JOURNAL OF SCIENCE OF THE HIROSHIMA UNIVERSITY, SER. A, VOL. 21, No. 2, JULY, 1957

On Some Properties of FC-groups

By

Noboru NISHIGÔRI

(Received Apr. 25, 1957)

1. Introduction.

In the investigation of the infinite groups, the groups in which all the classes of conjugate elements are finite have received an attention in recent years. Such groups were named "FC-group" by R. Baer and studied by R. Baer, B. H. Neumann, J. Erdös and F. Haimo. And it is known that these groups have many similar properties with abelian groups and finite groups.

In this paper, we shall investigate some properties of FC-groups. First, we require the necessary and sufficient conditions in order that a group be an FC-group. Next we prove that in FC-groups the division hull of an arbitrary subgroup, that is, the set of all elements for which there exists a positive integer m such that the m-th power of it is contained in the subgroup, is also subgroup. This is a common property of FC-groups with abelian groups and finite groups. Moreover we show that the simple FC-groups are finite groups. Finally we prove that some FC-groups decompose in the product of a finite number of its subgroups.

We use the following notations: When S and T are two subsets of a group G, $S \cap T$ is the intersection of S and T. And $P(S)$ means the set of all elements of S each of which has a finite order, that is, the periodic part of S. Next, let S be a subgroup of G. S' is the derived group of S. And the division hull of S in G is denoted by $D(S:G)$ which due to F. Haimo [3].[1] Moreover, if a and b are two elements of G, $[a, b]$ is the commutator of a and b.

2. Characterization of FC-groups.

If G is an FC-group, then its subgroups and factor groups are FC-groups. But the converse is not true. In this section we obtain the necessary and sufficient conditions in order that a group be an FC-group. And each of them is corresponding to the converse.

First, we prove a lemma.

LEMMA 1. *If a group G has a normal subgroup N such that $N \cap G'$ is*

1) The numbers in brackets refer to the references at the end of this paper.

a finite subgroup and the factor group G/N is an FC-group, then G is an FC-group.

PROOF. Let N be a normal subgroup of G which satisfies the conditions of the lemma.

Now we choose an element x of G arbitrarily and fix it. The set xN is an element of G/N and contains the element x. As the group G/N is an FC-group, the element xN has a finite number of conjugate elements. We can denote these conjugate elements as follows:

$$xN,\ g_1^{-1}xg_1N, \cdots,\ g_{n-1}^{-1}xg_{n-1}N \tag{1}$$

Let $g^{-1}xg\ (g \in G)$ be an element which is conjugate to x. Then we have:

$$g^{-1}xg \in g^{-1}xgN=(gN)^{-1}(xN)(gN),$$

that is, $g^{-1}xg$ is contained in one and only one coset of N in G, an element of the factor group G/N, in (1). Therefore any conjugate element to x in G is certainly contained in one coset in (1).

Now we assume that $g^{-1}xg$ is contained in the coset $g_i^{-1}xg_iN$ (where the suffix i is a fixed number). Then there exists an element n of N such that $g^{-1}xg=g_i^{-1}xg_in$. Therefore we have $[x, g_i]^{-1}[x, g]=n$, that is, $[x, g_i]^{-1}[x, g]$ is contained in N. On the other hand, $[x, g_i]^{-1}[x, g]$ is contained in G'. Thus $[x, g_i]^{-1}[x, g]$ is contained in $N \cap G'$.

Let H be the set $\{g;\ g^{-1}xg \in g_i^{-1}xg_iN,\ g \in G,\ i: \text{fixed}\}$. Then the number of the elements with the form $[x, g_i]^{-1}[x, g]$ $(g \in H)$ is finite, for by the assumption of the lemma, $N \cap G'$ is a finite group. But, as the suffix i is fixed, the element $[x, g_i]^{-1}$ is the fixed element of G. Therefore, when g runs over the elements of H, the number of the element $[x, g]$ is finite, that is, the number of conjugate elements to x which is contained in the coset $g_i^{-1}xg_iN$ is finite. By the same argument, the number of conjugate elements to x which is contained in each coset in (1) is finite. Therefore the number of conjugate elements to x in G is finite. Thus we have proved the lemma.

