JOURNAL OF SCIENCE OF THE HIROSHIMA UNIVERSITY, SER. A, VOL. 21, NO. 2, JULY, 1957

On a Locally Convex Space Introduced By J.S.E Silva

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(Received April 30, 1957)

Introduction

In the recent paper [8], J.S.E Silva has given a detailed discussion on the (LN^*) space, a certain type of an inductive limit of normed spaces, in view of applications to the theory of analytic functions and to the theory of distributions. His way of proving the properties of (LN^*) space, which are requisite to applications, is based rather on the consideration of the spectre defining the space than on the direct investigation of the (LN^*) space itself as a locally convex space.

In the present paper, we give certain necessary and sufficient conditions for a locally convex space to be an (LN^*). And basing on this characterization, though some of the results are due to Silva, we prove certain fundamental properties of (LN^*) spaces. Among other things it is shown that a closed subspace of an (LN^*) is an (LN^*), that an inductive limit of a countable number of (LN^*) is an (LN^*), and that the strong dual of a Schwartz (LF) space is nothing but a projective limit of a sequence of (LN^*) of certain type.

In most cases of this paper, notions and terminologies of Bourbaki [1] are used without explicit references.

§1. Preliminaries

Let E be a locally convex space over the complex numbers. If $A(\subset E)$ is a bounded closed disk (=convex circled subset), E_A denotes the normed space generated by A with the norm $\|x\|_A=\inf_{x\in\lambda A}|\lambda|$, $x\in E_A$. E is a (DF) *space* [5] if

(1) E has a fundamental sequence of bounded sets;

(2) if the intersection $\bigcap_k U_k$ of a sequence of closed convex circled neighbourhoods U_k of the origin absorbs every bounded set, $\bigcap_k U_k$ is a neighbourhood. E is said to satisfy the *strict Mackey condition* if for every bounded set $A\subset E$ there exists a bounded closed disk $B\supset A$ such that the topology on A induced by E is identical with the topology on A induced by E_B. E is called a *Schwartz space* [5] if

(1) for every equicontinuous set $A \subset E'$, there exists a neighbourhood U of the origin of F such that the topology on A induced by the strong E' is identical with the topology on A induced by E'_{U^0};

(2) every bounded subset of E is précompact.

It is known [5] that an (F) space E is a Schwartz space if and only if it is a Montel space satisfying the condition: every convergent sequence in the strong dual E' converges uniformly on a neighbourhood of the origin of E. The condition of separability (=existence of a countable dense subset) on E is superfluous, since Dieudonné [4] has shown that a Montel (F) space is necessarily separable.

Let $\{E_k\}$ (resp. $\{F_k\}$) be a sequence of locally convex spaces and $\iota_{k+1,k}$ (resp. $\pi_{k,k+1}$) be a continuous linear mapping of E_k into E_{k+1} (resp. of F_{k+1} into F_k), $k=1,2,\cdots$. Then $\{E_k, \iota_{k+1,k}\}$ (resp. $\{F_k, \pi_{k,k+1}\}$) is called an *inductive* (resp. *projective*) *spectre* [8]. Let E (resp. F) be its *inductive* (resp. *projective*) *limit space*. Then the canonical mapping of E_k into E (resp. of F into F_k) is denoted by ι_k (resp. π_k). If E_k is a vector subspace of E_{k+1} and $\iota_{k+1,k}$ is the continuous injection, $k=1,2,\cdots$, then E is called the *canonical inductive limit* of $\{E_k\}$, and if, in addition, $\iota_{k+1,k}$ is the isomorphic injection, E is called the *strict inductive limit*.

For a later purpose we mention here the following two propositions concerning the limit space. But previously it seems convenient to prepare the next lemma.

LEMMA 1. *Any linear mapping of a locally convex Hausdorff space into another is a weak homomorphism if and only if it is a homomorphism in the Mackey topology (=relatively strong topology), provided the Mackey topology of the image is identical with the Mackey topology of the range space* (cf. Bourbaki [1], no. 1229, p. 106, Ex. 3), b)).

