensional vectors whose components are p^{μ}. The equation (4.10) is an equation for normal components of variation of solutions and, as is predicted at the beginning, it coincides with (4.8).

Let $\Phi(t)$ be a fundamental matrix of (4.1) of which the column vectors are $\boldsymbol{X}\{\boldsymbol{\varphi}(t)\}$, $\boldsymbol{y}_{\nu}(t)$ $(\nu=2,3,\cdots,n)$, then there exists a regular constant matrix K such that

$$\Phi(t+\omega)=\Phi(t)K. \tag{4.12}$$

The characteristic roots of K are called *multipliers of solutions of* (4.1) and their logarithms divided by ω are called *characteristic exponents* of (4.1).

If we put $K=(k_\nu^\mu)$ and

$$\boldsymbol{y}_\nu=p_\nu(t)\boldsymbol{X}+\sum_\mu p_\nu^\mu(t)\boldsymbol{\xi}_\mu,$$

then, for $t=0$, the relation (4.12) is written as follows:

$$\begin{cases} \boldsymbol{X}\{\boldsymbol{\varphi}(0)\}=\boldsymbol{X}\{\boldsymbol{\varphi}(0)\}k_1^1+\sum_\mu \boldsymbol{y}_\mu(0)k_1^\mu, \\ p_\nu(\omega)\boldsymbol{X}\{\boldsymbol{\varphi}(0)\}+\sum_\mu p_\nu^\mu(\omega)\boldsymbol{\xi}_\mu(0) \end{cases}$$

$$=\boldsymbol{X}\{\boldsymbol{\varphi}(0)\}k_\nu^1+\sum_\mu [p_\mu(0)\boldsymbol{X}\{\boldsymbol{\varphi}(0)\}+\sum_\kappa p_\mu^\kappa(0)\boldsymbol{\xi}_\kappa(0)]k_\nu^\mu$$

$$=\{k_\nu^1+\sum_\mu k_\nu^\mu p_\mu(0)\}\boldsymbol{X}\{\boldsymbol{\varphi}(0)\}+\sum_\kappa \{\sum_\mu k_\nu^\mu p_\mu^\kappa(0)\}\boldsymbol{\xi}_\kappa(0),$$

consequently, from linear independence of $\boldsymbol{X}, \boldsymbol{y}_2,\cdots, \boldsymbol{y}_n$ and of $\boldsymbol{X}, \boldsymbol{\xi}_2,\cdots, \boldsymbol{\xi}_n$, follows

(4.13) $$k_1^1=1, \quad k_1^\nu=0;$$

(4.14) $$\begin{cases} \text{(i)} & p_\nu(\omega)=\sum_\mu k_\nu^\mu p_\mu(0)+k_\nu^1, \\ \text{(ii)} & p_\nu^\mu(\omega)=\sum_\kappa k_\nu^\kappa p_\kappa^\mu(0). \end{cases}$$

Now, $\det(p_\nu^\mu(0))\neq 0$, for, if $\det(p_\nu^\mu(0))=0$, there exist numbers c^ν not all vanishing such that

$$\sum_\nu c^\nu p_\nu^\mu(0)=0,$$

consequently it is valid that

$$\sum_\nu c^\nu \boldsymbol{y}_\nu(0)=\sum_\nu c^\nu p_\nu(0)\cdot \boldsymbol{X}\{\boldsymbol{\varphi}(0)\},$$

from which follows $c^\nu=0$ ($\nu=2,3,\cdots, n$) because of linear independence of $\boldsymbol{X}\{\boldsymbol{\varphi}(0)\}$, $\boldsymbol{y}_2(0),\cdots, \boldsymbol{y}_n(0)$. This is a contradiction. Then the matrix $(p_\nu^\mu(t))$ becomes a fundamental matrix of (4.10).

