JOURNAL OF SCIENCE OF THE HIROSHIMA UNIVERSITY, SER. A, VOL. 21, NO. 2, JULY, 1957

On a Theorem of Ehrenpreis

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(Received April 30, 1957)

Let $f(z_1,\cdots,z_n)$, $g(z_1,\cdots,z_n)$ be functions analytic in a neighbourhood of $(0,\cdots,0)$. The *cospectrum* of g is said to contain that of f at $(0,\cdots,0)$ if there exists a neighbourhood N of $(0,\cdots,0)$ so that, both f and g are analytic in N and for any $P\in N$ and non-negative integers $t_1,\cdots,t_n$, the conditions

$$[(\partial^{s_1+\cdots+s_n}/\partial z_1^{s_1}\cdots\partial z_n^{s_n})f](P)=0,$$

whenever $s_1\leqq t_1,\cdots,\ s_n\leqq t_n$ imply

$$[(\partial^{s_1+\cdots+s_n}/\partial z_1^{s_1}\cdots\partial z_n^{s_n})g](P)=0$$

for all $s_1\leqq t_1,\cdots,\ s_n\leqq t_n$. In a recent paper ([1], Theorem 2, p. 315) L. Ehrenpreis has proved the following theorem:

THEOREM. *Let $f(z_1,\cdots,z_n)$, $g(z_1,\cdots,z_n)$ be functions analytic in a neighbourhood of $(0,\cdots,0)$. Suppose that f is not identically zero and the cospectrum of g contains that of f at $(0,\cdots,0)$. Then g/f is an analytic function in a neighbourhood of $(0,\cdots,0)$.*

His way of proving the theorem is done by an application of his own generalized Weierstrass preparation theorem, which is of much interest in itself, but somewhat complicated in its proof. In this paper we shall prove the above theorem with the aid of the classical Weierstrass preparation theorem alone.

First, let us note that if the theorem is true then for any linear transformation L in the variables the cospectrum of $g\circ L$ contains that of $f\circ L$ at $(0,\cdots,0)$.

Since the result is clear when $n=1$, we shall prove this by induction on the number of variables, supposing that it is true for the cases that the number of variables is less than n. If $f(0,\cdots,0)\neq 0$, then the result is obvious; we assume $f(0,\cdots,0)=0$. Expressing $f(z_1,\cdots,z_n)$ in power series $f(z_1,\cdots,z_n)=\sum_{p_1,\cdots,p_n=0}^{\infty} a_{p_1,\cdots,p_n} z_1^{p_1}\cdots z_n^{p_n}$, let l denote the exponent of the maximal power of the product $z_1\cdots z_n$ which is contained commonly in each non-zero term and p be the minimum degree of the terms each of which has $(z_1\cdots z_n)^l$ as the maximal power contained in it. Then we may write $f(z_1,\cdots,z_n)$ in the form

$$f(z_1,\cdots,z_n)=(z_1\cdots z_n)^l f'(z_1,\cdots,z_n)$$

and also, by the hypothesis, $g(z_1,\cdots,z_n)$ in the form

$$g(z_1,\cdots,z_n)=(z_1\cdots z_n)^l g'(z_1,\cdots,z_n).$$

Let us write $f'(z_1,\cdots,z_n)$ in the form $f'(z_1,\cdots,z_n)=\sum_{i=0}^{\infty}A_i(z_1,\cdots,z_n)$ where $A_i(z_1,\cdots,z_n)$ is a homogeneous polynomial of degree i, and $q=p-ln$, then $A_q(z_1,\cdots,z_n)$ does not contain any positive power of $z_1\cdots z_n$ so that we may assume

$$A_q(z_1,\cdots,z_n)=a_0(z_2,\cdots,z_n)z_1^q+a_1(z_2,\cdots,z_n)z_1^{q-1}+\cdots+a_q(z_2,\cdots,z_n),$$