REMARK 1. We omit the obvious proof of this lemma. But in view of applications, two important special cases are pointed out, the case of a metrisable range space and the case of an onto mapping. The additional assumption made on the image of the mapping is superfluous in these cases.

PROPOSITION A. *Let F be the limit space of a projective spectre $\{F_k, \pi_{k,k+1}\}$ of locally convex Hausdorff spaces such that $\pi_{k,k+1}$ is a mapping of F_{k+1} onto F_k, transforming any bounded subset of F_{k+1} into a weakly relatively compact subset of F_k, $k=1,2,\cdots$. Then F is semi-reflexive and its strong dual F is the limit space of the inductive spectre $\{F'_k, {}^t\pi_{k,k+1}\}$ regarding not only the strong topology but the Mackey topology $\tau(F'_k, F_k)$* of F'_k, *$k=1,2,\cdots$. Every ${}^t\pi_{k,k+1}$ is univalent.*

PROOF. Since $\pi_{k,k+1}$ is onto, it is not difficult to see that ${}^t\pi_{k,k+1}$ is univalent and that F' is algebraically the inductive limit of $\{F'_k, {}^t\pi_{k,k+1}\}$. The semi-reflexivity of F is proved as follows. Let B be any bounded subset of F. Then $\pi_{k+1}(B)$ is bounded in F_{k+1}. Hence, owing to the assumption on the mapping $\pi_{k,k+1}$, $\pi_k(B)=\pi_{k,k+1}(\pi_{k+1}(B))$ is weakly relatively compact in $F_k, k=1,2,\cdots$. Thus, by the definition of the projective limit, it is not difficult to see that B is weakly relatively compact in F. Consequently, F is semi-reflexive. Next we show that the strong topology of F' is the inductive limit topology of the spectre $\{F'_k, {}^t\pi_{k,k+1}\}$ regarding the strong topology of F'_k. Let V be a closed disk neighbourhood of the limit topology. Then ${}^t\pi_k^{-1}(V)=V_k$ is a closed disk neighbourhood of the strong E'_k, and hence $A_k=V_k^0$ is a bounded closed disk in E_k. Owing to the relation $(\pi_{k,k+1}(A_{k+1}))^0$ ${}^t\pi_{k,k+1}^{-1}(A_{k+1}^0)={}^t\pi_{k,k+1}^{-1}(V_{k+1})=V_k$, it is not difficult to see that $\overline{\pi_{k,k+1}(A_{k+1})}=(\pi_{k,k+1}(A_{k+1}))^{00}=V_k^0=A_k$. Hence by the assumption on $\pi_{k,k+1}$, it is seen that A_k is weakly compact, $k=1,2,\cdots$. As a weakly continuous image of a weakly compact set, $\pi_{k,k+1}(A_{k+1})$ is weakly compact. Consequently $\pi_{k,k+1}(A_{k+1})$ $=\overline{\pi_{k,k+1}(A_{k+1})}=A_k$, $k=1,2,\cdots$. Let $A=\bigcap_k \pi_k^{-1}(A_k)$. Then A is a bounded disk in F. It is not difficult to see that $V=A^0$, namely V is a strong neighbourhood of F'. Thus the strong topology of F' is not weaker than the limit topology of $\{F'_k, {}^t\pi_{k,k+1}\}$ regarding the strong topology of $F'_k, k=1,2,\cdots$. The converse is obvious. The case of Mackey topology is almost evident. This completes the proof.

