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On the Role of Hamiltonians in the Relativistic Dynamics referred to the New Fundamental Group of Transformations

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§ 1. Introduction.

In the usual form of non-relativistic dynamics, the description of the dynamical system is done by the so-called instant form, and the non-trivial change of the system is caused by the *one* Hamiltonian. This corresponds to the fact that, in the non-relativistic dynamics, the transformation connecting two equivalent observers is Galilean transformations which *leave the instant of time $t=0$ invariant, and only the translation of time changes the instant.* On the other hand, in the relativistic dynamics, two equivalent coordinate systems are related by the special Lorentz transformations by which the instant of time $t=0$ is not remained invariant. Therefore, it is necessary to introduce extra Hamiltonians other than that of non-relativistic dynamics. Among the fundamental quantities P_μ, $M_{\mu\nu}$ which satisfy the following P.b. (Poisson bracket) relations:

$$\left.\begin{array}{l}[P_\mu, P_\nu]=0, \quad [M_{\mu\nu}, P_\rho]=-g_{\mu\rho}P_\nu+g_{\nu\rho}P_\mu \ (\mu, \nu=1,\cdots,4) \\ [M_{\mu\nu}, M_{\rho\sigma}]=-g_{\mu\rho}M_{\nu\sigma}+g_{\nu\rho}M_{\mu\sigma}-g_{\mu\sigma}M_{\rho\nu}+g_{\nu\sigma}M_{\rho\mu}\end{array}\right\} \quad (1.1)$$

with $\quad -g_{11}=-g_{22}=-g_{33}=g_{44}=1, \quad g_{\mu\nu}=0$ for $\mu\neq\nu$,

and leave the instant invariant, Dirac [1][†)] called the following four quantities:

$$P_4, \ M_{4i} \qquad (i=1, 2, 3) \qquad (1.2)$$

as Hamiltonians. The discussion about what roles the extra Hamiltonians M_{4i} other than the usual P_4 play is not yet dealt with at the present.

On the other hand, in the instant form, we are obliged to consider that the difference of the descriptions of a dynamical variable in two coordinate systems related by the special Lorentz transformations is caused by Hamiltonians. Therefore, the relativistic instant form has not a parallelism with non-relativistic dynamics. Thus, we are led to the following problem: Can we have then the new dynamical form having this parallelism? We investigated this problem in the previous paper [2], and attempted to construct the new dynamical form satisfying the requirement that the descriptions of a dynamical variable by two observers one of which

†) Numbers in brackets refer to the references at the end of the paper.

moves with uniform velocity to the other is trivial and the two observers are equivalent to describe the dynamical system. That is, we showed that such a new dynamical system can be obtained by taking the new group G_3 of transformations:

$$x'^{\lambda} = A^{\lambda}_{\mu} x^{\mu} \tag{1.3}$$

with
$$A^{\lambda}_{\mu} = \delta^{\lambda}_{\mu} + k^{\lambda}\frac{U_{\mu} - U'_{\mu}}{(kU')} + \frac{U'^{\lambda} - U^{\lambda}}{(kU)}k_{\mu} + \frac{(UU') - (UU)}{(kU)(kU')}k^{\lambda}k_{\mu}$$
$$k^{\lambda} = (d^h, 1), \quad k^{\lambda}k_{\lambda} = g_{\lambda\mu}k^{\lambda}k^{\mu} = 0 \qquad (\lambda, \mu = 1, \cdots, 4)$$
$$U^{\lambda} = (u^h/\sqrt{1-u^2/c^2},\ c/\sqrt{1-u^2/c^2}), \quad (h = 1, 2, 3)$$

in place of the special Lorentz transformations, as the relation connecting the two observers one of which moves with the uniform velocity u^h $(h=1, 2, 3)$ to the other, and adopting the front form as the dynamical form. (The transformations (1.3) leave the wave front $k_{\mu}x^{\mu}=0$ invariant.) And also, as the Hamiltonians we took the following three quantities [2]:

$$\lambda^{\mu}P_{\mu}, \quad \alpha^{\mu}\lambda^{\nu}M_{\mu\nu}, \quad \beta^{\mu}\lambda^{\nu}M_{\mu\nu} \tag{1.4}$$

with
$$\left.\begin{array}{l} \lambda^{\mu} = (-d^h, 1), \quad \lambda^{\mu}\lambda_{\mu} = 0, \quad \alpha^4 = \beta^4 = 0, \\ \lambda^{\mu}\alpha_{\mu} = \lambda^{\mu}\beta_{\mu} = \alpha^{\mu}\beta_{\mu} = 0, \quad \alpha^{\mu}\alpha_{\mu} = \beta^{\mu}\beta_{\mu} = 1. \end{array}\right\} \tag{1.5}$$

The P.b.'s between any two of the above three Hamiltonians are zero. It seems that this is a peculiar property of our new dynamical form in comparison with the instant form where this is not the case.

