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# On Splitting of Valuations in Extensions of Local Domains

By

Hazimu SATÔ

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**Introduction.** In their paper [1], S. Abhyankar and O. Zariski proved a theorem concerning splitting of valuations in extensions of local domains, and in the subsequent paper [2], Abhyankar generalized the theorem in the following form:

Let $(R, M)$ be a local domain of dimension $>1$, such that either (a) $R$ admits a nucleus, or (b) $R$ is regular and has the same characteristic as its residue field. Let $K$ be the quotient field of $R$ and $K^*$ a finite separable extension of $K$. Then there exist infinitely many real discrete valuations $v$ of $K$ with the following two properties: (1) $v$ has center $M$ in $R$, and (2) any $K^*$-extension $v^*$ of $v$ has degree 1 over $v$, or equivalently, $v$ has exactly $[K^*:K]$ distinct extensions to $K^*$.

The purpose of this paper is to generalize this result by removing any condition imposed upon $R$, and to give a considerably simpler proof than the original ones. We state this generalization as our

Theorem. *Let $(R, M)$ be a (general) local domain of dimension $d>1$ and let $K^*$ be a finite separable algebraic extension of the quotient field $K$ of $R$. Then there exist infinitely many real discrete valuations $v$ of $K$ having the following two properties:* (1) *$v$ has center $M$ in $R$, and* (2) *any $K^*$-extension $v^*$ of $v$ has degree* 1 *over $v$.*

Following Abhyankar closely, first we prove, in § 1, that there exist infinitely many real discrete valuations $v$ of $K$ such that $v$ has center $M$ in $R$ and $v$ has more than one extension to $K^*$ $(K \neq K^*)$, and secondly, with the aid of this result, our theorem will be proved in § 2.

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We shall use the following notations: If $v$ is a valuation of a field $K$, then $R_v$ will denote the valuation ring of $v$ and $M_v$ will denote the valuation ideal of $v$. If $R$ is a local ring with the maximal ideal $M$, then we shall express this by saying, "$(R, M)$ is a local ring".

## § 1. Splitting of valuations.

We shall prove the following:

**Proposition 1.** *Let $(R, M)$ be a local domain of dimension $d>1$ and let $K^*$ be a finite separable algebraic field extension of the quotient field $K$ of $R$ $(K\neq K^*)$. Then there exist infinitely many real discrete valuations $v$ of $K$ having the following two properties*: (1) *$v$ has center $M$ in $R$, and* (2) *$v$ has more than one extension to $K^*$.*

Our proof of Proposition 1 depends essentially on the following lemma which is a generalization of Lemms 3, p. 85, [1].

**Lemma 1.** *Let $(R, M)$ be a local domain of dimension $d>1$ with quotient field $K$ and residue field $k$. Let $\{x_1,\cdots,x_d\}$ be a system of parameters in $R$. Let $y_1=x_1$, $y_i=x_i/x_1$ for $i=2,\cdots,d$; and let $S=R[y_2,\cdots,y_d]$. Then $MS$ is a prime ideal of $S$, and also the radical of $y_1S$; $MS\cap R=M$, and $k=R/M$ can be canonically identified with a subfield of $S/MS$. Furthermore, the residues $\bar{y}_2,\cdots,\bar{y}_d$ modulo $MS$ of $y_2,\cdots,y_d$ are algebraically independent over $k$, and $S/MS$ can be canonically identified with the polynomial ring $k[\bar{y}_2,\cdots,\bar{y}_d]$ in $d-1$ independent variables.*