COROLLARY. *Let F be the limit space of a projective spectre $\{F_k, \pi_{k,k+1}\}$ of semi-reflexive locally convex Hausdorff spaces with metrisable strong duals, satisfying $\pi_{k,k+1}(F_{k+1})=F_k, k=1,2,\cdots$. Then F is semi-reflexive and $\{F'_k, {}^t\pi_{k,k+1}\}$ may be considered as a strict inductive spectre whose limit space is the strong dual F' of F. Moreover, if all F_k are (DF) spaces, then F' is an (LF) space.*

PROOF. Owing to the semi-reflexivity of $F_k, k=1,2,\cdots$, the mapping $\pi_{k,k+1}$ transforms any bounded subset of F_{k+1} into a weakly relatively compact subset of $F_k, k=1,2,\cdots$. Hence, if we show that ${}^t\pi_{k,k+1}$ is an isomorphism of F'_k into F'_{k+1}, the proof will be concluded. $\pi_{k,k+1}$ is a continuous linear mapping of F_{k+1} onto F_k, ${}^t\pi_{k,k+1}$ is a weak isomorphism of F'_k into F'_{k+1}. Thus, by Lemma 1 together with Remark 1, ${}^t\pi_{k,k+1}$ is an isomorphism in the Mackey topology, that is in the strong topology of the dual.

PROPOSITION B. *Let E be the inductive limit of the inductive spectre $\{E_k, \iota_{k+1,k}\}$ with the property: for any bounded subset $B\subset E$, there exist an E_k and a bounded subset $B_k\subset E_k$ such that $\iota_k(B_k)\supset B$. (For example, this is the case for the strict inductive limit.) Then the strong dual E' is the projective limit of the projective spectre $\{E'_k, {}^t\iota_{k+1,k}\}$.*

PROOF. In any case E' is algebraically the projective limit of $\{E'_k, {}^t\iota_{k+1,k}\}$. As for the topology, the strong topology of the dual is not weaker than that of the inductive limit. The converse may be verified as follows. Let B be the bounded subset of E. Then by the assumption concerning the bounded set, $B\subset\iota_k(B_k)$ for a bounded subset B_k of E_k. Thus $B^0\supset(\iota_k(B))^0$ $={}^t\iota_k^{-1}(B_k^0)$. This completes the proof.

We note that a locally convex space is a normed space if and only if it is a metrisable (DF) space, and that a canonical inductive limit space of normed spaces is characterized as a bornological (DF) space.

§2. Silva space

For completeness, we begin with the definitions of (LN^*) space and (M^*) space. In view point of our characterization given below, it seems appropriate to prefer the terminology *Silva space* to (LN^*) space and *dual Silva space* to (M^*) space.

DEFINITION 1. [8]. A locally convex space E is called a *Silva space* if it is a canonical inductive limit of an increasing sequence of normed spaces $\{E_k\}$ such that the injection $E_k \to E_{k+1}$ is compact, $k=1,2,\cdots$.

DEFINITION 2. [8]. A locally convex space F is called a *dual Silva space* if it is a projective limit of a projective spectre $\{F_k, \pi_{k,k+1}\}$, consisting of normed spaces F_k and compact mappings $\pi_{k,k+1}: F_{k+1} \to F_k$, $k=1,2,\cdots$.

First we give our basic theorem.

THEOREM 1. *Let E be a locally convex space. Then the following conditions* (1)–(4) *are equivalent:*

(1) *E is a Silva space;*

(2) *E is a Montel, Hausdorff (DF) space satisfying the strict Mackey condition;*

(3) *E is a Hausdorff (DF) space satisfying the following condition* $(*)$;

$(*)$ *For any bounded subset $A \subset E$, there exists a bounded closed disk $B \supset A$ such that A is relatively compact in E_B;*

(4) *E is a bornological Hausdorff space admitting a fundamental sequence of bounded sets, and satisfies the condition* $(*)$.