In the following, we shall investigate in what sense the Hamiltonians (1.2) in the instant form and (1.4) in our new dynamical form based on G_3 play the roles of Hamiltonian in connection with the precession of the spin. The usual Thomas precession based on the special Lorentz transformations or our spin precession obtained in the previous paper [3] based on G_3 appears as a pure relativistic effect, but the methods of these derivations seem to us to be not necessarily simple and plain. In this paper, we assume that (1.2) and (1.4) (of course, we must take the ones in the case involving the spin variables) play the same roles as the usual Hamiltonians and regard them as the generators for the infinitesimal change of spin variable. Therefore, we can conjecture that

$$\delta \boldsymbol{s} = [\boldsymbol{s}, H] \tag{16}$$

has the relation with the spin precession. In fact, we shall show that from the actual form of (1.6) the angular velocity of the ordinary spin precession

$$\boldsymbol{\omega}_0 = [\boldsymbol{u} \times \dot{\boldsymbol{u}}]/2c^2 \tag{1.7}$$

which is the sum of the Larmor precession and the Thomas one, can be obtained easily. Similarly, the spin precession in our new dynamical system based on G_3 is also obtained easily. These are the main aims of the present paper.

§ 2. The ordinary case based on the instant form.

In order to clear the physical meaning of our new method, we shall first show that the ordinary relativistic spin precession can be easily obtained by our new method.

For this purpose, we consider the case where the dynamical system is composed of the particle with spin. As the dynamical variable expressing the spin, we introduce the variable $\boldsymbol{s}$ with components (s_1, s_2, s_3) which satisfy the following P.b.'s

$$[s_1, s_2]=s_3\,, \quad [s_2, s_3]=s_1\,, \quad [s_3, s_1]=s_2 \tag{2.1}$$

which can be written in the unified form

$$[s_i, s_j]=-\epsilon_{ijk}s^k \quad (i, j, k=1, 2, 3) \tag{2.1'}$$

where ϵ_{ijk} is the Levi-Civita symbol, antisymmetric in all three indices and $\epsilon_{123}=\epsilon_{231}=\epsilon_{312}=1$.

When the particle has spin, P_μ and $M_{\mu\nu}$ satisfying the P.b. relations (1.1) can be expressed in the form

$$P_i=p_i, \quad M_{ij}=q_ip_j-q_jp_i-\epsilon_{ijk}s^k \tag{2.2}$$

$$P_4=H/c=\sqrt{m^2c^2-p^ip_i}\,, \quad M_{i4}=q_iH/c-\epsilon_{ijk}s^jp^k/(mc+H/c) \tag{2.3}$$

where q_i are three coordinates of the particle and p_i are conjugate momenta of q_i. The q_i and p_j satisfy the P.b.'s

$$[p_i, q_j] = g_{ij}\,. \tag{2.4}$$

It is easily confirmed by the straight forward calculations that (2.2) and (2.3) satisfy the relations (1.1).

Among our four Hamiltonians (2.3), only the three Hamiltonians M_{i4}, $(i=1, 2, 3)$ give the non-trivial change for spin variable $\boldsymbol{s}$. We shall write the linear combination of these three Hamiltonians by H_s, i.e.,

$$H_s = a^{j4}M_{j4}\,. \tag{2.5}$$

From the relation

$$\delta p^i = [p^i, H_s] = a^{i4}H/c$$

a^{i4} should be taken as

$$a^{i4} = c\delta p^i/H\,. \tag{2.6}$$

We shall now evaluate the infinitesimal change of $\boldsymbol{s}$ due to the Hamiltonian H_s,

$$\begin{aligned}\delta s_k=[s_k, H_S]&=-[s_k, \epsilon_{ilm}s^lp^m\delta p^i]c/H(mc+H/c)\\&=-\epsilon_{ilm}[s_k, s^l]p^m\delta p^ic/H(mc+H/c)\\&=\epsilon_k{}^{ln}s_n\epsilon_{lmi}p^m\delta p^ic/H(mc+H/c)\end{aligned}$$

which is written in the vector notation as

$$\delta\boldsymbol{s}=c[\boldsymbol{p}\times\delta\boldsymbol{p}]\times\boldsymbol{s}/H(mc+H/c)\,. \tag{2.7}$$