*Proof.* For a suitable number $n$, $M^n\subseteq(x_1,\cdots,x_d)R$, and from this $M^nS\subseteq(x_1,\cdots,x_d)S=y_1S$. We show that $y_1S\neq S$. Assume the contrary. Then there exists a polynomial $f(y_2,\cdots,y_d)=\sum f_{i_2,\cdots,i_d}\, y_2^{i_2}\cdots y_d^{i_d}$, with the coefficients $f_{i_2,\cdots,i_d}$ in $R$, such that $1=y_1f(y_2,\cdots,y_d)$. Multiplying both sides of this equation by a suitable high power $x_1^t$ of $x_1$, we get

$$x_1^t\in x_1(x_1,\cdots,x_d)^tR\subseteq M(x_1,\cdots,x_d)^tR.$$

This is a contradiction by virtue of the analytical independency of a sytsem of parameters. Hence $MS\cap R=M$, and if it is shown that $\bar{y}_2,\cdots,\bar{y}_d$ are algebraically independent over $k$, then the proof will be completed. Assume the contrary, then there exists a polynomial $f(y_2,\cdots,y_d)$ with the coefficients in $R$ (the same notation as above will be used), but not all in $M$, such that $f(y_2,\cdots,y_d)\in MS$, namely,

$$f(y_2,\cdots,y_d)=g(y_2,\cdots,y_d),$$

where $g(y_2,\cdots,y_d)$ is a polynomial in $y_2,\cdots,y_d$ with coefficients in $M$. Multiplying both sides of the above equation by a suitable high power $x_1^t$ of $x_1$ and setting

$$F_t(x_1,\cdots,x_d)=\sum f_{i_2,\cdots,i_d}\, x_1^{t-i_2-\cdots-i_d}x_2^{i_2}\cdots x_2^{i_2}\cdots x_d^{i_d},$$

we get

$$F_t(x_1,\cdots,x_d)\in M(x_1,\cdots,x_d)^t.$$

Since not all the coefficients of $F_t$ are in $M$, again by virtue of the analytical independency of a system of parameters, we have a contradiction.

**Corollary.** *There exist infinitely many maximal ideals of $S$ which contain $y_1$. Let $m$ be one of them, $R^*=S_m$, and $m^*=mR^*$. Then $(R^*, m^*)$ is a local domain of dimension $d$.*

Since $m$ contains $MS$ and, hence, $m/MS$ is a zero dimensional ideal in the polynomial ring $k[\bar{y}_2,\cdots,\bar{y}_d]$, it follows that $m/MS$ has a basis $\{\bar{z}_2,\cdots,\bar{z}_d\}$ of $(d-1)$ elements ([6], p. 541, Lemma 9). Therefore, as is easily seen, $(y_1, z_2,\cdots,z_d)S$ is $m$-primary. Hence rank $m\leqslant d$. On the other hand $m/MS$ is of rank $d-1$, and hence rank $m\geqslant d$. Therefore $\dim R^*=d$.

We are now in a position to prove Proposition 1.

*Proof of Proposition 1.* Let $\alpha$ be a primitive element of $K^*/K$ which is integral over $R$, and let

$$F(X)=X^n+f_1X^{n-1}+\cdots+f_n \ (f_i\in R)$$

be the minimal monic polynomial of $\alpha$ over $K$. Upon replacing, if necessary, $\alpha$ by $\alpha x (x\in M)$, we may assume that $f_i\in M$ for $i=1,\cdots,n$.

The difference of our proof from that of Theorem 1, p. 87, [1], is as follows: the authors of the article consider a quadratic transformation with respect to a regular system of parameters, while we apply a quadratic transformation with respect to a system of parameters to which a suitable coefficient $f_i$ belongs as a member. To do this we shall divide the argument into two cases. Let $p$ be the characteristic of $K$, then

*Case* 1, $p\neq 0$ and $n\equiv 0(p)$: Since $K^*/K$ is separable and $n\equiv 0(p)$, there exists at least one coefficient $f_i\neq 0$ such that $i\not\equiv 0(p)$. Let $e$ be the smallest value of $i$ such that $i\not\equiv 0(p)$ and $f_i\neq 0$. Put $f_e=x_1$ and take a system of parameters $\{x_1,\cdots,x_d\}$ in $R$ to which $x_1$ belongs. Then, by Corollary of Lemma 1, there exist infinitely many maximal ideals $m$ of $S$[1] such that $m$ contains $y_1S$ and hence also $MS$, where $y_1=x_1$. Let us fix one such $m$, and let $R^*=S_m$. Further, let $\{z_1,\cdots,z_d\}$ be a system of parameters in $R^*$, where $z_1=y_1$. Put $\beta=\alpha-z_2$. Then also $\beta$ is a primitive element of $K^*/K$, and the minimal monic polynomial $G(X)$ of $\beta$ over $K$ is given by

$$G(X)=F(X+z_2)=X^n+g_1X^{n-1}+\cdots+g_n \ (g_i\in R^*),$$

where $g_e=f_e=z_1$. Also $g_n=F(z_2)=z_2^n+f_1z_2^{n-1}+\cdots+f_n(f_i\in M)$.