PROOF. Ad (1)$\to$(2): Let $\{E_k\}$ be a defining sequence of E. We may assume that $\{E_k\}$ are all Banach spaces and the unit sphere B_k of E_k is a compact subset of E_{k+1}, $k=1,2,\cdots$, [8]. First we shall prove that E is a Hausdorff space. For this purpose it suffices to see that the one point set $\{0\}$ is a closed subset of E. Let $0 \neq x \in E$. Then there exists a positive number α_1 such that $\alpha_1 B_1 \not\ni x$. The subset $\alpha_1 B_1$ of E_2 is compact in E_2, so that a suitable positive number α_2 satisfies $\alpha_1 B_1 + \alpha_2 B_2 \not\ni x$. Continuing this process, we may obtain a sequence $\{\alpha_k\}$ of positive numbers such that $x \notin \sum_{k=1}^{n} \alpha_k B_k$, $n=1,2,\cdots$. $U = \sum_{k=1}^{\infty} \alpha_k B_k$ (algebraic sum) is a neighbourhood of the origin in E, with $x \notin U$. This shows that $\{0\}$ is closed. By the remark at the end of §1, E is a bornological (DF) space. Let A be a closed bounded subset of E. Then A is contained in the closure in E of a bounded subset of some E_k ([5], Théorème 9). Hence A is compact in E_{k+1} and consequently in E. It follows from this that E satisfies the strict Mackey condition and that E is a complete space ([5], Proposition 5, Corollaire 2). Thus E is a bornological complete space whose bounded sets are relatively compact, and in particular it is a Montel space.

Ad (2)$\rightarrow$(3): We need only to verify the condition $(*)$. Let A be a closed bounded subset of E. Then A is compact. By the strict Mackey condition, the topology of E and that of an E_B, B being a bounded closed disk, induce the same topology on A, as desired.

Ad (3)$\rightarrow$(4): We need only to show that E is bornological. Owing to the condition $(*)$, any closed bounded subset of E is compact and metrisable. Hence E is semi-reflexive and in the strong dual E' every bounded subset is equicontinuous ([5], Théorème 5), namely E is reflexive. Thus E is barreled (=tonnelé). Since E' is metrisable, we may conclude that E is bornological ([5], Théorème 7).

Ad (4)$\rightarrow$(1): Owing to the condition $(*)$, we may choose a fundamental sequence $\{B_k\}$ of bounded subsets in such a way that B_k is a compact disk in $E_{B_{k+1}}$, $k=1,2,\cdots$. Let $\tilde{E}$ be the limit space of the canonical inductive spectre $\{E_{B_k}\}$. Then $\tilde{E}$ is a Silva space, and except the topology $E=\tilde{E}$. But E and $\tilde{E}$ are both bornological spaces with the same bounded subsets. Thus $E=\tilde{E}$ topologically. Hence E is a Silva space. This completes the proof.

COROLLARY. *A Silva space is characterized as the strong dual of a Schwartz (F) space.*

PROOF. If E is a Silva space, then by the proof (3)$\rightarrow$(4) above mentioned, E is a Montel space whose strong dual E' is an (F) space. Since E is reflexive and E' is a Montel space, it is not difficult to see that E' satisfies the conditions (1) and (2) of the Schwartz space: (1) is nothing but the condition $(*)$ satisfied by E and (2) is the direct consequence from the fact that E' is a Montel space. Thus E' is a Schwartz (F) space. Conversely, let F be a Schwartz (F) space. Then F is a Montel space and hence reflexive. Thus it follows easily from the condition (3) of Theorem 1 that the strong dual F' is a Silva space. This completes the proof.

It is to be noted that a Silva space is complete and reflexive. On the other hand, metrisable locally convex space admitting a fundamental sequence of bounded sets and satisfying the condition $(*)$ is finite dimensional. Hence a metrisable Silva space is nothing but a finite dimensional Euclidean space [8].

REMARK 2. A locally convex space satisfying the condition $(*)$ is semi-reflexive, and its closed subspaces have the same property.