This shows that the direction of the spin is turned through an angle

$c|\boldsymbol{p}\times\delta\boldsymbol{p}|/H(mc+H/c)$. For the case of $u\ll c$, by putting $\boldsymbol{p}=m\boldsymbol{u}$ we obtain the angular velocity of spin precession

$$\boldsymbol{\omega}_0=[\boldsymbol{u}\times\dot{\boldsymbol{u}}]/2c^2\,. \tag{2.8}$$

Thus, by treating M_{i4}, given by (2.3), on the same footing with the usual Hamiltonian $P_4=H/c$, we are able to show that spin precession is obtained easily.

§ 3. The case of new dynamical form based on G_3.

By analogy of the preceding section, it is expected that the angular velocity of the spin precession in our new dynamical system based on G_3 will be obtained also by considering (1.4) as Hamiltonians.

When the particle has spin, P_μ in (1.4) has the same form as the one in the previous paper [2] (for simplicity, we take $f_\mu=0$),

$$P_\mu=p_\mu-\tfrac{1}{2}k_\mu(p^\lambda p_\lambda-m^2c^2)/(k^\rho p_\rho)\quad(\lambda,\mu=1,\cdots,4) \tag{3.1}$$

while $M_{\mu\nu}$ is altered into

$$M_{ij}=q_iP_j-q_jP_i-\epsilon_{ijk}s^k\qquad(i,j,k=1,2,3)$$
$$M_{i4}=q_iP_4-q_4P_i-\epsilon_{ijk}s^jP^k/(mc+P_4)$$

where P_μ is given by (3.1). It is easily seen that the above P_μ, $M_{\mu\nu}$ satisfy the P.b. relations (1.1).

The Hamiltonian giving non-trivial change of the spin variable is

$$H_s=c_1\alpha^\mu\lambda^\nu M_{\mu\nu}+c_2\beta^\mu\lambda^\nu M_{\mu\nu} \tag{3.3}$$

where c_1 and c_2 are infinitesimal scalar quantities and λ_μ, α_μ, β_μ are given by (1.5). We shall here introduce the following four-vector

$$\gamma^\mu=c_1\alpha^\mu+c_2\beta^\mu\,. \tag{3.4}$$

Using this γ^μ, we can express H_s (3.3) in the form

$$H_s=\gamma^\mu\lambda^\nu M_{\mu\nu}=-\gamma^id^jM_{ij}+\gamma^iM_{i4}=H_s^1+H_s^2 \tag{3.5}$$

where
$$H_s^1=\gamma^id^jM_{ij},\quad H_s^2=\gamma^iM_{i4}\qquad(i,j=1,2,3) \tag{3.6}$$

Adopting the following relation:

$$[s_k,M_{ij}]=g_{ik}s_j-g_{jk}s_i\,,$$

we obtain

$$\delta_1s_k=[s_k,H_s^1]$$
$$=-d_k(\gamma_js_j)+\gamma_k(d_js_j)$$

which can be written in the vector notation

$$\delta_1\boldsymbol{s}=[\boldsymbol{d}\times\boldsymbol{\gamma}]\times\boldsymbol{s} \tag{3.7}$$

On the other hand, $\delta_2\boldsymbol{s}=[\boldsymbol{s},H_s^2]$ can be evaluated by the similar procedure as in § 2,

$$\delta_2\boldsymbol{s}=[\boldsymbol{P}\times\boldsymbol{\gamma}]\times\boldsymbol{s}/(mc+P_4) \tag{3.8}$$

Hence, the total non-trivial change of spin is

$$\delta \boldsymbol{s}=[\boldsymbol{s}, H_s]=\delta_1 \boldsymbol{s}+\delta_2 \boldsymbol{s}$$
$$=[\{\boldsymbol{d}+\boldsymbol{P}/(mc+P_4)\}\times\boldsymbol{\gamma}]\times\boldsymbol{s} \tag{3.9}$$