Suppose now that $z_1^q\in g_nR^*(q\geqslant 1)$. Then, if $p^*$ is a minimal prime ideal of $g_nR^*$, $z_1^q\in p^*$, and hence $z_1\in p^*$. Since the radical of $z_1R^*$ is the prime ideal $MR^*$, $MR^*\subseteq p^*$. On the other hand $p^*$ is of rank 1; hence $MR^*=p^*$, that is, $g_n\in MR^*$. Since $f_i\in M$, it follows that $z_2^n\in MR^*$; hence $z_2\in MR^*$. This is a contradiction, because $\{z_1, z_2,\cdots,z_d\}$ is a system of parameters. Hence $g_e^q\notin g_nR^*$ for any positive integer $q$.

*Case* 2, $p=0$, or $p\neq 0$ and $n\not\equiv 0(p)$: Upon replacing at first $\alpha$ by $n\alpha$, secondarily $\alpha$ by $\alpha+(f_1/n)$, and finally again $\alpha$ by $n\alpha$, we may assume that

$$F(X)=X^n+f_2X^{n-2}+\cdots+f_n,$$

where $f_i\in M$ $(i=2,\cdots,n)$, $f_n=nf(f\in R)$.

Put $f_n=x_1$ and take a system of parameters $\{x_1,\cdots,x_d\}$ in $R$ to which $x_1$ belongs. We proceed as in case 1, and let $\beta=\alpha-z_2$, then the minimal monic polynomial $G(X)$ of $\beta$ is given by

1) Notations are the same as in Lemma 1.

$$G(X)=F(X+z_2)=X^n+g_1X^{n-1}+\cdots+g_n(g_i\in R^*),$$

where $g_1=nz_2$. Also $g_n=F(z_2)=z_2^n+f_2z_2^{n-2}+\cdots+f_n(f_n=x_1=y_1=z_1)$.

Assume now that $g_1^q\in g_nR^*(q\geqslant 1)$. Then, if $p^*$ is a minimal prime ideal of $g_nR^*$, $nz_2\in p^*$. Hence $z_2\in p^*$ or $n\in p^*$. (i) If $z_2\in p^*$, then $f_n=z_1\in p^*$; (ii) if $n\in p^*$, then also $z_1=f_n=nf\in p^*$. So, in both cases, $z_1\in p^*$. From this, as in case 1, we have a contradiction. Hence $g_1^q\notin g_nR^*$ for any positive integer $q$.

The above considerations show that, in any case, we can construct a defining equation $G(X)$ of $K^*/K: G(X)=X^n+g_1X^{n-1}+\cdots+g_n(g_i\in R^*)$ such that $g_e^q\notin g_nR^*(q\geqslant 1)$ for a suitable $e$. Fix an integer $q>(n/e)$, and let $u=g_ng_e^{-q}$. Then, if there exists a real discrete valuation $v$ of $K$ with center $m^*$ in $R^*$ such that $v(u)\geqslant 0$, the valuation $v$ has center $M$ in $R$ and has more than one extension to $K^*$. Because $v(u)\geqslant 0$ implies that

$$v(g_n)\geqslant qv(g_e)>\frac{n}{e}v(g_e),\text{ i.e., } v(g_e)<\frac{e}{n}v(g_n).$$ [2]

Since $v$ has center $m$ in $S$, it follows that distinct choices of the maximal ideal $m(\supset MS)$ of $S$ give us distict valuations $v$ of the required type.

