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# *On Algebras of Totally Geodesic Spaces (Lie triple systems)*

By

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It is known that, in the space of continuous group of transformations $G$ with affine connection without torsion, the curvature tensor is equal to $\frac{1}{4}C_{ij}{}^{s}C_{sk}{}^{l}$ ([2],[1] p. 64) and the condition in order that the linear family of infinitesimal transformations $X_1,\cdots,X_r$ ($r\leqq$ dimension of $G$) at a point 0 of $G$ generate a totally geodesic subspace is that $[[X_iX_j]X_k]$ $(i, j, k=1,\cdots,r)$ linearly depends to $X_1,\cdots,X_r$ ([2], p. 72, in more sharped form [5]). Therefore we can study a vector space which is closed with respect to a ternary composition $[[ab]c]$ as a tangent space of a totally geodesic subspace of $G$.

On the other hand, N. Jacobson has considered such a system algebraically and called it *Lie triple system* (this will be denoted as *L.t.s.*).

In this paper we shall define the abstract L.t.s. and prove the existence of the 1-to-1 imbedding of an abstract L.t.s. into a Lie algebra in the weaker assumption than that of N. Jacobson [3] (Standard imbedding in [4]) (§1 and §2). Next we shall consider the geometrical meaning of Theorem 2.1. §3 is concerned with a condition that a L.t.s. isomorphism should be extended to a enveloping Lie algebra isomorphism. In §4 for complex L.t.s. an analogous results of some properties for Lie algebra is proved. In §5 we classfy the 2-dimensional L.t.s. over the complex field.

## §1. Abstract Lie triple systems and some properties.

DEFINITION 1.1. An *abstract Lie triple system* $\mathfrak{T}$ is a vector space over a field $\Phi$ in which a ternary composition $[abc]$ is defined such that

(1) $[abc]$ is trilinear,

(2) $[aab]=0$,

(3) $[abc]+[bca]+[cab]=0$.

Lie algebra is an abstract L.t.s. relative to the ternary composition $[[ab]c]$. In the following, unless the contrary is explicitly stated, we assume that the vector space $\mathfrak{T}$ is finite-dimensional and the characteristic of $\Phi$ is 0. In this section we shall consider the abstract L.t.s. $\mathfrak{T}$ only, therefore we shall call $\mathfrak{T}$ L.t.s. simply.

A vector subspace $\mathfrak{V}$ of L.t.s. $\mathfrak{T}$ is called an ideal in $\mathfrak{T}$ if $[\mathfrak{V}\mathfrak{T}\mathfrak{T}]\subseteqq\mathfrak{V}$. Let $\mathfrak{V}$ be an ideal of L.t.s. $\mathfrak{T}$, then the factor space $\mathfrak{T}/\mathfrak{V}$ becomes an L.t.s.

1) Numbers in brackets refer to the references at the end of the paper.

(factor L.t.s.) by defining that the product of three elements $\bar{a}=a+\mathfrak{B}$, $\bar{b}=b+\mathfrak{B}$, $\bar{c}=c+\mathfrak{B}$ of $\mathfrak{T}/\mathfrak{B}$ is $[\bar{a}\bar{b}\bar{c}]=[abc]+\mathfrak{B}$. A homomorphism $f$ of a L.t.s. $\mathfrak{T}$ into a L.t.s. $\mathfrak{U}$ is a linear mapping which satisfies $f[abc]=[f(a), f(b), f(c)]$. If $\mathfrak{B}$ is an ideal of L.t.s. $\mathfrak{T}$ then $\mathfrak{T}$ is mapped homomorphically onto every factor system $\mathfrak{T}/\mathfrak{B}$. Conversely, assume that L.t.s. $\mathfrak{T}$ be mapped onto L.t.s. $\mathfrak{U}$ by homomorphism $f$ and $\mathfrak{B}$ the kernel of $f$, then $\mathfrak{B}$ is an ideal in $\mathfrak{T}$ and $\mathfrak{U}$ is isomorphic with the factor system $\mathfrak{T}/\mathfrak{B}$. ([4], p. 218)

PROPOSITION 1.1. *If $\mathfrak{A}$ is a subsystem and $\mathfrak{B}$ an ideal of L.t.s. $\mathfrak{T}$, then*

$$\mathfrak{A}/\mathfrak{A}\frown\mathfrak{B}\cong\mathfrak{A}+\mathfrak{B}/\mathfrak{B}.$$

