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On Pairs of Domains in R^n with Boundaries in Common

By

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In this paper, we are concerned with relations between pairs of domains in the Euclidean space, R^n, and their boundaries. The following question in connection with my paper(*) is raised: does there exist a pair of domains D_1, D_2 in R^n such that (1) D_1 and D_2 are simple, i.e. homeomorphic with the domain $x_1^2+x_2^2+\cdots+x_n^2<1$, (2) $D_1\cdot D_2=\phi$, (3) $\mathscr{F}(D_1)=\mathscr{F}(D_2)$(**) and (4) $\mathscr{I}(\overline{D_1+D_2})=D_1+D_2$? The answer is affirmative for $n\geqq 2$. That is,

THEOREM 1. *There exists a pair of domains in R^n having the above properties* (1)~(4), *for* $n\geqq 2$.

PROOF. We shall show a process of construction of such a pair. Let π be the (x_1, x_2)-plane in R^n. Let C_m and C'_m $(m=1,2,\cdots)$ be the semi-circles on π;

$$C_m \quad (x_1-1/2^{m+1})^2+x_2^2=(1-1/2^{m-1}+1/2^{m+1})^2 \text{ and } x_2\geqq 0,$$
$$C'_m \quad (x_1+1/2^{m+1})^2+x_2^2=(1-1/2^{m-1}+1/2^{m+1})^2 \text{ and } x_2\leqq 0.$$

Let $p_{m,i}$ be points of C_m $(1\leqq i\leqq 2^{m+5})$ such that $\angle p_{m,i-1}\,o_m\,p_{m,i}=\angle p_{m,j-1}\,o_m\,p_{m,j}$ $(2\leqq i, j\leqq 2^{m+5})$, where o_m is the center of C_m, and both $p_{m,1}$ and $p_{m,2^{m+5}}$ are on the axis of π, $x_2=0$. Now let $C_{m,i}$ be the circle being in contact with the segments $o_m p_{m,i}$, $o_m p_{m,i+1}$ at $p_{m,i}$, $p_{m,i+1}$ respectively. We shall denote $\bigcup_{m,i} C_{m,i}+\bigcup_{m,i} C'_{m,i}$ by E'.

Next we consider semi-circles, K_m and K'_m, in R^2;

$$K_m \quad (x-1/2^{m+1}+1/2^{m+2})^2+y^2=(1+1/2^{m+1}+1/2^{m+2})^2 \text{ and } y\geqq 0,$$
$$K'_m \quad (x+1/2^{m+1}-1/2^{m+2})^2+y^2=(1+1/2^{m+1}+1/2^{m+2})^2 \text{ and } y\leqq 0.$$

Let $q_{m,i}$ be points on K_m $(1\leqq i\leqq 2^{m+3})$ such that $\angle q_{m,i-1}\,\bar{o}_m\,q_{m,i}=\angle q_{m,j-1}\,\bar{o}_m\,q_{m,j}$ $(2\leqq i, j\leqq 2^{m+3})$, where $\bar{o}_m$ is the center of K_m, and both $q_{m,1}$ and $q_{m,2^{m+3}}$ belong to the x-axis, $y=0$. Let $K_{m,i}$ be the circle being in contact with the segments $\bar{o}_m q_{m,i}$, $\bar{o}_m, q_{m,i+1}$ at $q_{m,i}$, $q_{m,i+1}$ respectively. Now we shall define F_1, F_2 by $F_1=\bigcup_m K_m+\bigcup_m K'_m$, $F_2=\bigcup_{m,i} K_{m,i}+\bigcup_{m,i} K'_{m,i}$. Then we define F_3 in the following way. In place of $K_{m,i}$ (or $K'_{m,i}$) and F_2 take the similar figure to F_1 in such a way as it is contained in the closed disk whose boundary

(*) A. Tominaga: Note on fixed-point theorem, Jour. Sci. Hiroshima Univ. Ser. A, **21** (1957), 7-14.

(**) If M is a set in R^n, $\mathscr{F}(M)$ and $\mathscr{I}(M)$ mean the boundary and the interior of M respectively.

is $K_{m,i}$ (or $K'_{m,i}$) and $q_{m,i}$ (or $q'_{m,i}$), $q_{m,i+1}$ (or $q'_{m,i+1}$) correspond to (1,0), (−1,0) respectively. (Strictly speaking, this correspondence needs some modification.) By induction we obtain the limit figure, F.

Finally in place of $C_{m,i}$ (or $C'_{m,i}$) in E' take the similar figure to F in such a way as $p_{m,i}$ (or $p'_{m,i}$), $p_{m,i+1}$ (or $p'_{m,i+1}$) correspond to (1,0), (−1,0) in F respectively. Thus we have the figure E_1 in π. Let E_2 be the figure obtained by the projection of E_1 from the points $(0,0,1,0,\cdots,0)$ and $(0,0,-1,0,\cdots,0)$. Step by step, E_i can be given by the projection of E_{i-1} from $(\underbrace{0,\cdots,0}_{i},1,0,\cdots,0)$ and $(\underbrace{0,\cdots,0}_{i},-1,0,\cdots,0)$. The desired domains D_1, D_2 are just two bounded components of R^n-E_{n-1}. Q.E.D.

THEOREM 2. *There exists a sequence of domains in* R^n $(n\geqq 2)$, $D_1, D_2, \cdots$, *such that* (1) D_i*'s are simple,* (2) $D_i\cdot D_j=\phi$ *for* $i\neq j$, (3) $\mathcal{F}(D_i)=\mathcal{F}(D_{i+1})$ and (4) $\mathcal{J}(\overline{\bigcup_i D_i})=\bigcup D_i$.

As we can easily see in the figure constructed in Theorem 1, such pairs of domains must have the following property.

THEOREM 3. *Let* D_1, D_2 *be two domains in* R^n *such that* $D_1\cdot D_2=\phi$, $\mathcal{F}(D_1)=\mathcal{F}(D_2)$ and $\mathcal{J}(\overline{D_1+D_2})=D_1+D_2$. *Then* D_1+D_2 *separates* R^n.

PROOF. Suppose, on the contrary, that $\overline{D_1+D_2}$ does not separate R^n. Then neither $\overline{D}_1$ nor $\overline{D}_2$ separates R^n. For if $\overline{D}_1$ separates R^n, D_2 is contained in a component, C, of $R^n-\overline{D}_1$, since $\overline{D}_1\cdot D_2=\phi$ and D_2 is connected. From $\mathcal{J}(\overline{D_1+D_2})=D_1+D_2$, we have $\mathcal{J}(\overline{D}_2)=D_2$ and hence there exists a point, p, of C not belonging to $\overline{D}_2$. As $\overline{D_1+D_2}$ does not separate R^n, we can find a solid, containing $\overline{D_1+D_2}$, not separating R^n and which is approximated by solids containing it. There exists a solid, $S\supseteq\mathcal{J}(S)\supseteq\overline{D}$, not separating R^n, and a point, $q\in D_2$, which does not belong to S. Let q_0 be an arbitrary point such that $q_0\notin S+\overline{D}_2$ and α an arc in R_n-S joining q and q_0. Then $\mathcal{F}(D_2)\cdot\alpha\neq\phi$, which contradicts the fact $\mathcal{F}(D_1)=\mathcal{F}(D_2)$. Q.E.D.

As a consequence, we can modify Theorem 1 in (∗) as follows: *Let M be a plane continuum which does not separate* R^2 *and each 2-element has a.f.p.p. If f is a topological mapping of M onto itself, then f has a fixed point.*

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