As for the dual Silva space we mention the following

THEOREM 2. *A locally convex space is a dual Silva space if and only if it is a Schwartz (F) space.*

PROOF. (1) Dual Silva$\rightarrow$Schwartz (F): Let F be a dual Silva space defined by a projective spectre $\{F_k, \pi_{k,k+1}\}$. Clearly F is metrisable. Let B be a bounded subset of F. The continuous image $\pi_k(B)$ of B is relatively compact in F_k, $k=1,2,\cdots$. By means of the ordinary diagonal method, it is not difficult to see that B is relatively compact in F. From this and the metrisability of F, it follows that F is a Montel (F) space. Let $\{x_i'\}$ be a sequence from F' converging strongly to 0. Then $\{x_i'\}$ is bounded and hence equicontinuous, namely, contained in the polar U^0 of a neighbourhood U of the origin in F. We show that $\{x_i'\}$ converges uniformly on U, or what comes to the same thing, a subsequence of $\{x_i'\}$ converges uniformly on U. Assume that $U=\pi_k^{-1}(U_k)$, U_k being a neighbourhood of the origin in E_k, and take $y_i' \in F_k'$ with ${}^t\pi_k(y_i')=x_i'$, $i=1,2,\cdots$. Then $|<y_i', U_k>|=|<y_i', \pi_k(U)>|=|<x_i', U>|\leq 1$. Put $z_i'={}^t\pi_{k,k+1}(y_i')$. Owing to the compactness of ${}^t\pi_{k,k+1}$, $\{z_i'\}$ is strongly relatively compact in F_{k+1}'. Hence a convergent subsequence $\{z_{i_j}'\}$ can be found. Let the limit of $\{z_{i_j}'\}$ be z'. Then ${}^t\pi_{k+1}(z')=0$, and $<x_{i_j}', U>=<z_{i_j}'-z', \pi_{k+1}(U)>$ shows that $\{x_{i_j}'\}$ converges uniformly on U. Thus F is a Schwartz (F) space.

(2) Schwartz $(F)\to$Dual Silva: Let F be a Schwartz (F) space, and $\{U_k\}$ be a decreasing sequence of closed disk neighbourhoods such that $\bigcap_k U_k=\{0\}$. Put $\|x\|_k=\inf_{x\in\lambda U_k}|\lambda|$ and $N_k=\{x;\ \|x\|_k=0,\ x\in F\}$. Then $\{N_k\}$ is a decreasing sequence of closed subspaces. Let $F_k=\widehat{F/N_k}$, a Banach space, be the completion of F/N_k with respect to the norm induced by the semi-norm $\|x\|_k$. The extension to $F_{k+1}\to F_k$ of the canonical mapping $F/N_{k+1}\to F/N_k$ is denoted by $\pi_{k,k+1}$. By a suitable choice of $\{U_k\}$, we may assume that the closure in F_k of $\pi_{k,k+1}(U_{k+1})$ is compact [5]. Then the projective spectre $\{F_k, \pi_{k,k+1}\}$ defines a dual Silva space G. Owing to the definition of the neighbourhood of G, it is not difficult to see that F may be regarded as a dense linear subspace of G, and that the topology of G induces on F the original topology of F. Hence by the completeness of F, it follows that $F=G$. This completes the proof.

From this proof it is seen that a dual Silva space is a Montel separable space and that its defining sequence may be assumed to consist of Banach spaces.

COROLLARY. *The strong dual of a Silva space is a dual Silva space, and conversely.*

PROOF. Evident.

At any rate the strong dual of a metrisable Montel space is a Schwartz space ([5], p. 118). Hence a Silva space is a Schwartz space. Thus it is to be noted that the Silva space, the dual Silva space, their closed subspaces and their quotient spaces by closed subspaces are all separable [5].

PROPOSITION 1. *Let E be a Silva space and F be its closed subspace. Then F and E/F are Silva spaces.*

PROOF. The strong dual E' is a Schwartz (F) space. Hence $F=F^{00}$ is isomorphic with the strong dual $(E'/F^0)'$ [5]. Thus F is a Silva space, since E'/F^0 is a dual Silva space [5]. Let u be the canonical mapping $E\to E/F$ and let A be a bounded subset of E/F. Then there exists a bounded closed, hence compact subset B of E, such that $A\subset u(B)$ [5]. Owing to the condition

$(*)$ satisfied by E, B is relatively compact in E_C for a suitably chosen compact disk $C \supset B$. Then $u(B)$ and consequently A are relatively compact in $u(E)_{u(C)}$. Thus E/F satisfies the condition $(*)$. Hence E/F is a Silva space, since it is a (DF) space [5].