A finite sequence of subsystems of a L.t.s. $\mathfrak{T}$ : $\mathfrak{T}=\mathfrak{T}_0\supset\mathfrak{T}_1\supset\mathfrak{T}_2\supset\cdots\supset\mathfrak{T}_s=(0)$ is called a composition series of $\mathfrak{T}$ if every subsystem $\mathfrak{T}_i$ is a maximal proper ideal of $\mathfrak{T}_{i-1}$, $(1\leqq i\leqq s)$. A sequence of factor systems $\mathfrak{T}_0/\mathfrak{T}_1,\cdots,\mathfrak{T}_{s-1}/\mathfrak{T}_s$ is a composition factor series of $\mathfrak{T}$. Since now $dim\,\mathfrak{T}<\infty$ there is a composition series of $\mathfrak{T}$.

PROPOSITION 1.2. (Jordan-Hörder) *Any two composition factor series of L.t.s. are isomorphic in a suitable order.*

The set of the elements $c$ of L.t.s. $\mathfrak{T}$ which satisfy $[cab]=0$ for every $a$, $b$ in $\mathfrak{T}$ is called the center of $\mathfrak{T}$. $\mathfrak{T}$ is called to be *abelian* if $\mathfrak{T}$ coincides with its center.

Let $\mathfrak{T}$ be a L.t.s.. We form the sequence of subsets $\mathfrak{T}\equiv D^0(\mathfrak{T})$, $D^i(\mathfrak{T})\equiv[D^{i-1}(\mathfrak{T}), D^{i-1}(\mathfrak{T}), D^{i-1}(\mathfrak{T})]$, $i=1,2,\cdots$ . If there is an integer $s$ such that $D^s(\mathfrak{T})=(0)$ then $\mathfrak{T}$ is called to be *solvable.*

PROPOSITION 1.3. *L.t.s. $\mathfrak{T}$ is solvable if and only if the composition factors of $\mathfrak{T}$ are all abelian.*

The composition factor system of a solvable L.t.s. have the dimension 1.

## §2. 1-to-1 imbedding of abstract L.t.s. into Lie algebra.

PROPOSITION 2.1. *For abstract L.t.s. $\mathfrak{T}$ the following two conditions are equivalent.*

(4) $[[abc]de]+[[bad]ce]+[ba[cde]]+[cd[abe]]=0.$

(4′) $[[abc]dd]+[[bad]cd]+[ba[cdd]]+[cd[abd]]=0.$

PROOF. We shall prove (4′)⇒(4). Since $[\cdots]$ is trilinear, by replacing $d$ by $d+e$ in (4′) we have

(A) $[[abc]de]+[[abc]ed]+[[bad]ce]+[[bae]cd]$
$+[ba[cde]]+[ba[ced]]+[cd[abe]]+[ce[abd]]=0$

for all $a$, $b$, $c$, $d$, $e$ in $\mathfrak{T}$.

Denote by (B) the expression obtained from (A) by interchange $c$ and $d$. Then by subtracting (B) from (A) and by making use of identities (2), (3) we have (4).

For $a, b\in\mathfrak{T}$, let $D_{(a,b)}$ be the mapping: $x\rightarrow[abx]$ for all $x\in\mathfrak{T}$, then $D_{(a,b)}$ is a linear mapping of $\mathfrak{T}$. (4) also can be rewritten:

$$[ab[cde]]=[[abc]de]+[c[abd]e]+[cd[abe]].$$

Hence $D_{\langle a, b\rangle}$ satisfies the following:

$$D_{\langle a, b\rangle}[cde] = [(D_{\langle a, b\rangle} c), d, e] + [c, (D_{\langle a, b\rangle} d), e] + [c, d, (D_{\langle a, b\rangle} e)].$$

$D_{\langle a, b\rangle}$ is called an inner derivation of $\mathfrak{T}$.[2)]

THEOREM 2.1. *If an abstract L.t.s.* $\mathfrak{T}$ *satisfies the condition* (4), *then* $\mathfrak{T}$ *can be 1-to-1 imbedded into a Lie algebra* $\mathfrak{L}$ *such that the given composition* $[abc]$ *in* $\mathfrak{T}$ *coincides with the product* $[[ab]c]$ *defined in* $\mathfrak{L}$.