Now, the reasoning with which we concluded that $g_e^q\notin g_nR^*$ shows the following fact: every minimal prime ideal of $g_nR^*$ does not contain $g_e^q$. This implies that there exists a system of parameters $\{t_1,\cdots,t_d\}$ in $R^*$ to which $g_e^q$ and $g_n$ belong ($g_e^q$ non unit). Let $t_1=g_e^q$, $t_2=g_n$, and $S^*=R^*[t_2/t_1,\cdots,t_d/t_1]$, then $(t_2/t_1, m^*)\ S^*$ is a prime ideal of $S^*$. Hence our proof will be completed with the aid of the following lemma:

**Lemma 2.** *Let $R$ be a Noetherian integral domain with quotient field $K$ and let $A$ be an ideal of $R$ different from* (0) *and $R$. Then there exists a real discrete valuation $v$ of $K$ such that $R_v\supseteq R$ and $M_v\supseteq A$* ([1], p. 85, Lemma 1).[3]

## § 2. Main Theorem.

To prove the theorem mentioned in the introduction we shall begin with grouping up the following known results as lemmas.

**Lemma 3.** *Let $(R, M)$ be a local domain with quotient field $K$ and $K^*$ a Galois extension of $K$. Let $S$ be either the valuation ring of a valuation of $K$ or a local domain with quotient field $K$, and let $N$ be the maximal ideal of $S$. Let $\bar{M}$ and $\bar{N}$ be certain maximal ideals of the integral closures $\bar{R}$ and $\bar{S}$ in $K^*$ of $R$ and $S$, respectively. Let $R^*=\bar{R}_{\bar{M}}$, $M^*=\bar{M}R^*$, $S^*=\bar{S}_{\bar{N}}$ and $N^*=\bar{N}S^*$. Let $G$ and $H$ be the splitting groups of $M^*$ and $N^*$ over*

2) See [5], p. 298, Satz 3, and p. 303, Satz 5.

3) This will be proved by Lemma 1 and the Krull-Akizuki's theorem concerning the integral closure of a Noetherian integral domain of rank 1, without making any use of the existence theorem of valuations.

*$M$ and $N$, respectively. Assume that $R \subset S$, $N \cap R = M$, and $R^* \subset S^*$. Then* $H \subset G$ ([2], p. 221, Lemma 1).

**Lemma 4.** *The derived normal ring of a local domain of dimension 2 is also Noetherian* ([4], p. 301, Theorem 3).

**Lemma 5.** *Let $R$ be a normal integral domain with quotient field $K$ such that $R$ contains a unique maximal ideal $M$. Assume that $R$ is not the valuation ring of a valuation of $K$. Then there exists a valuation $w$ of $K$ which has center $M$ in $R$ and for which $R_w/M_w$ is of positive transcendence degree over $R/M$* ([7], p. 75, Theorem).

**Lemma 6.** *Let $(R, M)$ be a local domain of dimension $d$ with quotient field $K$ and let $w$ be a valuation of $K$ with center $M$ in $R$. Let $n$ and $r$ be respectively the $R$-dimension and the rank of $w$. Then:* (1) *$n+r \leq d$.* (2) *If $n+r=d$, then $w$ is discrete* ([3], p. 330, Theorem 1).

**Theorem.** *Let $(R, M)$ be a local domain of dimension $d>1$ and let $K^*$ be a finite separable algebraic field extension of the quotient field $K$ of $R$. Then there exist infinitely many real discrete valuations $v$ of $K$ having the following two properties:* (1) *$v$ has center $M$ in $R$, and* (2) *any $K^*$-extension $v^*$ of $v$ has degree* 1 *over $v$.*

*Proof.* It is sufficient to prove the assertion in the case when $K^*$ is a Galois extension of $K$; so we assume throughout our proof $K^*$ is Galois over $K$. The proof will now be divided into several steps.

*Step* 1: We show that it is sufficient to prove our assertion in the case when $R$ is a normal local domain of dimension 2.