PROPOSITION 2. *Let E and F be Silva spaces. Then the projective tensor product* $E \hat{\otimes} F$ (Grothendieck's notation [6], p. 30) *is also a Silva space.*

PROOF. $E \hat{\otimes} F$ is a Montel (DF) space together with E and F ([6], p. 45). Hence we need only to verify the condition $(*)$. Let A be a bounded subset of $E \hat{\otimes} F$. Then A is contained in the disk closure $(B \otimes C)^{00}$ of some compact disks $B \subset E$ and $C \subset F$ ([6], p. 43). Owing to the condition $(*)$ satisfied by E and F, B and C are compact, respectively, in E_K and F_L for some suitably chosen compact disks K and L such that $B \subset K \subset E$, $C \subset L \subset F$. Then $B \otimes C$ is précompact in $E_K \otimes F_L$ ([6], p. 45). Hence it is easy to see that $B \otimes C$ is relatively compact in the Banach space $(E \hat{\otimes} F)_D$, where $D = (K \otimes L)^{00}$ is a compact subset of the complete space $E \hat{\otimes} F$. Consequently, $(B \otimes C)^{00}$ is a compact subset of $(E \hat{\otimes} F)_D$. Thus $E \otimes F$ satisfies the condition $(*)$. This completes the proof.

PROPOSITION 3. *The direct product of two Silva spaces is also a Silva space.*

PROOF. Evident from (4) of Theorem 1.

PROPOSITION 4. *An inductive limit defined by a sequence of Silva spaces is a Silva space, provided it is a Hausdorff space. In particular, a strict inductive limit of a sequence of Silva spaces is a Silva space.*

PROOF. First we prove the proposition in the special case of the direct sum. Let H be the direct sum $\sum_{k=1}^{\infty} E_k$ of Silva spaces $\{E_k\}$. Then H is automatically a Hausdorff space. Clearly H is a bornological space admitting a fundamental sequence of bounded sets. Condition $(*)$ is verified as follows. Let A be a bounded subset of H. Then for a sufficiently large k, $E_1 + \cdots + E_k \supset A$. Since $E_1 + \cdots + E_k$ is a Silva space, condition $(*)$ holds in it, consequently in H. Theorem 1, (4) is now applicable. Next we prove the general case. Let E be any inductive limit Hausdorff space of a sequence $\{E_k\}$ of Silva spaces. Put $H = \sum_{k=1}^{\infty} E_k$, the direct sum of $\{E_k\}$, and put $\iota(x) = \iota_1(x_1) + \cdots + \iota_k(x_k)$ for $x \in H$, $x = x_1 + \cdots + x_k$, where ι_j is the canonical mapping $E_j \to E$. Then ι is a continuous mapping of H onto E. By the assumption that E is a Hausdorff space, the kernel N of ι is a closed subspace of the Silva space H. Hence H/N is a Silva space, algebraically isomorphic with the continuous image E. But, as is easily verified, the canonical mapping $E_k \to H/N$ is continuous, $k = 1, 2, \cdots$. Hence the topology of H/N is not stronger than the inductive limit topology of E. Thus E is topologically isomorphic with the Silva space H/N. This completes the proof of the first part. The second part is evident.

The next proposition is a special case of a Grothendieck's Théorème B ([6], p. 17). But in this case we may give a simple proof.

PROPOSITION 5. *Let E and F be Silva spaces. Then:*

(1) *Any continuous linear mapping of E onto F is a homomorphism;*

(2) *Any linear mapping of E into F is continuous provided that it has a closed graph.*

PROOF. Let u be the mapping in question.