PROOF. For $a, b \in \mathfrak{T}$, let $D_{\langle a, b\rangle}$ be an inner derivation of $\mathfrak{T}$. We define the addition of two linear mappings: $D_{\langle a, b\rangle} + D_{\langle c, d\rangle}$ and the scalar multiplication: $\alpha D_{\langle a, b\rangle}$, $\alpha \in \Phi$ as follows:

$$(D_{\langle a, b\rangle} + D_{\langle c, d\rangle})(x) = D_{\langle a, b\rangle}(x) + D_{\langle c, d\rangle}(x), \quad (\alpha D_{\langle a, b\rangle})(x) = \alpha(D_{\langle a, b\rangle}(x)).$$

The set of the inner derivations becomes a vector space $D$ over $\Phi$. Also, $D' = \{D_{\langle a, b\rangle} \mid D_{\langle a, b\rangle}(x) = 0$ for all $x$ in $\mathfrak{T}\}$ is a vector subspace of $D$. We denote by $\mathfrak{D}(\mathfrak{T})$ the factor space $D/D'$. The totality of linear mappings of vector space $\mathfrak{T}$ becomes an algebra, if the product of two linear mappings $A$ and $B$ is defined by the equation $(AB)(x) = A(Bx)$ for all $x$ in $\mathfrak{T}$. In this case, since

$$\begin{aligned}(D_{\langle a, b\rangle} D_{\langle c, d\rangle} - D_{\langle c, d\rangle} D_{\langle a, b\rangle})(x) &= [ab[cdx]] - [cd[abx]] \\ &= [[abc], d, x] - [[abd], c, x] \qquad \text{(by (4))} \\ &= (D_{\langle [abc], d\rangle} - D_{\langle [abd], c\rangle})(x),\end{aligned}$$

we have $D_{\langle a, b\rangle} D_{\langle c, d\rangle} - D_{\langle c, d\rangle} D_{\langle a, b\rangle} = D_{\langle [abc], d\rangle} - D_{\langle [abd], c\rangle} \in \mathfrak{D}(\mathfrak{T})$. It can be proved that the direct sum $\mathfrak{L} \equiv \mathfrak{T} \oplus \mathfrak{D}(\mathfrak{T})$ has a structure of Lie algebra. For $a, b, c, d \in \mathfrak{T}$, we define the product as follows:

$$\begin{aligned}&[a, b] = D_{\langle a, b\rangle}, \qquad [D_{\langle a, b\rangle}, c] = [abc], \\ &[c, D_{\langle a, b\rangle}] = -[abc], \quad [D_{\langle a, b\rangle}, D_{\langle c, d\rangle}] = D_{\langle [abc], d\rangle} - D_{\langle [abd], c\rangle}\end{aligned}$$

and for $u = a + \sum D_{\langle b, c\rangle}$ and $v = d + \sum D_{\langle e, f\rangle}$ $\qquad a, b, c, d, e, f \in \mathfrak{T}$

$$[u, v] = [a, d] + \sum [a, D_{\langle e, f\rangle}] + \sum [D_{\langle b, c\rangle}, d] + \sum [D_{\langle b, c\rangle}, D_{\langle e, f\rangle}].$$

The elements of $\mathfrak{D}(\mathfrak{T})$ can be written as $\sum [a, b]$, $a, b \in \mathfrak{T}$. The above defined product is bilinear. We shall prove that it is skew-symmetric and satisfies the Jacobi identity.

i) skew-symmetry: For $a, b, c, d \in \mathfrak{T}$

$$[a, b] + [b, a] = D_{\langle a, b\rangle} + D_{\langle b, a\rangle} = 0, \quad \text{(by (2))}.$$

$$[D_{\langle a, b\rangle}, c] + [c, D_{\langle a, b\rangle}] = [abc] - [abc] = 0.$$

$$\begin{aligned}&[D_{\langle a, b\rangle}, D_{\langle c, d\rangle}] + [D_{\langle c, d\rangle}, D_{\langle a, b\rangle}] \\ &\quad = D_{\langle a, b\rangle} D_{\langle c, d\rangle} - D_{\langle c, d\rangle} D_{\langle a, b\rangle} + D_{\langle c, d\rangle} D_{\langle a, b\rangle} - D_{\langle a, b\rangle} D_{\langle c, d\rangle} = 0.\end{aligned}$$

ii) Jacobi identity: For $a, b, c, d, e \in \mathfrak{T}$

$$[[ab]c] + [[bc]a] + [[ca]b] = 0 \quad \text{(by (3))}.$$

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2) As for the geometrical meaning of condition (4) see the end of this section, and for the meaning of $D_{\langle a, b\rangle}$ see [2], p. 65.