REMARK. From the above lemma, the following two propositions follow:

i) If G has a finite normal subgroup N such that the factor group G/N is an FC-group, then G is an FC-group.

ii) If the derived group G' of the group G is finite, then G is an FC-group.

THEOREM 1. *For a group G, the following three propositions are equivalent.*

1) *G is an FC-group.*

2) *G has a finitely generated normal FC-subgroup N such that the factor group $G/P(N)$ is an FC-group, where $P(N)$ is the periodic part of N.*

3) *The periodic part $P(G)$ of G contains the derived group G' and G has a finitely generated normal FC-subgroup N such that the factor group G/N is an FC-group.*

Proof. Let G be an FC-group. If g is an arbitrary element of G, then the number of its conjugate elements is finite and we denote them $g_1(=g), g_2, \cdots, g_m$. Let N be the subgroup which is generated by $g_1, g_2, \cdots, g_m$. Then N is a finitely generated normal FC-subgroup. And, as the periodic part $P(N)$ is a characteristic subgroup of N, it is a normal subgroup of G.[2] Hence there exists the factor group $G/P(N)$ and it is an FC-group. Therefore 2) follows from 1).

Let N be a finitely generated normal FC-subgroup such that the factor group $G/P(N)$ is an FC-group. As $G/P(N)$ is an FC-group, the derived group $(G/P(N))'$ of $G/P(N)$ is periodic. And it holds that $(G/P(N))'=G'\cdot P(N)/P(N)$. Therefore, for an arbitrary element c of G' there exists a positive integer n such that $(c\,P(N))^n=P(N)$. Then it follows that $c^n \in P(N)$ and hence c belongs to $P(G)$. Thus we have proved that G' is contained in $P(G)$.

Moreover, as $G/P(N)$ is an FC-group and $P(N)\subset N$, the factor group G/N is an FC-group. Therefore 3) follows from 2).

Finally we assume that $P(G)$ contains the derived group G' and G has a finitely generated normal FC-subgroup N such that the factor group G/N is an FC-group. As G' is a periodic group, we have $N\cap G'=P(N)\cap G'$. On the other hand, as N is a finitely generated FC-group, $P(N)$ is a finite group.[3] Therefore, from the lemma 1, we have that G is an FC-group. Thus 1) follows form 3).

3. Division hull of FC-groups.

In this section we shall investigate the properties of the division hull which is introduced in a group by F. Haimo [3] and P. Alexandroff [6]. In abelian groups and finite groups, the division hull of an arbitrary subgroup is also subgroup, but in general infinite groups the division hull need not be a subgroup. We show that in FC-groups, as well as abelian groups and finite groups, the division hull of an arbitrary subgroup is also a subgroup.

We define the division hull as follows which is due to F. Haimo.

Let K be a subgroup of a group G. By the *division hull* of K in G, $D(K:G)$, we mean the set of all $x \in G$ for which there exist positive integers $n=n(x)$ with $x^n \in K$.

Then we have the following lemmas:

Lemma 2. *Let G be a group with the periodic derived group G'. If K is an arbitrary normal subgroup of G, then $D(K:G)$ is a subgroup of G.*

Proof. Let x and y be two elements of $D(K:G)$. From the definition of the division hull, there exist the positive integers s and t such that $x^s \in K$ and $y^t \in K$. Then we have

2) A characteristic subgroup of a group means the subgroup which is invariant under all the automorphisms of the group.

3) cf. [4] Theorem 5·1

$$(xy)^{st}=c\cdot x^{st}y^{st} \quad (\text{where } c\in G')$$
$$=c\cdot k \quad (\text{where } k\in K)$$

Now, let n be the order of the element c. As K is a normal subgroup of G, we have that $(xy)^{stn}=(c\cdot k)^{n}=k'\in K$, that is, $xy\in D(K:G)$.

