JOURNAL OF SCIENCE OF THE HIROSHIMA UNIVERSITY, SER. A, VOL. 20, NO. 3, FEBRUARY, 1957

Numerical Determination of Periodic Solution of Nonlinear system

By

Minoru URABE

(Received Sept. 25, 1956)

Preface

In this paper, we seek for methods of numerical determination of a periodic solution of a nonlinear system—both a periodic and an autonomous system. Compared with the customary methods, our methods will suit to minute computation because the accuracy of computation can be easily raised as high as we desire and will fit with automatic computation because of their iterative character. Besides, our methods guarantee the existence of a periodic solution at the same time as its numerical determination. This is very important, because the customary methods—except for those by successive approximation which do not, in general, suit to numerical computation—give nothing concerning the existence of a periodic solution.

Due to the fact that the solution $\varphi_i(t, x)$ such that $\varphi_i(0, x)=x_i$ is periodic when and only when $\varphi_i(\omega, x)=x_i$ for certain $\omega>0$, for determination of a periodic solution, our methods bring the methods of numerical determination of the initial values x_i such that $\varphi_i(\omega, x)=x_i$. In the case of a periodic system, ω is given, but, in the case of an autonomous system, ω is also unknown, consequently, in the latter case, ω must be computed at the same time as the initial values x_i. In the case of an autonomous system, for determination of ω and x_i, our method presents an iterative process which converges rapidly.

In Chap. I, we deal with a periodic system and a one-parameter family of such systems. In Chap. II, we deal with an autonomous system and, in Chap. III, with a one-parameter family of autonomous systems. A one-parameter family of the systems is, in particular, convenient for seeking for periodic solutions by our methods. On a one-parameter family of autonomous systems, the two-dimensional case has already been discussed by the present writer [1][1]. In this paper, the general case is discussed in parallel but in connection with our methods for numerical computation of periodic solutions, and one property on the stability of a periodic solution produced by deformation of continuum of periodic solutions is added. Lastly,

1) Numbers in the crotchets refer to the references listed at the end of the paper.

in the Appendix, the formulas of Chaps. II and III are calculated on the two dimensional systems and the formulas convenient for actual computation are shown.

Chapter I Periodic system

1.1 Periodic solution of a periodic system

Given

$$\frac{dx_i}{dt}=X_i(x_1, x_2,\cdots, x_n, t) \qquad (i=1, 2,\cdots, n), \tag{1.1.1}$$

where $X_i(x, t)$'s are

(i) analytic and periodic with respect to t with period $\omega>0$,

(ii) analytic with respect to x_j in a certain domain Ω of n-dimensional Euclidean space[1].

Let $\varphi_i(t, x)$ $(i=1, 2,\cdots, n)$ be a solution of (1.1.1) such that

$$\varphi_i(0, x) = x_i \qquad (i=1, 2,\cdots, n). \tag{1.1.2}$$

Then the periodic solution of (1.1.1) with period ω is a solution corresponding to the initial values x_i such that

$$\varphi_i(\omega/2, x)=\varphi_i(-\omega/2, x) \qquad (i=1, 2,\cdots, n). \tag{1.1.3}$$

The equations (1.1.3) can be numerically solved by means of Newton's method when the approximate solution is known, in other words, the true periodic solution of (1.1.1) is found when the approximately periodic solution is known.

Let the approximate solution of (1.1.3) and its first correction be $x_i^{(0)}$ and $\delta x_i^{(0)}$ respectively. Then, by Newton's method, if

$$\det.\left|\frac{\partial\varphi_i(\omega/2, x^{(0)})}{\partial x_j^{(0)}}-\frac{\partial\varphi_i(-\omega/2, x^{(0)})}{\partial x_j^{(0)}}\right|\neq 0, \tag{1.1.4}$$

$\delta x_i^{(0)}$'s are calculated from

$$\Delta x_i^{(0)}+\sum_{j=1}^{n}\left(\frac{\partial\varphi_i(\omega/2, x^{(0)})}{\partial x_j^{(0)}}-\frac{\partial\varphi_i(-\omega/2, x^{(0)})}{\partial x^{(0)}}\right)\delta x_j^{(0)}=0 \qquad (i=1, 2,\cdots, n), \tag{1.1.5}$$

where

$$\Delta x_i^{(0)}=\varphi_i(\omega/2, x^{(0)})-\varphi_i(-\omega/2, x^{(0)}). \tag{1.1.6}$$

---

1) For applicability of Newton's method, the condition of analyticity of the functions $X_i(x, t)$'s is too strong, but, for numerical solution of the differential equations, sufficient smoothness of $X_i(x, t)$'s is necessary, consequently, in the present paper, for simplicity, we have assumed analyticity of $X_i(x, t)$'s.