$$[[a, b], D_{\langle c, d\rangle}]+[[b, D_{\langle c, d\rangle}], a]+[[D_{\langle c, d\rangle}, a], b]$$
$$=[D_{\langle a, b\rangle}, D_{\langle c, d\rangle}]-D_{\langle [cdb], a\rangle}+D_{\langle [cda], b\rangle}=[D_{\langle a, b\rangle}, D_{\langle c, d\rangle}]+[D_{\langle c, d\rangle}, D_{\langle a, b\rangle}]=0.$$
$$[[D_{\langle a, b\rangle}, D_{\langle c, d\rangle}], e]+[[D_{\langle c, d\rangle}, e], D_{\langle a, b\rangle}]+[[e, D_{\langle a, b\rangle}], D_{\langle c, d\rangle}]$$
$$=[[abc]de]-[[abd]ce]-[ab[cde]]+[cd[abe]]=0. \qquad \text{(by (4))}.$$

For $u=[a, b]$, $v=[c, d]$, $w=[e, f]$, $a, b, c, d, e, f \in \mathfrak{T}$

$$[[uv]w]+[[vw]u]+[[wu]v]$$
$$=D_{\langle a, b\rangle}D_{\langle c, d\rangle}D_{\langle e, f\rangle}-D_{\langle c, d\rangle}D_{\langle a, b\rangle}D_{\langle e, f\rangle}-D_{\langle e, f\rangle}D_{\langle a, b\rangle}D_{\langle c, d\rangle}+D_{\langle e, f\rangle}D_{\langle c, d\rangle}D_{\langle a, b\rangle}$$
$$+Q+R=0,$$

where $Q$ and $R$ are obtained by the cyclic permutations of the pairs $(a, b)$, $(c, d)$ and $(e, f)$ from the four previous terms. Hence $\mathfrak{T}\oplus\mathfrak{D}(\mathfrak{T})$ is a Lie algebra and $[abc]=[[ab]c]$. Thus this theorem is proved.

The abstract L.t.s. $\mathfrak{T}$ which satisfies (4) is called a *Lie triple system.* W. G. Lister called the Lie algebra $\mathfrak{D}(\mathfrak{T})$ of skew-symmetric mappings the *algebra of inner derivations of* $\mathfrak{T}$ [4].

Next we shall consider the geometrical meaning of Theorem 2.1. In the space with affine connection, if the torsion tensor $S_{jk}^{\cdot\cdot l}=0$[3] and the covariant derivative of the curvature tensor $\nabla_m R_{ijk}^{\cdot\cdot\cdot l}=0$, then this connection is called to be *symmetric* in the sense of E. Cartan. In this case we have

$$R_{ijk}^{\cdot\cdot\cdot l}+R_{jik}^{\cdot\cdot\cdot l}=0,$$
$$R_{ijk}^{\cdot\cdot\cdot l}+R_{jki}^{\cdot\cdot\cdot l}+R_{kij}^{\cdot\cdot\cdot l}=0,$$

and the Ricci's formula becomes

$$\nabla_e\nabla_f R_{ijk}^{\cdot\cdot\cdot l}-\nabla_f\nabla_e R_{ijk}^{\cdot\cdot\cdot l}=R_{efm}^{\cdot\cdot\cdot l}R_{ijk}^{\cdot\cdot\cdot m}-R_{efi}^{\cdot\cdot\cdot m}R_{mjk}^{\cdot\cdot\cdot l}-R_{efj}^{\cdot\cdot\cdot m}R_{imk}^{\cdot\cdot\cdot l}-R_{efk}^{\cdot\cdot\cdot m}R_{ijm}^{\cdot\cdot\cdot l}=0.$$

Above identities are nothing but the conditions (2), (3) and (4) in coefficient terms. Let $X_1, \cdots, X_r$ be the base of a tangent space $T_p$ of a point $p$ of the space, then we define the trilinear composition $[X_iX_jX_k]$ by $[X_iX_jX_k]=R_{ijk}^{\cdot\cdot\cdot l}X_l$. Since $T_p$ is L.t.s. which satisfies (2), (3) and (4), $T_p$ can be imbedded in a Lie algebra, that is, we can say that *any space with affine connection which is symmetric in the sense of E. Cartan is realizable by a totally geodesic subspace in a group space without torsion.* ([2], p. 90). It is clear that the totally geodesic subspace in a group space without torsion is a symmetric space of E. Cartan since the ambient group space is symmetric. Thus L.t.s. characterizes the symmetric space intrinsically.