And it is clear that there exist the unit and inverse elements in $D(K:G)$. Therefore, $D(K:G)$ is a subgroup of G.

Next, in the following lemma, we shall show that in the above lemma $D(K:G)$ is a normal subgroup.

LEMMA 3. *Let G be a group and K be a subgroup of G. And let $D(K:G)=L$ be a subgroup of G. Then L is a normal subgroup of G if and only if for any element $x\in K$ and $g\in G$ there exists a positive integer m such that $g^{-1}x^{m}g\in K$.*

PROOF. Let L be a normal subgroup of G. And let x and g be an arbitrary element of K and G respectively. From $x\in K$ and $K\subset L$, it holds $x\in L$. As L is normal in G, $g^{-1}xg$ is contained in L. Therefore there exists a positive integer m such that $g^{-1}x^{m}g\in K$.

Conversely we assume that there exists a positive integer m such that $g^{-1}x^{m}g\in K$ for $x\in K$ and $g\in G$. If y be an arbitrary element of L, then there exists a positive integer s such that $y^{s}\in K$. By the assumption, for any element g of G, there exists a positive integer t such that $g^{-1}(y^{s})^{t}g\in K$. Therefore $g^{-1}yg\in L$, that is, N is a normal subgroup of G.

LEMMA 4. *Let G be a group and K be an arbitrary subgroup of G. And let S be an arbitrary finitely generated subgroup of G and L be an arbitrary subgroup of S. Then $D(K:G)$ is a subgroup of G if and only if $D(L:S)$ is a subgroup of S.*

PROOF. We assume that the division hull of an arbitrary subgroup of G is a subgroup of G. From the assumption of the lemma, L is a subgroup of G. Hence $D(L:G)$ is a subgroup of G. Therefore, if x and y are two elements of $D(L:S)$, then there exists a positive integer l such that $(xy)^{l}\in L$. That is, xy belongs to $D(L:S)$. And there exist the unit and inverse elements in $D(L:S)$. Consequently $D(L:S)$ is a subgroup of S.

Conversely we assume that the division hull of any subgroup L of an arbitrary finitely generated subgroup S is a subgroup. If x and y are two elements of $D(K:G)$, then there exist the positive integers s and t such that $x^{s}\in K$ and $y^{t}\in K$. We take as S the subgroup of G which is generated by x and y, and as L the subgroup of S which is generated by x^{s} and y^{t}. From the assumption, $D(L:S)$ is a subgroup of S. Therefore, since x and y belong to $D(L:S)$, there exists a positive integer l such that $(xy)^{l}\in L$. On the other hand K contains L. Therefore $(xy)^{l}\in K$, that is, xy belongs to $D(K:G)$. And there exist the unit and inverse elements in $D(K:G)$. Thus we have proved that $D(K:G)$ is a subgroup of G.

Lemma 5. *Let G be a finitely generated FC-group. If K is an arbitrary subgroup of G, then $D(K:G)$ is a subgroup of G.*

Proof. As G is a finitely generated FC-group, it contains in its center a torsion-free abelian subgroup A which has a finite index m in G[4]. If $g_1, g_2, \cdots, g_m$ are the representatives of the cosets of A in G, then it holds that $(g_i g_j)^m = g_i^m g_j^m$ $(1 \leqq i, j \leqq m)$.[5] For, it holds that $(g_i g_j)^m = c \cdot g_i^m g_j^m$ $(c \in G')$. On the other hand, as $(g_i g_j)^m$, g_i^m and g_j^m belong to A, c belongs to A. Hence, c belongs to $A \cap G'$. But, as G is an FC-group, G' is a periodic group. Therefore it holds $A \cap G' = e$. Consequently we have $c = e$.

From the above proposition, for any elements a and b of G, it holds that $(ab)^m = a^m \cdot b^m$.

