JOURNAL OF SCIENCE OF THE HIROSHIMA UNIVERSITY, SER. A, VOL. 21, NO. 2, JULY, 1957

Existence Theorems of Valuations Centered in a Local Domain with Preassigned Dimension and Rank

By

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(Received April 25, 1957)

Introduction. In his recent work, S. Abhyankar developed systematically certain aspects of valuations centered in a local domain [2]. Among his results, we are particularly interested in the following theorem [2, theorem 1 p. 330].

Let $(\mathfrak{o}, \mathfrak{m})$ be a local domain of rank d with quotient field Σ and let v be a valuation of Σ with center $\mathfrak{m}$ in $\mathfrak{o}$. Let ρ, $\bar{r}$ and r be respectively the $\mathfrak{o}$-dimension, the rational rank and the rank of v. Then: (1) $\rho+\bar{r}\leqq d$, (2) if $\rho+\bar{r}=d$, then v is an integral direct sum and R_v/M_v is finitely generated over $\mathfrak{o}/\mathfrak{m}$, (3) if $\rho+r=d$, then v is discrete and R_v/M_v is finitely generated over $\mathfrak{o}/\mathfrak{m}$, (4) if $\rho=d-1$, then v is real discrete and R_v/M_v is finitely generated over $\mathfrak{o}/\mathfrak{m}$.[1)]

However, it seems not to be known, for any integers r and ρ such that $r+\rho\leqq$ rank of $\mathfrak{o}$ $(1\leqq r,\ \rho\geqq 0)$, conversely, whether there exists a valuation v of Σ such that which has ceter $\mathfrak{m}$ in $\mathfrak{o}$ and whose rank and $\mathfrak{o}$-dimension are equal to preassigned integers r and ρ respectively.

In section 1, we shall give an affirmative answer to this question dealing with the possible individual cases. In section 2, we shall prove a theorem concerning the existence of a sequence of valuations with preassigned centers.

Our existence theorems cover the following fundamental theorem, due to O. Zariski [7, Theorem 5, p. 501].

Theorem. Given an arbitrary descending chain $W_0\supseteqq W_1\supseteqq\cdots\supseteqq W_{\sigma-1}$ of irreducible subvarieties of V^r and given any set of integers $\rho_0, \rho_1, \cdots, \rho_{\sigma-1}$ such that $r-1\geqq\rho_0>\rho_1>\cdots>\rho_{\sigma-1}$, $\rho_i\geqq$ dimension of W_i, there exists a *sequence of valuations* $v_0, v_1, \cdots, v_{\sigma-1}$ such *that:*

(1) v_i is of dimension ρ_i, of rank $i+1$ and its center is W_i;

(2) v_i is compounded with v_{i-1}.

The writer expresses here his hearty thanks to Mr. M. Yoshida and Mr. H. Sato for their valuable suggestions during the preparation of this paper.

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1) For the definitions and notations, see §1 and also his paper [2].

§1. Existence of valuations with the preassigned dimension and rank.

We start with the following definition.

DEFINITION. Let $\mathfrak{o}$ be a Noetherian domain with quotient field Σ, and let $\mathfrak{p}$ be a prime ideal in $\mathfrak{o}$ and v a valuation of Σ. Then v is said to have center $\mathfrak{p}$ in $\mathfrak{o}$ provided that $R_v \supseteq \mathfrak{o}$ and $M_v \cap \mathfrak{o} = \mathfrak{p}$ where R_v and M_v denote the valuation ring and the valuation ideal of v respectively.

By the $\mathfrak{o}$-dimension of v, we mean the transcendence degree of R_v/M_v over the quotient field of $\mathfrak{o}/\mathfrak{p}$.

We need the following two lemmas. The first one is known as Krull-Akizuki's theorem [4, Theorem 3, p. 29]. The second one will play an essential rôle in our theory.