## §3. Extension of L.t.s. isomorphism to Lie isomorphism of enveloping Lie algebras.

THEOREM 3.1. *Let $\mathfrak{T}$ and $\mathfrak{T}'$ be L.t.s. with Lie algebras of inner derivations $\mathfrak{D}(\mathfrak{T})$ and $\mathfrak{D}(\mathfrak{T}')$ respectively. If $\mathfrak{T}'$ is L.t.s. isomorphic to $\mathfrak{T}$ then $\mathfrak{D}(\mathfrak{T}')$ is Lie isomorphic to $\mathfrak{D}(\mathfrak{T})$.*

PROOF. From that the mapping $a(\in\mathfrak{T})\rightarrow a'(\in\mathfrak{T}')$ is 1-to-1, it follows that the mapping $D_{\langle a, b\rangle}\rightarrow D_{\langle a', b'\rangle}$ is 1-to-1, because $D_{\langle a', b'\rangle}=0$ implies $[abx]'$

3) We use the notations of J. A. Schouten [6].

$=[a'b'x']=0$ for all $x$ in $\mathfrak{T}$, i.e., $D_{(a,b)}=0$. It is clear that this correspondence is linear, and that

$$\begin{aligned}([D_{(a,b)}, D_{(c,d)}](x))' &= ([[abc]dx]-[[abd]cx])' \\ &= [[a'b'c']d'x']-[[a'b'd']c'x'] \\ &= [D_{(a',b')}, D_{(c',d')}](x').\end{aligned}$$

Hence $\mathfrak{D}(\mathfrak{T})\cong\mathfrak{D}(\mathfrak{T}')$ by the mapping $D_{(a,b)}\to D_{(a',b')}$.

Let $\mathfrak{T}$ be a L.t.s. and $\mathfrak{L}=\mathfrak{T}+[\mathfrak{T}\mathfrak{T}]$ an enveloping Lie algebra of $\mathfrak{T}$. Then the Lie algebra of inner derivations of $\mathfrak{T}$ is homomorphic to a Lie algebra $[\mathfrak{T},\mathfrak{T}]$ and the kernel of this homomorphism is in the center of $\mathfrak{L}$.

THEOREM 3.2. *Let $\mathfrak{L}$ and $\mathfrak{L}'$ be an enveloping Lie algebras of L.t.s. $\mathfrak{T}$ and $\mathfrak{T}'$ respectively, and denote by $\mathfrak{D}(\mathfrak{T})$ and $\mathfrak{D}(\mathfrak{T}')$ Lie algebras of inner derivations of $\mathfrak{T}$ and $\mathfrak{T}'$ respectively. When $[\mathfrak{T},\mathfrak{T}]\cong\mathfrak{D}(\mathfrak{T})$ and $[\mathfrak{T}',\mathfrak{T}']\cong\mathfrak{D}(\mathfrak{T}')$, $\mathfrak{T}\cong\mathfrak{T}'$* (L.t.s. isomorphic) *implies* $\mathfrak{L}\cong\mathfrak{L}'$ (Lie isomorphic).

PROOF. Since $[\mathfrak{T},\mathfrak{T}]\cong\mathfrak{D}(\mathfrak{T})\cong\mathfrak{D}(\mathfrak{T}')\cong[\mathfrak{T}',\mathfrak{T}']$ we have $[\mathfrak{T},\mathfrak{T}]\cong[\mathfrak{T}',\mathfrak{T}']$ by the mapping $[a,b]\to[a',b']$, $a,b\in\mathfrak{T}$. For $e\in[\mathfrak{T},\mathfrak{T}]$, $c\in\mathfrak{T}$ we can write $e=\sum[a,b]$, $a,b\in\mathfrak{T}$. Therefore $[ec]'=\sum[[ab]c]'=\sum[[a'b']c']=[e'c']$. Hence we have $\mathfrak{L}\cong\mathfrak{L}'$

Theorem 3.2 is not true in general. For instance, L.t.s. $\mathfrak{T}$ with the base $X_1=\sqrt{-1}\left(x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}\right)$, $X_2=\frac{\partial}{\partial x}$ and L.t.s. $\mathfrak{U}$ with the base $Y_1=x\frac{\partial}{\partial x}$, $Y_2=\frac{\partial}{\partial x}$ are isomorphic solvable L.t.s. by the mapping $X_i\to Y_i$ $(i=1,2)$. But the enveloping Lie algebras can not be isomorphic.