Proof of (1): With no loss of generalities we may assume that u is univalent. Then ${}^tu$ is a weak isomorphism of F' onto a weakly dense subset ${}^tu(F')$ of E'. Lemma 1 together with Remark 1 yields us that ${}^tu$ is an isomorphism in the Mackey topology, and consequently in the strong topology, since F' and E' are (F) spaces. Thus ${}^tu(F')=E'$, and owing to the reflexivity of the spaces, u is an isomorphism. This completes the proof of (1).

Proof of (2): Let the graph $G=\{(x, u(x)); x \in E\}$ of u be closed in the Silva space $F\times F$. Then G is a Silva space and the projection v of G onto E is a continuous linear mapping and hence isomorphism by (1). Thus $u=w\circ v^{-1}$ is continuous where w is the projection of G into F. This completes the proof of (2).

Silva [8] has shown that in any (LN^*) a subset A is closed if it has a closed intersection with each bounded closed subset. In general this statement does not hold in an (LF) space even if A is a vector subspace ([5], p. 93). The next proposition is essentially due to Silva [8], but we will give here a somewhat simpler proof.

PROPOSITION 6. *For any subset A of a Silva space E, following statements* (1)–(3) *are equivalent:*

(1) *A is closed;*

(2) *For any bounded subset $B\subset E$, $B\cap A$ is closed in B;*

(3) *A is sequentially closed.*

PROOF. Ad (1)$\rightarrow$(2): Clear. Ad (2)$\rightarrow$(3): Let $A\ni x_n\rightarrow x$ in E. Then there exists a closed bounded disk B such that $\{x, x_1, x_2, \cdots\}\subset B$. By the strict Mackey condition satisfied by E, the topology of E induces a norm topology on B. Hence in this norm topology $B\cap A\ni x_n\rightarrow x$ on B. Thus $x\in B\cap A$ since $B\cap A$ is closed in B.

Ad (3)$\rightarrow$(1): We may reduce the problem in the following form: if $0\notin A$, there exists a neighbourhood U of 0 such that $U\cap A=\phi$. This is proved as follows. Choose a fundamental sequence $\{B_k\}$ of bounded sets consisting of closed bounded disks. There exists a positive number α_1 such that $(\alpha_1B_1)\cap A=\phi$. If $(\alpha_1B_1+\alpha B_2)\cap A\neq\phi$ for every positive number α, there exists a sequence $\{x_k\}$ of elements $x_k\in\left(\alpha_1B_1+\frac{1}{k}B_2\right)\cap A$, $k=1,2,\cdots$. Since $\{x_k\}$ is bounded in the Montel space E, we may assume that $\{x_k\}$

converges to an element x. By the sequential closedness of A and by the compactness of $\alpha_1 B_1+\frac{1}{k}B_2$, it follows that $x\in\left(\alpha_1 B_1+\frac{1}{k}B_2\right)\cap A$ for every $k=1,2,\cdots$. Consequently $x\in(\alpha_1 B_1)\cap A$, a contradiction. Hence it holds that $(\alpha_1 B_1+\alpha_2 B_2)\cap A=\phi$ for a positive number α_2. Continuing this process, we may obtain a sequence $\{\alpha_k\}$ of positive numbers such that $\left(\sum_{k=1}^{n}\alpha_k B_k\right)\cap A=\phi$, $n=1,2,\cdots$, that is, $U\cap A=\phi$ where $U=\sum_{k=1}^{\infty}\alpha_k B_k$ (algebraic sum). The disk U is a neighbourhood, since it absorbs bounded sets. This completes the proof.

As for the dual Silva space we mention the following facts most of which are easy consequences of Grothendieck [5], [6]. A closed subspace of a dual Silva space and its quotient space by a closed subspace are dual Silva spaces [5]. A projective tensor product and a direct product of two dual Silva spaces are also dual Silva spaces ([6], p. 43, p. 48).