LEMMA 1. *Let $(\mathfrak{o}, \mathfrak{m})$ be a local domain of rank* 1[2] *and let Σ be its quotient field. Denote by $\bar{\mathfrak{o}}$ the integral closure of $\mathfrak{o}$ in Σ. Then $\bar{\mathfrak{o}}$ is Noetherian and there exist in $\bar{\mathfrak{o}}$ a finite number of proper prime ideals $\bar{\mathfrak{m}}_i$ $(i=1, \cdots, s)$. $\bar{\mathfrak{o}}_{\bar{\mathfrak{m}}_i}$ is a discrete rank* 1 *valuation ring and its residue field is a finite algebraic extension of $\mathfrak{o}/\mathfrak{m}$. Furthermore $\bar{\mathfrak{o}} = \bigcap_{i=1}^{s} \bar{\mathfrak{o}}_{\bar{\mathfrak{m}}_i}$.*

The special case of the next lemma, where $\mathfrak{o}$ is regular and $\{\omega_1, \cdots, \omega_d\}$ is a regular system of parameters, was given in [1].

LEMMA 2. *Let $(\mathfrak{o}, \mathfrak{m})$ be a local domain of rank $d>1$ and let $\{\omega_1, \cdots, \omega_d\}$ be a system of parameters in $\mathfrak{o}$. Put $\omega=\omega_1$, $y_i=\frac{\omega_i}{\omega}$ $(i=2, \cdots, d)$ and form a ring $\mathfrak{o}'=\mathfrak{o}[y_2, \cdots, y_d]$. Then $\mathfrak{m}'=\mathfrak{m}\mathfrak{o}'$ is a proper prime ideal of rank* 1 *and is the radical of the principal ideal $(\omega_1, \cdots, \omega_d)\mathfrak{o}'=\omega\mathfrak{o}'$. $k=\mathfrak{o}/\mathfrak{m}$ can be canonically identified with a subfield of $\mathfrak{o}'/\mathfrak{m}'$. Furthermore, the residues $\bar{y}_2, \cdots, \bar{y}_d$ modulo $\mathfrak{m}'$ of $y_2, \cdots, y_d$ are algebraically independent over k and $\mathfrak{o}'/\mathfrak{m}'$ can be canonically identified with a polynomial ring $k[\bar{y}_2, \cdots, \bar{y}_d]$ in $d-1$ independent variables.*

For the proof, see [7, Lemma 1, p. 70].

By these lemmas, we obtain the following:

PROPOSITION 1. *Let $(\mathfrak{o}, \mathfrak{m})$ be a local domain of rank d with quotient field Σ. Then, there exists a valuation v of Σ which satisfies the following conditions.*

i) *v has center $\mathfrak{m}$ in $\mathfrak{o}$.*
ii) *rank of $v=1$.*
iii) *$\mathfrak{o}$-dimension of $v=d-1$.*

2) In this note, we shall denote by $(\mathfrak{o}, \mathfrak{m})$ a local ring with maximal ideal $\mathfrak{m}$, and call the rank of $\mathfrak{m}$ the rank of $(\mathfrak{o}, \mathfrak{m})$ (usually called the dimension).

iv) *discrete.*
v) R_v/M_v *is finitely generated over* $\mathfrak{o}/\mathfrak{m}$.

PROOF. The case $d=1$ is settled in Lemma 1. So we proceed to the case when $d>1$. In this case, with the same notations as in Lemma 2, $\mathfrak{o}'_{\mathfrak{m}'}$ is a local domain of rank 1 with quotient field Σ and $\mathfrak{o}'_{\mathfrak{m}'}/\mathfrak{m}'\mathfrak{o}'_{\mathfrak{m}'}$ is a purely transcendental extension of degree $d-1$ over $\mathfrak{o}/\mathfrak{m}$. To complete our proof, it will suffice to apply Lemma 1 to $\mathfrak{o}'_{\mathfrak{m}'}$.