Further, from (4.13), K is of the form $K=\begin{pmatrix} 1 & K_1 \\ 0 & K_2 \end{pmatrix}$ where K_1 is a 1-($n-1$) matrix and K_2 is an ($n-1$)-($n-1$) matrix. Then, from (ii) of (4.14), the characteristic roots of K_2 become the multipliers of solutions of (4.10). Now, as is seen from the form of K, the characteristic roots of K_2 are those left by excluding 1 once from the characteristic roots of K, namely from the multipliers of solutions of the initial variational equation (4.1).

Thus, summarizing the above, we have

Theorem 2. *The equation* (4.8) *for normal variation of solutions coincides with the equation* (4.10) *for normal components of variation of solutions.*

The multipliers of solutions of these equations are those left by excluding 1 *once from the multipliers of solutions of the initial variational equation* (4.1).

The latter half of the above theorem can also be stated as follows:

The characteristic exponents of the equation (4.8) *or* (4.10) *are those left by excluding an integral multiple of* $2\pi i/\omega$ *once from the characteristic exponents of the initial variational equation* (4.1).

5 Orbital stability

When all but one of the characteristic exponents of the variational equation (4.1) have negative real parts, by Theorem 2, all the characteristic exponents of (4.8) have negative real parts. Then, as is well known [3], for the solution $\boldsymbol{\rho}=\boldsymbol{\rho}(t)$ of (4.7) such that $\|\boldsymbol{\rho}(0)\|$ is sufficiently small, $\|\boldsymbol{\rho}(t)\|$ remains small in $0\leqq t<\infty$ and there exist the positive constants R and α such that

$$\|\boldsymbol{\rho}(t)\|\leqq Re^{-\alpha t} \quad \text{in} \quad 0\leqq t<\infty. \tag{5.1}$$

Therefore we have the well known

Theorem 3. *When all but one of the characteristic exponents of the variational equation* (4.1) *have negative real parts, the periodic solution* $\boldsymbol{x}=\boldsymbol{\varphi}(t)$ *is asymptotically orbitally stable.*

Further, when the assumption of this theorem is valid, for $0<t'<t''$, from (4.5) follows

$$\begin{aligned}&\{\tau(t'')-t''\}-\{\tau(t')-t'\}\\&=\int_{t'}^{t''}\left(\frac{\|\boldsymbol{X}\|^2+\sum_\nu\rho^\nu\boldsymbol{X}^*\dfrac{d\boldsymbol{\xi}_\nu}{dt}}{\boldsymbol{X}^*\boldsymbol{X}'}-1\right)dt\\&=\int_{t'}^{t''}\frac{\sum_\nu\rho^\nu\left(\boldsymbol{X}^*\dfrac{d\boldsymbol{\xi}_\nu}{dt}-\boldsymbol{X}^*A(t)\boldsymbol{\xi}_\nu\right)+o(\|\boldsymbol{\rho}\|)}{\boldsymbol{X}^*\boldsymbol{X}'}\,dt.\end{aligned}$$

Consequently, for certain positive constant R_1, it holds that

$$|\{\tau(t'')-t''\}-\{\tau(t')-t'\}|\leqq R_1\int_{t'}^{t''}e^{-\alpha t}\,dt<\frac{R_1}{\alpha}e^{-\alpha t'}.$$

The right-hand side of this inequality tends to zero as $t'\to+\infty$, therefore there exists a constant t_0 such that

$$\tau(t)-t\to t_0 \quad \text{as} \quad t\to+\infty. \tag{5.2}$$

Now, since

$$\boldsymbol{x}\{\tau(t)\}-\boldsymbol{x}(t+t_0)=\int_{t+t_0}^{\tau(t)}\boldsymbol{X}\{\boldsymbol{x}(\tau)\}\,d\tau,$$

by Schwartz's inequality, it holds that

$$\|\boldsymbol{x}\{\tau(t)\}-\boldsymbol{x}(t+t_0)\|\leqq|\tau(t)-(t+t_0)|\cdot\text{bound of }\|\boldsymbol{X}\{\boldsymbol{x}(\tau)\}\|.$$