We shall here determine $\boldsymbol{\gamma}$. From the relation

$$\delta P_i=[P_i, H_s]=\gamma^\mu\lambda^\nu(g_{\mu i}P_\nu-g_{\nu i}P_\mu)$$
$$=\gamma_i(\lambda^\nu P_\nu)+d_i(\gamma^j P_j) \tag{3.10}$$

and $(\gamma_i d_i)=0$, we obtain

$$\gamma^j P_j = d_i\delta P_i \tag{3.11}$$

By inserting this into the right hand side of (3.10), γ_i is obtained

$$\gamma_i=\{\delta P_i-d_i(d_j\delta P_j)\}/(\lambda^\nu P_\nu) \tag{3.12}$$

By using this γ_i, (3.9) becomes

$$\delta\boldsymbol{s}=[\{\boldsymbol{d}+\boldsymbol{P}/(mc+P_4)\times\delta\boldsymbol{P}]\times\boldsymbol{s}/(\lambda^\nu P_\nu)$$
$$+(d_j\delta P_j)[\boldsymbol{d}\times\boldsymbol{P}]\times\boldsymbol{s}/(mc+P_4)(\lambda^\nu P_\nu) \tag{3.13}$$

Here, we shall assume $u\ll c$, and then $\boldsymbol{P}=m\boldsymbol{u}$, $P_4=mc$. From this equation, we can obtain the angular velocity of the spin precession in our new dynamical system

$$\boldsymbol{\omega}=[(\boldsymbol{d}+\boldsymbol{u}/2c)\times\dot{\boldsymbol{u}}]/c+(d\cdot\dot{u})[\boldsymbol{d}\times\boldsymbol{u}]/2c^2. \tag{3.14}$$

Since the sign of our angular velocity is reverse to that of $\boldsymbol{\omega}_s$ obtained in [3]:

$$\boldsymbol{\omega}_s=-[(\boldsymbol{d}+\boldsymbol{u}/2c)\times\dot{\boldsymbol{u}}]/c-(d\cdot\dot{u})[\boldsymbol{d}\times\boldsymbol{u}]/c^2 \tag{3.15}$$

and the magnitude of the last term of (3.14) is half of the last term of $\boldsymbol{\omega}_s$, it seems that there is discrepancy between our new method of deriving the spin precession and the previous one. However, taking into account the relations (3.11) and (3.12), we can show that the last term of (3.14) is of order $u^2\dot{u}/c^3$ which has been neglected in our approximation. And further, by remembering the fact that there is Larmor precession ($\boldsymbol{\omega}_s$ is only a part of the angular velocity of total spin precession), the discrepancy will be disappear within our approximation procedure.

§ 4. Concluding remarks.

By regarding (1.2) and (1.4) as the generators of non-trivial changes of dynamical variables for the instant and front form respectively, we showed that the ordinary spin precession and our new precession based on G_3 can be obtained easily. This fact shows that *(1.2) and (1.4) play indirectly the same role as the usual Hamiltonian H.* Hence, when we treat the relativistic dynamical problem by one Hamiltonian H, we must add to H some term which gives the same effect as one that the extra Hamiltonians (such as (1.2) or (1.4)) cause the non-trivial changes of dynamical variables. For instance, when there is spin precession whose angular velocity is $\boldsymbol{\omega}$, we must add

$$\delta H_s = \boldsymbol{\omega}\cdot\boldsymbol{s}$$

to H in order that $\delta\boldsymbol{s}=[\boldsymbol{s}, \delta H_s]$ gives spin precession: $\boldsymbol{\omega}\times\boldsymbol{s}$. In fact, in this case, we have

$$\delta\boldsymbol{s}=[\boldsymbol{s}, \delta H_s]=[\boldsymbol{s}, \boldsymbol{\omega}\cdot\boldsymbol{s}]=\boldsymbol{\omega}\times\boldsymbol{s}$$

since

$$\begin{aligned}\delta s_i&=[s_i, \delta H_s]=[s_i, -\omega^l s_l]\\&=-\omega^l[s_i, s_l]=\omega^l\epsilon_{ilm}s^m\,.\end{aligned}$$

References

[1] P. A. M. Dirac: Forms of relativistic dynamics. Rev. Mod. Phys. Vol. **21** (1949), 392.

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[3] T. Shibata and T. Kimura: Spin-orbit interaction energy of an electron based on the new fundamental group of transformations. This Journal Vol. **20** (1956), 37.