Now, let x and y be two elements of $D(K:G)$. Then there exist the positive integers s and t such that $x^s \in K$ and $y^t \in K$. Therefore we have that $(xy)^{mst} = x^{mst} y^{mst} \in K$. That is, xy belongs to $D(K:G)$. And there exist the unit and inverse elements in $D(K:G)$. Thus we have proved that $D(K:G)$ is a subgroup of G.

From the lemmas 4 and 5, we have the following theorem:

Theorem 2. *Let G be an FC-group. If K is an arbitrary subgroup of G, then $D(K:G)$ is a subgroup of G.*

4. Some Properties of FC-groups.

In this section, we investigate the another properties of FC-groups which are concerned with the structure of FC-groups.

First, for simple FC-groups we have the following theorem:

Theorem 3. *Simple FC-groups are finite groups.*

Proof. Let G be a simple FC-group. We prove this theorem by dividing in the two cases:

Case 1. $P(G) = e$: As G is an FC-group, $P(G)$ contains G'. Therefore $G' = e$, that is, G is an abelian group. As G is simple, G is a cyclic group with a prime order.

Case 2. $P(G) \neq e$: Let $g (\neq e)$ be an element of $P(G)$ and $g_1 (= g), g_2, \cdots, g_n$ be conjugate elements of g. Then each of $g_1, g_2, \cdots, g_n$ has a finite order in G. And, a subgroup N of G which is generated by $g_1, g_2, \cdots, g_n$ is a normal subgroup of G and is not e. But, as G is simple, we have $N = G$. On the other hand, as N is a finitely generated FC-group and each of its generators has a finite order, N is a finite group[6]. Thus we have proved that G is a finite group.

Next, we consider some decomposition of FC-groups by using the division hulls.

Here, we define a decomposition of a group as follows:

4) cf. [4] Lemma 4.1.
5) J. Erdös has proved this relation by an another method. (cf. [2]).
6) cf. [4] Corollary 5.12; [2] Theorem 4.5.

A group G has a *decomposition* $G_1 \circ G_2 \circ \cdots \circ G_n$ into a product of its subgroups $G_1, G_2, \cdots, G_n$, if the following conditions are satisfied:

1) $G = G_1 \cdot G_2 \cdots G_n$,

2) $G_i \cap (G_1 \cdot G_2 \cdots G_{i-1} \cdot G_{i+1} \cdots G_n) = e$.

Then we have a lemma.

LEMMA 6. *Let G be an FC-group and L be a division hull which is a proper normal subgroup of G. If G/L is the direct sum of a finite number of cyclic groups, then G has a decomposition $A_1 \circ A_2 \circ \cdots \circ A_n \circ L$, where A_i ($i = 1, 2, \cdots, n$) is an infinite cyclic subgroup. And in this decomposition every element of G is uniquely expressible as a product of elements of subgroups A_i and L.*

PROOF. First, we prove that the factor group G/L is a torsion-free abelian group. Now let L be a division hull of some subgroup K of G. Since G is an FC-group and $L = D(K : G)$, it holds that $L \supset P(G) \supset G'$. Hence G/L is an abelian group. If an element $gL (\neq L)$ of G/L has a finite order, then we have $(gL)^m = L$ for some positive integer m, that is, $g^m \in L$. Hence there exists a positive integer n such that $(g^m)^n \in K$. That is, g belong to L. This is a contradiction. Therefore, G/L is a torsion-free abelian group.

Next, we show that any element g which is mapped on one element $(\neq L)$ of G/L under natural mapping of G onto G/L has an infinite order in G. For, if an element g which is mapped on the one element $(\neq L)$ (a coset of L in G) of G/L has a finite order, then the coset has a finite order in G/L. It is a contradiction.