PROPOSITION 2. *Let* $(\mathfrak{o}, \mathfrak{m})$ *be a local domain of rank* d *with quotient field* Σ. *Then, for any integers* r *and* ρ *such that* $r+\rho=d$ $(r\geqq 1$ *and* $\rho\geqq 0)$, *there exists a valuation* v *of* Σ *satisfying the following conditions.*
i) v *has center* $\mathfrak{m}$ *in* $\mathfrak{o}$.
ii) *rank of* $v=r$.
iii) $\mathfrak{o}$*-dimension of* $v=\rho$.
iv) *discrete.*
v) R_v/M_v *is finitely generated over* $\mathfrak{o}/\mathfrak{m}$.

PROOF. The case $r=1$ has been settled in Prop. 1. So we proceed by induction on r.

Take a prime ideal $\mathfrak{p}$ in $\mathfrak{o}$ such that rank $\mathfrak{p}=1$ and rank $\mathfrak{o}/\mathfrak{p}=d-1$. Then, by Lemma 1, there exists a discrete rank 1 valuation v' of Σ such that v' has center $\mathfrak{p}\mathfrak{o}_{\mathfrak{p}}$ in $\mathfrak{o}_{\mathfrak{p}}$ and $R_{v'}/M_{v'}$ is a finite algebraic extension of the quotient field Σ_1 of $\mathfrak{o}/\mathfrak{p}$.

On the other hand, since $\mathfrak{o}/\mathfrak{p}$ is of rank $d-1$, by our inductive assumption, we can find a discrete rank $r-1$ valuation v_1 of Σ_1 which has center $\mathfrak{m}/\mathfrak{p}$ in $\mathfrak{o}/\mathfrak{p}$ and R_{v_1}/M_{v_1} is a finitely generated extension of transcendence degree ρ of $\mathfrak{o}/\mathfrak{m}$. Since $R_{v'}/M_{v'}$ is a finite algebraic extension of Σ_1, this valuation v_1 can be extended to a valuation v'_1 of $R_{v'}/M_{v'}$ in such a way that its discreteness and rank remain unchanged. Compounding v' and v'_1, we get a composite valuation v of Σ, which will be easily seen to satisfy all our conditions.

PROPOSITION 3. *Let* $(\mathfrak{o}, \mathfrak{m})$ *be a local domain of rank* d *with quotient field* Σ. *Then, there exists a valuation* v *of* Σ *which satisfies the following conditions.*
i) v *has center* $\mathfrak{m}$ *in* $\mathfrak{o}$.
ii) *rank of* $v=1$.
iii) *rational rank of* $v=d$.
iv) R_v/M_v *is a finite algebraic extension of* $\mathfrak{o}/\mathfrak{m}$.

PROOF. (I) We first consider the case when $\mathfrak{o}$ is an unramified complete regular local ring, i.e. (1) $\mathfrak{o}=k\{x_1,\cdots,x_d\}$, or (2) $\mathfrak{o}=R\{x_2,\cdots,x_d\}$, where k is a field and R is a complete discrete rank 1 valuation ring whose maximal ideal is generated by a prime integer p.

To define a valuation of Σ, it is enough to define the value of any element f in $\mathfrak{o}$. Take a system of rationally independent positive real numbers $\tau_1=1, \tau_2, \cdots, \tau_d$ and define the value of f as follows:

Case (1) $v(f(x_1, \cdots, x_d))=$ the exponent of the first non zero term of $f(t^{\tau_1}, \cdots, t^{\tau_d})$ as a power series in a variable t.

Case (2) $v(f(x_2, \cdots, x_d))=\min\{\lambda+\text{value of } a_\lambda\}$, where a_λ is the coefficient of $f(t^{\tau_2}, \cdots, t^{\tau_d})$ as a power series in t:

$$f(t^{\tau_2}, \cdots, t^{\tau_d})=\sum_\lambda a_\lambda t^\lambda.$$

Then, we see that thus defined valuation has k or R/pR as its residue field and satisfies all our requirements.

(II) Next, we consider the case when $\mathfrak{o}$ is a complete local domain. It is well known, in virtue of the structure theorem of complete local rings, due to I.S. Cohen, that $\mathfrak{o}$ is a finite module over an unramified complete regular local ring $\mathfrak{o}_0$ with the same rank and the same residue field [3, Theorem 16, p. 90]. We denote by Σ_0 a quotient field of $\mathfrak{o}_0$. Since Σ is finite algebraic over Σ_0, any valuation of Σ_0, constructed in (I), can be extended to a valuation v of Σ, preserving its rank and rational rank. Then we see easily that this valuation v satisfies all our conditions.