But, when $\| \boldsymbol{\rho}(0) \|$ is sufficiently small, $\| \boldsymbol{\rho}(t) \|$ remains small in $0 \leqq t < \infty$, namely C' lies near C, consequently $\| \boldsymbol{X}\{\boldsymbol{x}(\tau)\} \|$ remains bounded. Then, from (5.2) follows

$$\lim_{t \to +\infty} \| \boldsymbol{x}\{\tau(t)\} - \boldsymbol{x}(t+t_0) \| = 0.$$

On the other hand, from (5.1) and (4.2), follows

$$\lim_{t \to +\infty} \| \boldsymbol{x}\{\tau(t)\} - \boldsymbol{\varphi}(t) \| = 0.$$

Therefore we have

$$\lim_{t \to +\infty} \| \boldsymbol{x}(t+t_0) - \boldsymbol{\varphi}(t) \| = 0,$$

or

$$\lim_{\tau \to +\infty} \| \boldsymbol{x}(\tau) - \boldsymbol{\varphi}(\tau - t_0) \| = 0.$$

Thus we have also

Theorem 4[1]. *When all but one of the characteristic exponents of the variational equation* (4.1) *have negative real parts, the periodic solution* $\boldsymbol{x} = \boldsymbol{\varphi}(t)$ *is not only asymptotically orbitally stable, but, as* $t \to +\infty$, *all the neighboring solution* $\boldsymbol{x} = \boldsymbol{x}(\tau)$ *of the periodic solution tend to the periodic solution itself except for certain differences of the phases.*

6 Perturbation of an autonomous system

In this paragraph, let us show how the perturbation problem of an autonomous system can be discussed referring to the moving orthonormal system.

Let the perturbed system of (2.1) be

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{X}(\boldsymbol{x}, \varepsilon) \tag{6.1}$$

where $\boldsymbol{X}(\boldsymbol{x}, \varepsilon)$ is continuously differentiable with respect to $\boldsymbol{x}$ and ε for $\boldsymbol{x} \in G$ and $|\varepsilon| < \delta$. Assume that the system (6.1) coincides with (2.1) when $\varepsilon = 0$ and that (2.1) has a closed path $C : \boldsymbol{x} = \boldsymbol{\varphi}(t)$ lying in G with positive period ω.

As in the paragraph 4, with respect to a moving orthonormal system along C, any point of a path C' of (6.1) lying near C is represented by

$$\boldsymbol{x} = \boldsymbol{x}(\tau) = \boldsymbol{\varphi}(t) + \sum_{\nu=2}^{n} \rho^{\nu}(t, \varepsilon) \boldsymbol{\xi}_{\nu}(t) \tag{6.2}$$

and this satisfies the equation

$$\frac{d\boldsymbol{x}}{d\tau} = \boldsymbol{X}(\boldsymbol{x}, \varepsilon). \tag{6.3}$$

1) Cf. [3].

Conseqently, as in the paragraph 4, we have:

$$\frac{d\tau}{dt}=\frac{\|\boldsymbol{X}\|^2+\sum_\nu \rho^\nu \boldsymbol{X}^* \frac{d\boldsymbol{\xi}_\nu}{dt}}{\boldsymbol{X}^*\boldsymbol{X}'}, \tag{6.4}$$

$$\frac{d\rho^\nu}{dt}=R^\nu(\boldsymbol{\rho}, t, \varepsilon) \tag{6.5}$$

$$=\frac{\|\boldsymbol{X}\|^2+\sum_\mu \rho^\mu \boldsymbol{X}^* \frac{d\boldsymbol{\xi}_\mu}{dt}}{\boldsymbol{X}^*\boldsymbol{X}'}\cdot \boldsymbol{\xi}_\nu^* \boldsymbol{X}'-\sum_\mu \rho^\mu \boldsymbol{\xi}_\nu^* \frac{d\boldsymbol{\xi}_\mu}{dt},$$