with $a_q(z_2,\cdots,z_n)\not\equiv 0$. Now, fixing z_1, we consider a non-singular linear transformation $L: z_i=\sum_{j=2}^{n}c_{ij}\zeta_j$, $i=2,3,\cdots,n$ such that it transforms $a_q(z_2,\cdots,z_n)$ into the form ζ_n^q+lower powers of ζ_n, and we write $A_i(z_1,\cdots,z_n)=B_i(z_1,\zeta_2,\cdots,\zeta_n)$ and $f'(z_1,\cdots,z_n)=F'(z_1,\zeta_2,\cdots,\zeta_n)$. Then

$$B_i(0,\cdots,0,\zeta_n)=\begin{cases}0 & \text{for } i<q\\ \zeta_n^q & \text{for } i=q\\ \text{higher power of } \zeta_n & \text{for } i>q\end{cases}$$

and $F'(0,\cdots,0,\zeta_n)=\zeta_n^q+$higher powers $\not\equiv 0$.

Therefore, by the Weierstrass preparation theorem [2], there exist a neighbourhood M of $(0,\cdots,0)$ and analytic functions $\Omega(z_1,\zeta_2,\cdots,\zeta_n)$, $H(z_1,\zeta_2,\cdots,\zeta_n)$, $E_i(z_1,\zeta_2,\cdots,\zeta_{n-1})$, $i=1,2,\cdots,q$ and $K_j(z_1,\zeta_2,\cdots,\zeta_{n-1})$, $j=1,2,\cdots,q$ such that, throughout M, F' and G' can be expressed in the following forms

(1) $\Omega(z_1,\zeta_2,\cdots,\zeta_n)\,F'(z_1,\zeta_2,\cdots,\zeta_n)$
$$=\zeta_n^q+E_1(z_1,\zeta_2,\cdots,\zeta_{n-1})\,\zeta_n^{q-1}+\cdots+E_q(z_1,\zeta_2,\cdots,\zeta_{n-1})$$

(2) $H(z_1,\zeta_2,\cdots,\zeta_n)\,F'(z_1,\zeta_2,\cdots,\zeta_n)-G'(z_1,\zeta_2,\cdots,\zeta_n)$
$$=K_1(z_1,\zeta_2,\cdots,\zeta_{n-1})\zeta_n^{q-1}+\cdots+K_q(z_1,\zeta_2,\cdots,\zeta_{n-1})$$

where $g'(z_1,\cdots,z_n)=G'(z_1,\zeta_2,\cdots,\zeta_n)$, $\Omega(0,\cdots,0)\neq 0$ and $E_i(0,\cdots,0)=0$ for $i=1,2,\cdots,q$. Let now

(3) $(z_1\cdots z_n)^l=z_1^l\{J_0(\zeta_2,\cdots,\zeta_{n-1})\zeta_n^r+\cdots+J_r(\zeta_2,\cdots,\zeta_{n-1})\}$,

and

(4) $f(z_1,\cdots,z_n)=F(z_1,\zeta_2,\cdots,\zeta_n)$ and $g(z_1,\cdots,z_n)=G(z_1,\zeta_2,\cdots,\zeta_n)$,

where $J_0(\zeta_2,\cdots,\zeta_{n-1})$ is not identically zero, and we put the right sides of (1), (2) and (3) $P_{z_1}(\zeta_2,\cdots,\zeta_n)$, $Q_{z_1}(\zeta_2,\cdots,\zeta_n)$ and $\Phi_{z_1}(\zeta_2,\cdots,\zeta_n)$ respectively. Then we may rewrite the above (1), (2) in the following forms

(1′) $\Omega(z_1,\zeta_2,\cdots,\zeta_n)\,F(z_1,\zeta_2,\cdots,\zeta_n)=\Phi_{z_1}(\zeta_2,\cdots,\zeta_n)\,P_{z_1}(\zeta_2,\cdots,\zeta_n)$

and

(2′) $H(z_1,\zeta_2,\cdots,\zeta_n)F(z_1,\zeta_2,\cdots,\zeta_n)-G(z_1,\zeta_2,\cdots,\zeta_n)=\Phi_{z_1}(\zeta_2,\cdots,\zeta_n)Q_{z_1}(\zeta_2,\cdots,\zeta_n)$.