(III) General case. Let $\hat{\mathfrak{o}}$ be a completion of $\mathfrak{o}$ and fix a minimal prime ideal $\mathfrak{p}$ in $\hat{\mathfrak{o}}$ such that rank $\hat{\mathfrak{o}}=$ rank $\hat{\mathfrak{o}}/\mathfrak{p}$. Then, $\mathfrak{o}$ may be considered as a subring of $\hat{\mathfrak{o}}/\mathfrak{p}$, because of the fact $\mathfrak{p}\cap\mathfrak{o}=(0)$. Since $\hat{\mathfrak{o}}/\mathfrak{p}$ is a complete local damain, by (II), there exists a valuation v_1 of a quotient field Σ_1 of $\hat{\mathfrak{o}}/\mathfrak{p}$ such that v_1 has center $\mathfrak{m}\hat{\mathfrak{o}}/\mathfrak{p}$ in $\hat{\mathfrak{o}}/\mathfrak{p}$, rank $v_1=1$, rational rank $v_1=\bar{r}$ and R_{v_1}/M_{v_1} is finite algebraic over $(\hat{\mathfrak{o}}/\mathfrak{p})/(\mathfrak{m}\hat{\mathfrak{o}}/\mathfrak{p})$. Denote by v the contraction of v_1 to Σ. Then, v satisfies all our conditions. In fact, since $1\leqq$ rank $v\leqq$ rank $v_1=1$ and $\mathfrak{o}/\mathfrak{m}\subseteq R_v/M_v\subseteq R_{v_1}/M_{v_1}$, it is enough to show that the value group of v is the same as that of v_1. Let $\hat{x}$ be any element of $\hat{\mathfrak{o}}$, $\hat{x}=\lim x_n$ $(x_n\in\mathfrak{o})$ in an $\mathfrak{m}\hat{\mathfrak{o}}$-adic topology. We may assume $\hat{x}-x_n\in\mathfrak{m}^n\hat{\mathfrak{o}}$. Since v_1 is of rank 1, for a sufficient large integer σ, $v_1(x^*-x_\sigma)>v_1(\mathfrak{m}^\sigma\hat{\mathfrak{o}}+\mathfrak{p}/\mathfrak{p})=\min\{v_1(y^*)$; y^* runs over all elements of $\mathfrak{m}^\sigma\hat{\mathfrak{o}}+\mathfrak{p}/\mathfrak{p}\}$, where x^* means the residue of $\hat{x}$ modulo $\mathfrak{p}$. Hence $v_1(x^*)=v(x_\sigma)$, which complete the proof.

THEOREM 1. *Let* $(\mathfrak{o}, \mathfrak{m})$ *be a local domain of rank* d *with quotient field* Σ. *For any integers* r, $\bar{r}$ *and* ρ *such that* $\rho+\bar{r}=d$, $\bar{r}\geqq r\geqq 1$ *and* $\rho\geqq 0$, *there exists a valuation* v *of* Σ *which has following properties.*

i) v *has center* $\mathfrak{m}$ *in* $\mathfrak{o}$.

ii) *rank of* $v=r$.

iii) *rational rank of* $v=\bar{r}$.

iv) $\mathfrak{o}$-*dimension of* $v=\rho$.

v) R_v/M_v *is finitely generated over* $\mathfrak{o}/\mathfrak{m}$.