where

$$\boldsymbol{X}'=\boldsymbol{X}(\boldsymbol{\varphi}+\sum_\mu \rho^\mu \boldsymbol{\xi}_\mu, \varepsilon)$$

$$=\boldsymbol{X}+\sum_\mu \rho^\mu A(t)\boldsymbol{\xi}_\mu+\varepsilon \frac{\partial \boldsymbol{X}}{\partial \varepsilon}\bigg|_{x=\varphi(t),\, \varepsilon=0}+o(\|\boldsymbol{\rho}\|+|\varepsilon|).$$

Consequently the equations (6.5) are written in the vector form as follows:

$$\frac{d\boldsymbol{\rho}}{dt}=\boldsymbol{R}(\boldsymbol{\rho}, t, \varepsilon)=\Xi(t)\boldsymbol{\rho}+\varepsilon\boldsymbol{\eta}(t)+o(\|\boldsymbol{\rho}\|+|\varepsilon|), \tag{6.6}$$

where $\Xi(t)$ is the same as that defined in the paragraph 4 and $\boldsymbol{\eta}(t)$ is an $(n-1)$ dimensional vector whose components are $\boldsymbol{\xi}_\nu^* \frac{\partial \boldsymbol{X}}{\partial \varepsilon}\Big|_{x=\varphi(t),\, \varepsilon=0}$.

Let $\boldsymbol{\rho}=\boldsymbol{\rho}(t, \boldsymbol{c}, \varepsilon)$ be a solution of (6.6) such that $\boldsymbol{\rho}(0, \boldsymbol{c}, \varepsilon)=\boldsymbol{c}$, then, by the assumption on differentiability of $\boldsymbol{X}(\boldsymbol{x}, \varepsilon)$, $\boldsymbol{\rho}(t, \boldsymbol{c}, \varepsilon)$ becomes continuously differentiable with respect to $\boldsymbol{c}$ and ε for small $\|\boldsymbol{c}\|+|\varepsilon|$ in a finite interval of t, consequently, because of $\boldsymbol{\rho}(t, 0, 0)=0$, it can be written in the form

$$\boldsymbol{\rho}(t, \boldsymbol{c}, \varepsilon)=G(t)\boldsymbol{c}+\varepsilon\boldsymbol{r}(t)+o(\|\boldsymbol{c}\|+|\varepsilon|), \tag{6.7}$$

where $G(t)$ is an $(n-1)$-$(n-1)$ matrix and $\boldsymbol{r}(t)$ is an $(n-1)$ dimensional vector. Substituting (6.7) into (6.6), we have

$$\frac{dG(t)}{dt}=\Xi(t)G(t), \tag{6.8}$$

$$\frac{d\boldsymbol{r}(t)}{dt}=\Xi(t)\boldsymbol{r}(t)+\boldsymbol{\eta}(t). \tag{6.9}$$

Since $\boldsymbol{\rho}(0, \boldsymbol{c}, \varepsilon)=\boldsymbol{c}$, from (6.7) follows

$$G(0)=E, \quad \boldsymbol{r}(0)=0, \tag{6.10}$$

where E is a unit matrix. Then $G(t)$ becomes a regular solution of (6.8), in other words, a fundamental matrix of (4.8), and $\boldsymbol{r}(t)$ is sought for as

$$\boldsymbol{r}(t)=G(t)\int_0^t G^{-1}(s)\,\boldsymbol{\eta}(s)\,ds.$$

The necessary and sufficient condition that a path C' may be closed is that there holds

$$\rho(p\omega, \boldsymbol{c}, \varepsilon)=\boldsymbol{c} \tag{6.11}$$

for some positive integer p. From (6.7), the condition (6.11) is written as follows:

$$\{G(p\omega)-E\}\boldsymbol{c}+\varepsilon \boldsymbol{r}(p\omega)+o(\|\boldsymbol{c}\|+|\varepsilon|)=0. \tag{6.12}$$