COROLLARY. *Using the notations of Theorem, if $\mathfrak{T}$ and $\mathfrak{T}'$ are semi-simple* (without solvable ideal) *L.t.s. then $\mathfrak{T}\cong\mathfrak{T}'$ implies $\mathfrak{L}\cong\mathfrak{L}'$.*

PROOF. Let $K$ be the kernel of homomorphism of $[\mathfrak{T},\mathfrak{T}]$ onto $\mathfrak{D}(\mathfrak{T})$ then $K$ is a solvable ideal of $\mathfrak{L}$. Since any enveloping Lie algebra of a semi-simple L.t.s. is semi-simple ([4], p. 222), we have $K=(0)$. Hence $[\mathfrak{T},\mathfrak{T}]\cong\mathfrak{D}(\mathfrak{T})$ and similarly we have $[\mathfrak{T}',\mathfrak{T}']\cong\mathfrak{D}(\mathfrak{T}')$.

The above corollary means that when the tangent spaces of two totally geodesic spaces $S$ and $S'$ have the structure of semi-simple L.t.s. the projective transformations between $S$ and $S'$ imply the isomorphism of the enveloping group spaces.

## §4. Infinitesimal group, Characteristic equation.

Now let $X_1,\cdots,X_r$ be the base of L.t.s. $\mathfrak{T}$, then $[X_iX_jX_k]=C_{ijk}^{\cdot\cdot\cdot l}X_l$, $C_{ijk}^{\cdot\cdot\cdot l}\in\Phi$ $(1\leq i,j,k,l\leq r)$. We call the constants $C_{ijk}^{\cdot\cdot\cdot l}$ the structure constants of $\mathfrak{T}$. In this section it is assumed that the base field $\Phi$ of $\mathfrak{T}$ is the complex field.

PROPOSITION 4.1. *Let $\mathfrak{T}$ be a L.t.s. and $C_{ijk}^{\cdot\cdot\cdot l}$ the structure constants of $\mathfrak{T}$. Then the vector space generated by all the operators $E_{ij}=e^aC_{ija}^{\cdot\cdot\cdot b}\frac{\partial}{\partial e^b}$ $(1\leq i,j\leq r)$ has the structure of a Lie algebra.*

PROOF. $[E_{ij}, E_{kl}]=\left[e^a C_{ija}^{\cdot\cdot\cdot b}\frac{\partial}{\partial e^b},\quad e^c C_{klc}^{\cdot\cdot\cdot d}\frac{\partial}{\partial e^d}\right]$

$$=e^a(C_{ija}^{\cdot\cdot\cdot b}C_{klb}^{\cdot\cdot\cdot c}-C_{kla}^{\cdot\cdot\cdot b}C_{ijb}^{\cdot\cdot\cdot c})\frac{\partial}{\partial e^c}$$

$$=e^a(C_{kli}^{\cdot\cdot\cdot b}C_{bja}^{\cdot\cdot\cdot c}-C_{klj}^{\cdot\cdot\cdot b}C_{bia}^{\cdot\cdot\cdot c})\frac{\partial}{\partial e^c} \qquad \text{(by (4))}$$

$$=C_{kli}^{\cdot\cdot\cdot b}E_{bj}-C_{klj}^{\cdot\cdot\cdot b}E_{bi}.$$

PROPOSITION 4.2. *In Proposition 4.1. $det\,|e^j e^k C_{ijk}^{\cdot\cdot\cdot l}-\delta_i^l\omega|$ is an invariant of the localgroup generated by $E_{ij}$.*

Proof is similar to that of E. Cartan ([1], p. 27) by using (4).