PROOF. The case $r=1$, $\rho=0$ has just been treated in Proposition 3. Next, we treat the case $r=1$, $\rho>0$. Let $\mathfrak{q}$ be the $\mathfrak{m}$-primary ideal generated by a system of parameters $\omega_1, \cdots, \omega_d$ and form a ring $\mathfrak{o}'=\mathfrak{o}\left[\frac{\omega_2}{\omega_1}, \cdots, \frac{\omega_d}{\omega_1}\right]$. Then, by Lemma 2, the ideal $\mathfrak{m}_1=\left(\mathfrak{m}\mathfrak{o}', \frac{\omega_2}{\omega_1}, \cdots, \frac{\omega_{d-\rho}}{\omega_1}\right)$ in $\mathfrak{o}'$ is a prime ideal and $\mathfrak{o}'/\mathfrak{m}_1$ can be canonically identified with a polynomial ring $k[\bar{y}_{d-\rho+1}, \cdots, \bar{y}_d]$ in ρ independent variables over k (with the same notations as in Lemma 2). $\mathfrak{o}'_{\mathfrak{m}_1}$ is a local domain of rank $d-\rho$. Therefore, by Proposition 3, we can find a rank 1 rational rank $\bar{r}$ valuation v of Σ such that R_v/M_v is a finite algebraic extension of $\mathfrak{o}'_{\mathfrak{m}_1}/\mathfrak{m}_1\mathfrak{o}'_{\mathfrak{m}_1}=k(\bar{y}_{d-\rho+1}, \cdots, \bar{y}_d)$. This valuation v satisfies our conditions.

We have proved the theorem in the case $r=1$. So we proceed by induction on r.

Let $\mathfrak{p}$ be a prime ideal in $\mathfrak{o}$ such that rank $\mathfrak{p}=1$ and rank $\mathfrak{o}/\mathfrak{p}=d-1$. Since $\mathfrak{o}_{\mathfrak{p}}$ is rank 1, there exists a discrete rank 1 valuation v' of Σ such that v' has center $\mathfrak{p}\mathfrak{o}_{\mathfrak{p}}$ in $\mathfrak{o}_{\mathfrak{p}}$ and $R_{v'}/M_{v'}$ is finite algebraic over the quotient field Σ_1 of $\mathfrak{o}_1=\mathfrak{o}/\mathfrak{p}$. By our inductive assumption, there is a rank $r-1$ rational rank $\bar{r}-1$ valuation v_1 of Σ_1 such that v_1 has center $\mathfrak{m}_1=\mathfrak{m}/\mathfrak{p}$ in $\mathfrak{o}_1$, $\mathfrak{o}_1$-dim of v_1 is ρ and R_{v_1}/M_{v_1} is finitely generated over $\mathfrak{o}_1/\mathfrak{m}_1$. This v_1 may be extended to a valuation v_1' of $R_{v'}/M_{v'}$ preserving its rank and rational rank. Compounding v' and v_1' we obtain a valuation v of Σ, which has all our required properties.

COROLLARY. *Let $(\mathfrak{o}, \mathfrak{m})$ be a local domain of rank d and let Σ be its quotient field. For any integers r and ρ such that $r+\rho \leqq d$ $(1 \leqq r,\ 0 \leqq \rho)$, there exists a valuation v of Σ which has rank r, $\mathfrak{o}$-dimension ρ and center $\mathfrak{m}$ in $\mathfrak{o}$.*

§ 2. Sequence of valuations with preassigned centers.

LEMMA 3. *Let Σ be a field and v a valuation of Σ and let $\Sigma(x)$ be a simple transcendental extension of Σ. Then v can be extended to a valuation of $\Sigma(x)$ in such a way that whose residue field remains unchanged.*

PROOF. Consider an ordered abelian group Γ' which contains the value group Γ of v as a subgroup and contains rationally independent elements with respect to Γ. Take an element $\tau>0$ of Γ' which is rationally independent with respect to Γ and define

$$v'(a_0x^n+\cdots+a_n)=\min_{i=0,\cdots,n}\{v(a_i)+(n-i)\tau\},$$

where $a_i \in \Sigma$ and

$$v'\left(\frac{f}{g}\right)=v'(f)-v'(g), \quad \text{where} \quad f, g \in \Sigma[x].$$

Then, we see easily that v' is a valuation of $\Sigma(x)$.

