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A Note on Principal Ideals

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In his paper ([1], §9) M. Nagata proved the following interesting properties concerning prime ideals of principal ideals of Noetherian integral domains: (1) Let R be a Noetherian integral domain and $\mathfrak{p}$ a prime ideal of R. Then, if $\mathfrak{p}$ is a prime ideal of aR where a is a non zero element of $\mathfrak{p}$, $\mathfrak{p}$ is also a prime ideal of bR for any non zero element b of $\mathfrak{p}$. (2) Let R be a local domain with maximal ideal $\mathfrak{m}$, a be a non zero element of $\mathfrak{m}$, and b be an element of $aR:\mathfrak{m}$. When R is of dimension 1, it is assumed that a is irreducible and that $aR:\mathfrak{m} \neq R$. Then b is integral over aR.

These theorems played important roles in his proof of the following theorem: The derived normal ring of a Noetherian integral domain is a Krull ring. The purpose of this note is to give a simple proof of these theorems in the more general case when R is a Noetherian ring ([2], §4). Our proof is based on the following fact: In a Noetherian ring, a prime ideal $\mathfrak{p}$ is a prime ideal of an ideal $\mathfrak{a}$ if and only if $\mathfrak{p}=\mathfrak{a}:(p)$ for some $p \notin \mathfrak{a}$.

We shall now begin with

Lemma 1. *Let R be a commutative ring and let a, b, c, d be elements of R. Assume that a is a non zero divisor, then, if $ad=bc$, $aR:bR \subseteq cR:dR$.*

Proof. Let x be any element of $aR:bR$, then $ay=bx$ $(y \in R)$; hence $ayc=bxc=axd$; since a is a non zero divisor, we have $cy=dx$; that is, $x \in cR:dR$.

Remark. If R is an integral domain and a, c non zero elements, then, from $ad=bc$, it follows that $aR:bR=cR:dR$.

Hereafter R will always denote a Noetherian ring.

Proposition 1. *Let $\mathfrak{p}$ be a prime ideal (isolated or embedded) of aR where a is a non zero divisor of R. Assume that c is a non zero divisor of R which belongs to $\mathfrak{p}$, then $\mathfrak{p}$ is also a prime ideal (isolated or embedded) of cR* ([2], Lemma 2, p. 299).

Proof. Since $\mathfrak{p}$ is a prime ideal of aR, $\mathfrak{p}=aR:bR$ for some $b \notin aR$; hence $cb=ad$ $(d \in R)$; consequently, from Lemma 1, $\mathfrak{p}=aR:bR=cR:dR$, and this shows that $\mathfrak{p}$ is a prime ideal of cR. The assertion quoted in parenthesis follows easily from Krull's Primidealkettensatz.

Corollary. *Let a, b be non zero divisors of R such that aR, bR have the same radical. Then aR, bR have the same associated prime ideals.*

Proposition 2. *Let $\mathfrak{p}$ be a prime ideal of R which contains a non zero divisor a of R. Assume that $R_{\mathfrak{p}}$ is not a real discrete valuation ring. Then, for any element b such that $b \in aR:\mathfrak{p}$, b is integral over aR.*

Proof. Let p be any element of $\mathfrak{p}$, then $bp=ar$ $(r \in R)$. Assume that $r \notin \mathfrak{p}$, then $\mathfrak{p} \subseteq aR:bR \subseteq pR:rR \subseteq \mathfrak{p}$; hence $\mathfrak{p}=pR:rR$ and this shows that $\mathfrak{p}$ is an isolated primary component in a normal decomposition of pR. Consequently $R_{\mathfrak{p}}$ is a real discrete valuation ring because $\mathfrak{p}$ contains a non zero divisor, and this contradicts with our assumption. So $b\mathfrak{p} \subseteq a\mathfrak{p}$; hence b is integral over aR ($\mathfrak{p}$ contains a non zero divisor).

Corollary. *Let a be a non zero divisor of R and aR integrally closed. Then $R_{\mathfrak{p}}$ is a real discrete valuation ring for any prime ideal $\mathfrak{p}$ of aR* ([3], Theorem 7, p. 73).

References

[1] Nagata, M., *Basic theorems on general commutative rings*, Memoirs Kyoto Univ., Ser. A, **29** (1955), 59–77.

[2] ———, *On the derived normal rings of Noetherian integral domains*, ibid., 293–303.

[3] Northcott, D.G., *Ideal theory*, Camb. Tract No. **42** (1953).

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