PROPOSITION 7. *Let F be a projective limit of a projective spectre $\{F_k, \pi_{k,k+1}\}$ of dual Silva spaces such that $\pi_{k,k+1}(F_{k+1})=F_k$ for every $k=1,2,\cdots$. Then F is a dual Silva space.*

PROOF. Clearly F is a Montel (F) space and consequently a reflexive space. It is not difficult to see that $\{F'_k, {}^t\pi_{k,k+1}\}$ is an inductive spectre of Silva spaces such that ${}^t\pi_{k,k+1}$ is an isomorphism of F'_k into F'_{k+1} (by a well-known property of (F) spaces [3]). Hence $\{F'_k, {}^t\pi_{k,k+1}\}$ is a strict inductive spectre with the strong dual F' as its limit space; details are omitted. F' is a Silva space by Proposition 4. Thus $F=(F')'$ is a dual Silva space. This completes the proof.

The rest of this paragraph is devoted to the Schwartz (LF) space and its strong dual. The space $(\mathfrak{D})$, the space of the infinitely continuously differentiable complex-valued functions with compact supports defined on R^n [7], is a Schwartz (LF) space. The strong dual $(\mathfrak{D}')$ of the space $(\mathfrak{D})$, the space of the distributions on R^n, is the strong dual of a Schwartz (LF) space [7], [9].

PROPOSITION 8. *Let E be a locally convex space. Then the following statements* (1) *and* (2) *are equivalent*:

(1) *E is a Schwartz (LF) space;*

(2) *E is the strict inductive limit of a sequence $\{E_k\}$ of Schwartz (F) spaces ($=$dual Silva spaces).*

In this case E is a complete, bornological, Montel space. Accordingly, E is reflexive.

PROOF. Ad (1)$\rightarrow$(2): At any rate, E is a strict inductive limit of a sequence $\{E_k\}$ of (F) spaces. Each E_k is a closed subspace of the Schwartz space E [3]. Hence E_k is also a Schwartz space [5]. The converse (2)$\rightarrow$(1) is obvious [5]. The rests are well-known properties of the (LF) space, of the bornological space and of the Schwartz space [2], [3], [5].

PROPOSITION 9. *Let F be a locally convex space. Then the following statements* (1) *and* (2) *are equivalent*:

(1) *F is the strong dual of a Schwartz* (LF) *space*;

(2) *F is the projective limit of a projective spectre $\{F_k, \pi_{k,k+1}\}$ of Silva spaces such that $\pi_{k,k+1}(F_{k+1})=F_k$ for every $k=1,2,\cdots$.*

In this case F is a complete Schwartz space satisfying the strict Mackey condition. Accordingly, F is a Montel space, hence reflexive.

PROOF. Ad (1)→(2): Let F be the strong dual E' of a Schwartz (LF) space E. E is the strict inductive limit of a sequence $\{E_k\}$ of Schwartz (F) spaces. Denote the injection $E_k \to E_{k+1}$ by $\iota_{k+1,k}$. Then $\{E_k', {}^t\iota_{k+1,k}\}$ forms a projective spectre of Silva spaces such that ${}^t\iota_{k+1,k}(E_{k+1}')=E_k'$ for every $k=1,2,\cdots$ [3]. By Proposition B, E' is topologically the projective limit of $\{E_k', {}^t\iota_{k+1,k}\}$. This completes the proof of (1)→(2).

Ad (2)→(1): Let F be the projective limit of a projective spectre $\{F_k, \pi_{k,k+1}\}$ of the stated kind. By Corollary of Proposition A, F is semi-reflexive and the strong dual F' is the strict inductive limit of $\{F_k'\}$, that is, a Schwartz (LF) space. Hence by Proposition B, the strong dual of F' is the projective limit of the projective spectre $\{F_k'', {}^{tt}\pi_{k,k+1}\}=\{F_k, \pi_{k,k+1}\}$ namely the space F. This proves (2)→(1).

It follows easily from (2) that F is complete. And as a closed subspace of a Schwartz space namely the direct product of $\{F_k\}$, F is a Schwartz space. Since it is shown in the proof of (2)→(1) that F is reflexive and that F' is a Schwartz space, F satisfies the strict Mackey condition [5]. This completes the proof.

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