Set $f(x)=a_0x^n+\cdots+a_n$ and $g(x)=b_0x^m+\cdots+b_m$ with $a_i, b_j \in \Sigma$. If $v'\left(\frac{f}{g}\right)=0$, obviously we have $v'(f)=v'(a_{i_0}x^{n-i_0})=v'(b_{i_0}x^{n-i_0})=v'(g)$ for some i_0, hence $v(a_{i_0})=v(b_{i_0})$. From this follows $v'\left(\frac{f}{g}-\frac{a_{i_0}}{b_{i_0}}\right)>0$, which shows v and v' have the same residue fields.

LEMMA 4. *(Under the same assumptions and notations as in Lemma 3.) v can be extended to a valuation v' of $\Sigma(x)$ unaltering the value group such that $R_{v'}/M_{v'}$ is a simple transcendental extension of R_v/M_v.*

PROOF. In this case, for $f(x)=a_0x^n+\cdots+a_n$ and $g(x)\in\Sigma[x]$, we define $v'(f(x))=\min_{i=0,\cdots,n} v(a_i)$ and $v'\left(\frac{f(x)}{g(x)}\right)=v'(f(x))-v'(g(x))$. To see that v' is a valuation of $\Sigma(x)$, it is enough to show that $v'(f(x)\cdot g(x))=v'(f(x))+v'(g(x))$, for any $f(x)$ and $g(x)$ in $\Sigma[x]$, because other axioms of valuations are trivially satisfied by v'.

Set $g(x)=b_0x^m+\cdots+b_m$, $f(x)\cdot g(x)=c_0x^{n+m}+\cdots+c_{n+m}$ with $b_i, c_j\in\Sigma$ and set $\alpha=\min v(a_i)$ and $\beta=\min v(b_j)$. Let i_0 (resp. j_0) be the smallest i (resp. j) such that $v(a_i)=\alpha$ (resp. $v(b_j)=\beta$). Then we have

$$v'(f\cdot g)=\min v(c_i)=v(c_{i_0+j_0})=v(a_{i_0}b_{j_0})=\alpha+\beta=v'(f)+v'(g).$$

Now, we denote by $\bar{x}$ the residue of x modulo $M_{v'}$, then $\bar{x}$ is transcendental over R_v/M_v. In fact, if $\bar{x}$ is algebraic, there exists an equation of the following form:

$$\bar{x}^s+\bar{a}_1\bar{x}^{s-1}+\cdots+\bar{a}_s=0,$$

where $\bar{a}_i$ is the residue of $a_i\in R_v$ modulo M_v. Hence, $x^s+a_1x^{s-1}+\cdots+a_s\in M_{v'}$, i.e. $v'(x^s+a_1x^{s-1}+\cdots+a_s)>0$, which contradicts our definition of v'.

The only thing that remains to be shown is that $R_{v'}/M_{v'}=R_v/M_v(\bar{x})$. If $v'\left(\frac{f}{g}\right)=0$, we have $v'(f)=v'(g)$, i.e. $\alpha=v(a_{i_0})=v(b_{j_0})=\beta$. Let $a_i'=\frac{a_i}{a_{i_0}}(i=0,\cdots,n)$, $b_j'=\frac{b_j}{b_{j_0}}(j=0,\cdots,m)$, $f'(x)=a_0'x^n+\cdots+a_n'$ and $g'(x)=b_0'x^m+\cdots+b_m'$. Then $\bar{g}'\overline{\left(\frac{f}{g}\right)}=\bar{g}'\overline{\left(\frac{f'}{g'}\right)}=\bar{f}'$, hence $\overline{\left(\frac{f}{g}\right)}=\frac{\bar{f}'}{\bar{g}'}\in R_v/M_v(\bar{x})$, q.e.d.

The following theorem is a generalization of Zariski's theorem which we mentioned in the introduction.