Now, from

$$\frac{d\varphi_i(t, x)}{dt} = X_i\{\varphi(t, x), t\}$$

follows that

$$\frac{d}{dt}\frac{\partial \varphi_i}{\partial x_j} = \sum_{k=1}^{n} \frac{\partial X_i(\varphi, t)}{\partial \varphi_k} \cdot \frac{\partial \varphi_k}{\partial x_j}.$$

Consequently, if we put

$$\text{(1.1.7)} \qquad \frac{\partial X_i(x, t)}{\partial x_k} = X_{ik}(x, t) \qquad (i, k = 1, 2, \cdots, n),$$

then, for each j $(j=1, 2, \cdots, n)$, $\begin{pmatrix} \partial\varphi_1/\partial x_j \\ \partial\varphi_2/\partial x_j \\ \vdots \\ \partial\varphi_n/\partial x_j \end{pmatrix}$ is a solution of the so-called variational equations

$$\text{(1.1.8)} \qquad \frac{d\xi_i}{dt} = \sum_{k=1}^{n} X_{ik}(\varphi, t)\xi_k \qquad (i=1, 2, \cdots, n).$$

If we write $\partial\varphi_i/\partial x_j$ as φ_{ij}, then, since $\varphi_{ij}(0, x)=\delta_{ij}$ where δ_{ij} is a Kronecker's delta, a set of $\begin{pmatrix} \varphi_{1j} \\ \varphi_{2j} \\ \vdots \\ \varphi_{nj} \end{pmatrix}$ $(j=1,2,\cdots, n)$ forms a fundamental system of solutions and it is found from (1.1.8) by numerical integration.

Put

$$\text{(1.1.9)} \qquad \varphi_{ij}(\omega/2, x) - \varphi_{ij}(-\omega/2, x) = \psi_{ij}(x)$$

and

$$\text{(1.1.10)} \qquad \| \psi_{ij}(x) \|^{-1} = \| \Psi_{ij}(x) \|$$

when $\| \psi_{ij}(x) \|^{-1}$ exists. Then, from (1.1.5) follows

$$\text{(1.1.11)} \qquad \delta x_j^{(0)} = -\sum_{i=1}^{n} \Psi_{ji}(x^{(0)}) \Delta x_i^{(0)} \qquad (j=1, 2, \cdots, n)$$

provided that (1.1.4) holds, namely there holds

$$\text{(1.1.12)} \qquad \det. |\psi_{ij}(x^{(0)})| \neq 0.$$

By (1.1.11), we compute $\delta x_j^{(0)}$ and put

$$x_i^{(0)} + \delta x_i^{(0)} = x_i^{(1)}.$$

Replacing $x_i^{(0)}$ by $x_i^{(1)}$ and repeating the above process, we get $x_i^{(2)}$ and so on. Then, as is well known, the sequences $\{x_i^{(m)}\}$ $(m=0, 1, 2, \cdots)$ converge to the true solution x_i of the equation (1.1.3). In this process, as is well known, the quantities $\Psi_{ji}(x)$ do not need to be computed at each step, but the correction $\delta x_j^{(p)}$ of the $p+1$-th step can be computed by

$$\delta x_j^{(p)} = -\sum_{i=1}^{n} \Psi_{ji} \cdot \Delta x_i^{(p)}, \tag{1.1.13}$$

where

$$\Delta x_i^{(p)} = \varphi_i(\omega/2, x^{(p)}) - \varphi_i(-\omega/2, x^{(p)})$$

and Ψ_{ji}'s are constants—which we may prefer arbitrarily—sufficiently near to $\Psi_{ji}(x^{(0)})$. By this modification, we can facilitate actual computation in great deal, though convergence of the above iteration process loses rapidity a little in general.

Remark 1. The reason why we considered the condition (1.1.3) instead of

$$\varphi_i(\omega, x) = x_i \qquad (i=1, 2, \cdots, n)$$

lies in making the errors caused by numerical integration of differential equations as small as possible. Of course, if convenient for computation, instead of (1.1.3), we may consider the condition

$$\varphi_i(\omega_1, x) = \varphi_i(\omega_2, x) \qquad (i=1, 2, \cdots, n)$$

for $\omega_1, \omega_2 \doteqdot \omega/2$ such that $\omega_1 - \omega_2 = \omega$.