For two elements $e$ and $f$ of $r$-dimensional L.t.s. $\mathfrak{T}$ corresponds a linear transformation of $\mathfrak{T}: D_{(e,f)}: x\to[efx]$. Denote by $\alpha_1, \alpha_2, \cdots, \alpha_s$ $(s\leqq r)$ the distinct roots with multiplicity $\nu_k$ of the characteristic equation $det\,|e^i f^j C_{ijk}^{\cdot\cdot\cdot l}-\delta_k^l\omega|=0$, where $C_{ijk}^{\cdot\cdot\cdot l}$ is the structure constants of $\mathfrak{T}$. Then $\mathfrak{T}$ decomposes to a direct sum of root spaces: $\mathfrak{T}=\mathfrak{V}_{\alpha_1}\oplus\mathfrak{V}_{\alpha_2}\oplus\cdots\oplus\mathfrak{V}_{\alpha_s}$, $(D_{(e,f)}-\alpha_k E)^{\nu_k}\mathfrak{V}_{\alpha_k}=(0)$. We say that the element $x$ of $\mathfrak{V}_\alpha$ belongs to root $\alpha$.

PROPOSITION 4.3. *If $x_\alpha$, $x_\beta$ and $x_\gamma$ belong to roots $\alpha$, $\beta$ and $\gamma$ respectively, then $[x_\alpha x_\beta x_\gamma]$ belongs to a root $\alpha+\beta+\gamma$ and $[x_\alpha x_\beta x_\gamma]=0$ when $\alpha+\beta+\gamma$ is not a root.*

PROOF. Put $D_{(e,f)}\equiv D_0$, $[x_\alpha x_\beta x_\gamma]\equiv x$ then we have

$$D_0 x=[(D_0 x_\alpha)x_\beta x_\gamma]+[x_\alpha(D_0 x_\beta)x_\gamma]+[x_\alpha x_\beta(D_0 x_\gamma)],$$

hence the proposition is clear.

## §5. Classification of 2-dimensional L.t.s.

LEMMA 5.1. *If $a$ and $b$ is a base of 2-dimensional L.t.s. then we obtain $[abb]=\alpha a+\beta b$, $[baa]=\beta a+\gamma b$, $\alpha, \beta, \gamma\in\Phi$.*

PROOF. Suppose that $[abb]=\alpha a+\beta b$, $[baa]=\delta a+\gamma b$.
Since it holds by using (4) that

$$[[baa]bb]+[[abb]ab]=0,$$

we obtain $(\delta-\beta)[abb]=0$. Therefore, if $\delta\neq\beta$ we have $\alpha=\beta=0$ and from (4) $[[baa]ba]=0$. This implies $\delta=0$ which contradicts our assumption.

PROPOSITION 5.1. *Any 2-dimensional L.t.s. over the complex field can be reduced by the suitable base transformations to one of the following* (i), (ii) *and* (iii).

$$\text{(i)}\begin{cases}[abb]=0\\ [baa]=0\end{cases}\text{(abelian)}\quad \text{(ii)}\begin{cases}[abb]=a\\ [baa]=0\end{cases}\text{(solvable)}\quad \text{(iii)}\begin{cases}[abb]=a\\ [baa]=b\end{cases}\text{(simple).}$$

PROOF. From Lemma 5.1 it follows that for the base $a$ and $b$

$$\text{(A)}\qquad [abb]=\alpha a+\beta b,\quad [baa]=\beta a+\gamma b.$$

case 1.1: $\alpha=0$ and $\beta=0$

If $\gamma=0$ it is the type (i).

If $\gamma\neq 0$ by the transformation $a'=b$, $b'=\frac{1}{\sqrt{\gamma}}a$ (A) reduces to the type (ii).

case 1.2: $\alpha=0$ and $\beta\neq 0$

If $\gamma=0$, by the transformation $a'=\frac{\beta a+b}{\sqrt{-2\beta}}$, $b'=\frac{b-\beta a}{\sqrt{2\beta}}$ we have the type (iii).

If $\gamma\neq 0$, by the transformation $a'=\frac{1}{\sqrt{\gamma}}a$, $b'=\frac{1}{\sqrt{-\gamma}}\left(a+\frac{\gamma}{\beta}b\right)$ (A) reduces to the type (iii).

case 2.1: $\alpha\neq 0$ and $\alpha\gamma-\beta^2=0$. By the transformation $a'=\alpha a+\beta b$, $b'=\frac{1}{\sqrt{\alpha}}b$ (A) reduces to the type (ii).

case 2.2: $\alpha\neq 0$ and $\alpha\gamma-\beta^2\neq 0$. By the transformation $a'=\frac{\alpha a+\beta b}{\sqrt{\alpha(\alpha\gamma-\beta^2)}}$, $b'=\frac{1}{\sqrt{\alpha}}b$ (A) reduces to the type (iii).

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