THEOREM 2. *Let $(\mathfrak{o}, \mathfrak{m})$ be a local domain of rank d with quotient field Σ and $\mathfrak{p}$ be a prime ideal in $\mathfrak{o}$ of rank d_1. Let ρ and ρ_1 be integers such that $d>\rho$, $d_1>\rho_1$ and $\rho_1+\text{rank } \mathfrak{o}/\mathfrak{p}>\rho$. Then, for any valuation v_1 of Σ which has center $\mathfrak{p}$ in $\mathfrak{o}$ and $\mathfrak{o}$-dimension ρ_1, there exists a valuation v of Σ, compounded with v_1, which has center $\mathfrak{m}$ in $\mathfrak{o}$ and is of rank r_1+1 ($r_1=$ rank of v_1) and of $\mathfrak{o}$-dimension ρ.*

PROOF. Set $\mathfrak{o}_0=\mathfrak{o}/\mathfrak{p}$ and denote by Σ_0 and L the quotient field of $\mathfrak{o}_0$ and the residue field of v_1 respectively. Then, by our assumption, we can find elements $x_1,\cdots,x_{\rho_1}$ in L which are algebraically independent over Σ_0 such that L is algebraic over $\Sigma_0(x_1,\cdots,x_{\rho_1})$.

In the case $\rho \geqq \rho_1$, by Theorem 1, there exists a valuation v' of Σ_0 which has center $\mathfrak{m}_0 = \mathfrak{m}/\mathfrak{p}$ in $\mathfrak{o}_0$, rank 1 and $\mathfrak{o}_0$-dimension $\rho - \rho_1$. We apply Lemma 4 to the simple transcendental extension $\Sigma_0(x_1, \cdots, x_i)/\Sigma_0(x_1, \cdots, x_{i-1})$ for $i = 1, \cdots, \rho_1$ succesively. Then v' can be extended to a valuation v'' of $\Sigma_0(x_1, \cdots, x_{\rho_1})$ whose residue field has a transcendence degree ρ_1 over $R_{v'}/M_{v'}$. This v'' can be extended to a valuation of L. We denote thus obtained valuation by $\bar{v}$. Then, our construction, $\bar{v}$ has the following properties:

(*) i) rank $\bar{v} = 1$, ii) $\mathfrak{o}$-dimension of $\bar{v} = \rho$ and iii) $\bar{v}$ has center $\mathfrak{m}_0$ in $\mathfrak{o}_0$.

If $\rho < \rho_1$ we start from the valuation v' of Σ_0 whose $\mathfrak{o}_0$-dimension is 0 in place of $\rho - \rho_1$ in the preceding case. Then, similarly, by Lemma 3 and 4, we also obtain a valuation $\bar{v}$ of L which satisfies the condition (*).

In either case, we can find a valuation $\bar{v}$ of L which satisfies the condition (*). Compounding v_1 with $\bar{v}$, we get a valuation v of Σ and this v satisfies all conditions of our theorem.

COROLLARY. *Let* $\mathfrak{m} = \mathfrak{p}_0 \supseteqq \mathfrak{p}_1 \supseteqq \cdots \supseteqq \mathfrak{p}_s$ *be an arbitrary descending chain of prime ideals in a local domain* $(\mathfrak{o}, \mathfrak{m})$ *of rank* $d(=d_0)$ *with quotient field* Σ, *and let* $\rho_0, \rho_1, \cdots, \rho_s$ *be any set of integers such that* $d_i = \text{rank } \mathfrak{p}_i > \rho_i$ $(i = 0, \cdots, s)$ *and* $\rho_i + \text{rank } \mathfrak{p}_{i-1}/\mathfrak{p}_i > \rho_{i-1}$ $(i = 1, \cdots, s)$. *Then, there exists a sequence of valuations* $v_0, \cdots, v_s$ *of* Σ *such that*

(1) v_i *has center* $\mathfrak{p}_i$ *in* $\mathfrak{o}$.

(2) *rank* $v_i = s + 1 - i$.

(3) $\mathfrak{o}$-*dimension of* $v_i = \rho_i$.

(4) v_{i-1} *is compounded with* v_i.

PROOF. Starting with a rank 1 valuation v_s of Σ whose $\mathfrak{o}$-dimension$=\rho_s$, by Theorem 2, we can construct a sequence of valuations $v_{s-1}, v_{s-2}, \cdots, v_0$ which satisfies our conditions.

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