Let $\{x_1, \cdots, x_d\}$ be a system of parameters in $R$ and let $S=R[x_2/x_1, \cdots, x_d/x_1]$. Then, by Lemma 1, $P=(x_2/x_1, M)S$ is a prime ideal of rank 2 of $S$. Consider the derived normal ring $\bar{R}$ of the quotient ring $S_P$ and let us denote by $\bar{M}$ a maximal ideal of rank 2 of $\bar{R}$ which lies over $PS_P$. Then the quotient ring $\bar{R}_{\bar{M}}$ is a normal local domain of dimension 2 by virtue of Lemma 4, and $\bar{M}\bar{R}_{\bar{M}} \cap R = M$. Hence any valuation $v$ of $K$ which has center $\bar{M}\bar{R}_{\bar{M}}$ in $\bar{R}_{\bar{M}}$ has also center $M$ in $R$. $(\bar{R}_{\bar{M}}, \bar{M}\bar{R}_{\bar{M}})$ will do the job of $(R, M)$.

*Step* 2: In step 3 it will be shown that if $(R, M)$ is a normal local domain of dimension 2, then there exists a normal local domain $(S, N)$ of dimension 2, such that $R \subseteq S$, $N \cap R = M$, and that $N$ splits in $K^*(K^* \neq K)$. We assume this to be true and prove the theorem applying induction to $[K^* : K]$.

The theorem is obvious for $[K^* : K]=1$. Let now $[K^* : K]=n>1$, and suppose that our assertion is true whenever $[K^* : K]<n$. Let $\bar{M}$ be a

maximal ideal in the integral closure $\bar{R}$ of $R$ in $K^*$. Let $R^*=\bar{R}_{\bar{M}}$, $M^*=\bar{M}R^*$ ($(R^*, M^*)$ is of dimension 2), and $K_s$ be the splitting field of $M^*$ over $M$. By the assumption stated above, we can assume that $K_s \neq K$. Let $R_s=R^*\cap K_s$ and $M_s=M^*\cap K_s$. Then $(R_s, M_s)$ is a normal local domain of dimension 2. Therefore, by our induction hypothesis, there exist infinitely many real discrete valuations $v_s$ of $K_s$ with center $M_s$ in $R_s$ such that, if $v^*$ is an extension of $v_s$ to $K^*$, then $v^*$ is of degree 1 over $v_s$. Let $v_s$, $v^*$ be such a pair, and let $v$ be the $K$-restriction of $v_s$. Then $v$ has center $M$ in $R$, and, as is easily seen, the splitting field of $v^*$ over $v$ is $K^*$ (Lemma 3), and hence $v$ has exactly $[K^*:K]$ distinct extentions to $K^*$. The infinitely many choices of $v_s$ give us infinitely many valuations $v$ of the required type.

*Step* 3: In this final step, we shall prove the assertion mentioned at the beginning of the step 2, and if it is done, then our proof of the theorem will be completed. In this step it will be assumed that $K^* \neq K$.

Let $(R, M)$ be a normal local domain of dimension 2. First we show that there exists a valuation $w$ of $K$ such that $w$ has center $M$ in $R$, the $R$-dimension of $w$ is zero, the rank of $w$ is two, and $w$ splits in $K^*$. Let us remember the proof of Proposition 1. Since now $R$ is of dimension 2, $(t_2/t_1, m^*)S^*$ ($t_1=g_e^q$, $t_2=g_n$) (the same notations as in the proof of Proposition 1) is a maximal ideal of $S^*$. Put $P^*=(t_2/t_1, m^*)S^*$. Then the quotient ring $S^*_{P^*}$ is a local domain of dimension 2. Let $w_1$ be a real discrete valuation of $K$ having center $P^*S^*_{P^*}$ in $S^*_{P^*}$ and $S^*_{P^*}$-dimension one, then $w_1$ has center $M$ in $R$ and splits in $K^*$. Hence, let $w$ be a valuation of $K$ which is compounded with $w_1$, then $w$ is of the required type.