Let G/L be the direct sum of the finite number of cyclic groups $C_i = \{a_i L\}$ ($i = 1, 2, \cdots, n$). If G_i is the inverse image of C_i under the natural mapping, then G_i is a normal subgroup of G and we have $G_i = \{a_i\} \cdot L$ where $\{a_i\}$ is an infinite cyclic group with generator a_i, because $a_i L$ and a_i has an infinite order in G/L and in G respectively.

And, as G is contained in $G_1 \cdot G_2 \cdots G_n$, G is represented as a product of subgroups A_i and L, that is, $G = G_1 \cdot G_2 \cdot \cdots \cdot G_n = A_1 \cdots A_n \cdot L$, where $A_i = \{a_i\}$ ($i = 1, 2, \cdots, n$).

Further we can show that $A_i \cap (A_1 \cdots A_{i-1} \cdot A_{i+1} \cdots A_n \cdot L) = e$, where $i = 1, 2, \cdots, n+1$ and $A_{n+1} = L$. For if $g (\neq e)$ is an element $A_i \cap (A_1 \cdots A_{i-1} \cdot A_{i+1} \cdots A_n \cdot L)$, then it holds that $g = b_i = b_1 \cdots b_{i-1} \cdot b_{i+1} \cdots b_{n+1}$ (where $b_i \in A_i$). Therefore we have $b_i L = b_1 L \cdots b_{i-1} L \cdot b_{i+1} L \cdots b_n L$. This contradicts that G/L is a direct sum of C_i. Thus G has a decomposition $A_1 \circ A_2 \circ \cdots \circ A_n \circ L$.

Moreover, let g be an element of G. If $g = b_1 \cdot b_2 \cdots b_{n+1} = \bar{b}_1 \cdot \bar{b}_2 \cdots \bar{b}_{n+1}$ ($b_i, \bar{b}_i \in A_i$), we have $gL = b_1 L \cdot b_2 L \cdots b_n L = \bar{b}_1 L \cdot \bar{b}_2 L \cdots \bar{b}_n L$. But, as the factor group G/L is a direct sum of C_i we have $b_i L = \bar{b}_i L$. Hence it holds that $b_i = \bar{b}_i$, because $A_i \cap L = e$. Therefore, every element of G is uniquely expressible as a product of elements of subgroups A_i and L. Thus we have proved the lemma.

THEOREM 4. *Let G be a finitely generated FC-group. Then G has the decomposition $A_1 \circ A_2 \circ \cdots \circ A_n \circ F$, where A_i $(i=1,2,\cdots,n)$ is an infinite cyclic group and F is a finite group. And every element of G is uniquely expressible as a product of elements of subgroups A_i and F.*

PROOF. As G is a finitely generated FC-group, $P(G)$ is the finite normal subgroup $D(e:G) \neq G$. If we put $P(G)=F$, then G/F is a finitely generated torsion-free abelian group. Therefore G/F is a direct sum of a finite number of the infinite cyclic groups. From the lemma 6 we have proved the theorem 4.

In conclusion I wish to express my thanks to Prof. *K.* Morinaga for his kind guidance in the course of the work.

References

[1] R. Baer, *Finiteness properties of groups*, Duke Math. J., **15** (1948), pp. 1021–1032.

[2] J. Erdös, *The theory of groups with finite classes of conjugate elements*, Acta Math. Acad. Sci. Hungar., **5** (1954), pp. 45–58.

[3] F. Haimo, *The FC-chain of a group*, Canad. J. Math., **5** (1953), pp. 498–511.

[4] B. H. Neumann, *Groups with finite classes of conjugate elements*, Proc. London Math. Soc., (3) **1** (1951), pp. 178–187.

[5] ———, *Groups with finite classes of conjugate subgroups*, Math. Z., **63** (1955), pp. 76–96.

[6] P. Alexandroff and H. Hopf, *Topologie*, I, Berlin, Springer, 1935.

[7] A. G. Kurosh, *The theory of groups*, (translated from the Russian), New York, vol. 1:1955, vol. 2:1956.

Shinonome Branch,
Hiroshima University.