On the other hand, by the hypothesis of induction and the remark made at the beginning of the proof, there exists a neighbourhood N of $(0,\cdots,0)$

in $(\zeta_2,\cdots,\zeta_n)$-space so that, for a suitable small fixed $z_1^0\neq 0$ the cospectrum of $G(z_1^0,\zeta_2,\cdots,\zeta_n)$ contains that of $F(z_1^0,\zeta_2,\cdots,\zeta_n)$ in N, and $(z_1^0, N)\subset M$. Thus from (1′) and (2′) the cospectrum of $\Phi_{z_1^0}(\zeta_2,\cdots,\zeta_n)Q_{z_1^0}(\zeta_2\cdots,\zeta_n)$ contains that of $\Phi_{z_1^0}(\zeta_2,\cdots,\zeta_n)P_{z_1^0}(\zeta_2,\cdots,\zeta_n)$ in N.

It is sufficient to show that $Q_{z_1}(\zeta_2,\cdots,\zeta_n)\equiv 0$. Assume the contrary, then we can find the first term $K_s(z_1,\zeta_2,\cdots,\zeta_{n-1})$ not identically zero in $Q_{z_1}(\zeta_2,\cdots,\zeta_n)$ and $z_1^0\neq 0$ such that $K_s(z_1^0,\zeta_2,\cdots,\zeta_{n-1})\,J_0(\zeta_2,\cdots,\zeta_{n-1})\not\equiv 0$, and therefore we may choose sufficiently small $\zeta_2^0,\cdots,\zeta_{n-1}^0$ such that $K_s(z_1^0,\zeta_2^0,\cdots,\zeta_{n-1}^0)$ $J_0(\zeta_2^0,\cdots,\zeta_{n-1}^0)\neq 0$ and $(\zeta_2^0,\cdots,\zeta_{n-1}^0,\zeta_n^{(i)})\in N$ for $i=1,\cdots,q$ denoting $\zeta_n^{(i)}, i=1,\cdots,q$ the roots of the equation $P_{z_1^0}(\zeta_2^0,\cdots,\zeta_{n-1}^0,\zeta_n)=0$ counting multiplicity. And $\zeta_n^{(j)}, j=q+1,\cdots,q+t$ also denote the roots of the equation $\Phi_{z_1^0}(\zeta_2^0,\cdots,\zeta_{n-1}^0,\zeta_n)=0$ counting multiplicity each of which is one of the $\zeta_n^{(i)}, i=1,\cdots,q$. Then by the above relation of cospectra $\zeta_n^{(i)}, i=1,2,\cdots,q+t$ are some of the roots of the equation $\Phi_{z_1^0}(\zeta_2^0,\cdots,\zeta_{n-1}^0,\zeta_n)\,Q_{z_1^0}(\zeta_2^0,\cdots,\zeta_{n-1}^0,\zeta_n)=0$ and there is no root of $\Phi_{z_1^0}(\zeta_2^0,\cdots,\zeta_{n-1}^0,\zeta_n)=0$ among $\zeta_n^{(i)}, i=1,\cdots,q+t$ except for $\zeta_n^{(j)}, j=q+1,\cdots,q+t$. Therefore all of $\zeta_n^{(i)}, i=1,2,\cdots,q$ are the roots of $Q_{z_1^0}(\zeta_2^0,\cdots,\zeta_{n-1}^0,\zeta_n)=0$. This is a contradiction.

Bibliography

[1] L. Ehrenpreis, *Mean Periodic Functions* Amer. Journ. of Math. **77** (1955), 293–328.

[2] S. Bochner and W. T. Martin, *Several Complex Variables*,, Princeton 1948.