Next we construct a sequence of normal local domains of dimension 2, which begins with $R$ and proceeds along $w$, as follows. Let $x_1, \cdots, x_t$ be an ideal basis of $M$, $w(x_1)=\min\{w(x_1), \cdots, w(x_t)\}$, and $S=R[x_2/x_1, \cdots, x_t/x_1]$. Then $S\subseteq R_w$. Consider the derived normal ring $\bar{R}$ of the quotient ring $S_N$, where $N=S\cap M_w$, and put $\bar{M}=\bar{R}\cap M_w$. Then it is easily seen, by Lemmas 4, 5 and 6, that the quotient ring $\bar{R}_{\bar{M}}$ is a normal local domain of dimension 2. We put $R_1=\bar{R}_{\bar{M}}$ and $M_1=\bar{M}\bar{R}_{\bar{M}}$. If this procedure is repeated, then a strictly ascending sequence of normal local domains $(R_i, M_i)$ of dimension 2 with the common quotient field $K$ will be obtained: $R=R_0\subset R_1\subset\cdots$, $R_i\cap M_w=M_i$ ($i=0, 1, \cdots$).

Let $R_u=\bigcup R_i$ and $M_u=\bigcup M_i$, then $R_u$ is a normal integral domain and contains a unique maximal ideal $M_u$. We show that $R_u$ is the valuation ring of a valuation of $K$. Assume the contrary, then there exists, by Lemma 5, a valuation $w'$ of $K$ which has center $M_u$ in $R_u$, and for which $R_{w'}/M_{w'}$ is of positive transcendence degree over $R_u/M_u$. This valuation $w'$, therefore, must be a real discrete valuation by Lemma 6, and the sequence $R_0\subset R_1\subset\cdots$ may also be regarded as proceeding along $w'$. Hence, by the same method as the one which Zariski gave in §9 of [7], we will reach a contradiction. Now that $R_u$ is the valuation ring of a valuation of $K$, it must be that $R_u=R_w$ and $M_u=M_w$, because $R_u\subseteq R_w$ and $M_u\subseteq M_w$.

Let $\bar{R}_i$ be the integral closure of $R_i$ in $K^*$, and let $w^*$ be a $K^*$-extension of $w$. Let $\bar{M}_i=\bar{R}_i\cap R_{w^*}$, $R_i^*=(\bar{R}_i)_{\bar{M}_i}$, and $M_i^*=\bar{M}_iR_i^*$. By Lemma 3, then, it is easily seen that, for some integer $n$, the splitting fields of $M_i^*$ over $M_i$ are all equal for $i\geqq n$, and that if we denote this common splitting field by $K_s$, then $K_s\subseteq K_s^*$, where $K_s^*$ is the splitting field of $w^*$ over $w$. Suppose, if possible, that $K_s\neq K_s^*$. Then $w$ has a $K^*$-extension $\bar{w}$ different from $w^*$, such that $\bar{w}$ has center $M_i^*$ in $R_i^*$ for all $i$. Therefore, $\bigcup R_i^*$ is not a valuation ring, and hence there exists a valuation $t^*$ of $K^*$ such that $t^*$ has center $M_i^*$ in $R_i^*$ and the $R_i^*$-dimension of $t^*$ is one for all $i$ (hence real discrete). Then, the $K$-restriction $t$ of $t^*$ has center $M_u$ in $R_u$, and the $R_u$-dimension of $t$ is 1. This is a contradiction, because $R_u\subseteq R_t$, $M_u\subseteq M_t$ and, hence, $R_u=R_t$. Therefore $K_s=K_s^*(\neq K)$.

## References

[1] Abhyankar, S. and Zariski, O., *Splitting of valuations in extensions of local domains*, Proc. Nat. Acad. Sci., **41** (1955), 84–90.

[2] Abhyankar, S., *Splitting of valuations in extensions of local domains*, ibid., 220–223.

[3] ———, *On the valuations centered in a local domain*, Amer. J. Math., **68** (1956), 321–348.

[4] Nagata, M., *On the derived normal rings of Noetherian integral domains*, Memoirs Kyoto Univ., **29** (1955), 293–303.

[5] Ostrowski, A., *Untersuchungen zur arithmetischen Theorie der Körper*, Math. Zeit., **39** (1934), 269–404.

[6] Zariski, O., *Foundations of a general theory of birational correspondences*, Trans. Amer. Math. Soc., **53** (1943), 490–542.

[7] ———, *Applicazioni geometriche della teoria delle valuazioni*, Rend. di Mat. (Roma), **13** (1954), 51–88.

Department of Mathematics,
Hiroshima University.