Consequently, if

$$\det\{G(p\omega)-E\}\neq 0, \tag{6.13}$$

there exists a unique solution $\boldsymbol{c}=\boldsymbol{c}(\varepsilon)$ of (6.12) continuously differentiable in ε and vanishing with ε, namely, in the neighborhood of C, there exists a unique closed path C_p' of the perturbed system (6.1) with period

$$\int_0^{p\omega} \frac{\|\boldsymbol{X}\|^2+\sum\limits_{\nu}\rho^{\nu}(t, \boldsymbol{c}(\varepsilon), \varepsilon)\boldsymbol{X}^*\dfrac{d\xi_\nu}{dt}}{\boldsymbol{X}^*\boldsymbol{X}\{\boldsymbol{\varphi}+\sum\limits_{\nu}\rho^{\nu}(t, \boldsymbol{c}(\varepsilon), \varepsilon)\xi_\nu, \varepsilon\}}dt \tag{6.14}$$

near $p\omega$.

Since $G(t)$ is a fundamental matrix of (4.8) satisfying (6.10) it is valid that

$$G(p\omega)=G^p(\omega),$$

consequently, if (6.13) is valid, it holds that

$$\det\{G(\omega)-E\}\neq 0. \tag{6.15}$$

Then, as before, there exists a unique closed path C_1' of the perturbed system, but, for this C_1', evidently (6.11) holds, consequently C_1' coincides with C_p', in other words, in this case, in the neighborhood of C, there does not exist a closed path C_p' other than C_1'. When (6.15) holds, but (6.13) does not hold, it may happen that, in the neighborhood of C, there exists C_p' other than C_1'.

Now, from Theorem 2, the characteristic roots of $G(\omega)$ are the multipliers left by excluding 1 once from the multipliers of solutions of the initial variational equation (4.1). Thus we have

Theorem 5. *If all but one of the multipliers of solutions of the initial variational equation* (4.1) *differ from* 1, *in the neighborhood of* C, *there exists a unique closed path of the perturbed system with period near* ω, *and further, if any integral power of all but one of the multiplies of solutions of the initial variational equation* (4.1) *deffers from* 1, *there dose not exist a closed path of the perturbed system other than that with period near* ω. *If p-th powers of some of the multipliers left by excluding* 1 *once from those of solutions of the initial variational equation* (4.1) *become* 1, *in the neighbor-*

hood of C, there may exist a closed path of the perturbed system with period near $p\omega$ other than that with period near ω.

Remark When $n=2$, the condition that a path C' may be closed becomes

$$\boldsymbol{p}(\omega, \boldsymbol{c}, \varepsilon)=\boldsymbol{c},$$

as is seen from the portrait of paths in a plane. Consequently it is seen that, if one of the multipliers of solutions of the initial variational equation differs from 1, there exists only a unique closed path of the perturbed system with period near ω.

References

[1] H. Poincaré, *Sur les courbes définies par les équations différentielles*, J. d. Math., **2** (1886), 151–217.

[2] For instance, cf.:

V. V. Nyemyitski aud V. V. Styepanov, *Qualitative theory of differential equations*, (1947) (In Russian);

N. Levinson, *Small periodic perturbations of an autonomous system with a stable orbit*, Annals of Math., **52** (1950), 727–738;

M. Urabe, *Infinitesimal deformation of cycles*, J. Sci. Hiroshima Univ., Ser. A, **18** (1954), 37–53;

M. Urabe, *Numerical determination of periodic solution of nonlinear system*, J. Sci. Hiroshima Univ., Ser. A, **20** (1957), 125–148.

[3] For instance, cf.:

E. A. Coddington and N. Levinson, *Theory of ordinary differential equations*, McGraw-Hill; New York (1955).

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