Remark 2. When $\varphi_i(t, x)$ is a periodic solution, the condition (1.1.4) is equivalent to

$$\det. \left| \frac{\partial \varphi_i(\omega, x^{(0)})}{\partial x_j^{(0)}} - \delta_{ij} \right| \neq 0.$$

For since $\{\|\varphi_{ij}\|\}$ $(j=1, 2, \cdots, n)$ is a fundamental system of solutions of (1.1.8) such that $\varphi_{ij}(0)=\delta_{ij}$, it is valid that

$$\|\varphi_{ij}(t+\omega)\| = \|\varphi_{ij}(t)\| \cdot \|\varphi_{ij}(\omega)\|,$$

from which follows

$$\|\varphi_{ij}(-\omega/2)\|^{-1} \cdot \|\varphi_{ij}(\omega/2)\| = \|\varphi_{ij}(\omega)\|.$$

1.2 Periodic solutions of a one-parameter family of periodic systems

Given a family of the systems

$$\frac{dx_i}{dt} = X_i(x, t, \lambda) \qquad (i=1, 2, \cdots, n) \tag{1.2.1}$$

like (1.1.1) depending analytically on a parameter λ for $a<\lambda<b$. In this

paragraph, we show that, if a periodic solution is known for $\lambda=a_0$, then, making use of a perturbation method, by means of the method of the preceding paragraph, under general conditions, the periodic solutions of (1.2.1) can be computed successively for $\lambda=a_0, a_1, a_2, \cdots, a_N$ where $a<a_0<a_1<a_2<\cdots<a_N<b$ or $a<a_N<a_{N-1}<\cdots<a_2<a_1<a_0<b$.

If the given system like (1.1.1) is imbedded in a family like (1.2.1), namely, for certain $\lambda=\lambda_0$ $(a<\lambda_0<b)$,

$$X_i(x, t, \lambda_0)=X_i(x, t) \qquad (i=1, 2, \cdots, n),$$

then the method of this paragraph could be used effectively when the certain general condition is satisfied.

By the assumption, for sufficiently small $|\lambda-a_0|$, the functions $X_i(x, t, \lambda)$'s are expanded as follows:

$$X_i(x, t, \lambda)=X_i(x, t, a_0)+\varepsilon X_i^{(1)}(x, t, a_0)+\cdots, \tag{1.2.2}$$

where

$$\varepsilon=\lambda-a_0,$$

consequently the solution $\varphi_i(t, x, \lambda)$ $(i=1, 2, \cdots, n)$ of (1.2.1) such that $\varphi_i(0, x, \lambda)=x_i$ is expanded as follows:

$$\varphi_i(t, x, \lambda)=\varphi_i(t, x, a_0)+\varepsilon\varphi_i^{(1)}(t, x, a_0)+\cdots, \tag{1.2.3}$$

where

$$\begin{cases} \varphi_i(0, x, a_0)=x_i, \\ \varphi_i^{(1)}(0, x, a_0)=\varphi_i^{(2)}(0, x, a_0)=\cdots=0. \end{cases} \tag{1.2.4}$$

Substituting (1.2.3) into (1.2.1), on account of (1.2.2), we have:

$$\begin{cases} \dfrac{d\varphi_i(t, x, a_0)}{dt}=X_i\{\varphi(t, x, a_0), t, a_0\}, \\ \dfrac{d\varphi_i^{(1)}(t, x, a_0)}{dt}=\sum_{k=1}^{n} X_{ik}(\varphi, t, a_0)\varphi_k^{(1)}+X_i^{(1)}(\varphi, t, a_0), \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots, \end{cases} \tag{1.2.5}$$

where $X_{ik}(x, t, \lambda)=\partial X_i(x, t, \lambda)/\partial x_k$. From the first of (1.2.4) and that of (1.2.5), $\varphi_i(t, x, a_0)$ is a solution of (1.2.1) for $\lambda=a_0$ such that $\varphi_i(0, x, a_0)=x_i$. Now, by the assumption, for $\lambda=a_0$, a periodic solution of (1.2.1) is known, so we suppose that a known periodic solution is $\varphi_i(t, x^{(0)}, a_0)$. Comparing (1.1.8) with the second of (1.2.5), we see that the solution of the second of (1.2.5) is expressed as follows:

$$\varphi_i^{(1)}(t, x, a_0)=\sum_{j=1}^{n}\varphi_{ij}(t, x, a_0)u_j(t, x, a_0) \tag{1.2.6}$$

where

$$\varphi_{ij}(t, x, a_0) = \partial\varphi_i(t, x, a_0)/\partial x_j\,, \tag{1.2.7}$$

because $\{\|\varphi_{ij}\|\}$ $(j=1, 2, \cdots, n)$ forms a fundamental system of solutions of (1.1.8). Substituting (1.2.6) into the second of (1.2.5), we have

$$\sum_{j=1}^{n} \varphi_{ij} \frac{du_j}{dt} = X_i^{(1)}(\varphi, t, a_0)\,,$$

consequently, from (1.2.4) follows

$$u_j(t, x, a_0) = \int_0^t \sum_{i=1}^{n} \Phi_{ji}(s, x, a_0) X_i^{(1)}\{\varphi(s, x, a_0), s, a_0\}\, ds\,,$$

or

$$\varphi_i^{(1)}(t, x, a_0) = \sum_{j=1}^{n} \varphi_{ij} \int_0^t \sum_{k=1}^{n} \Phi_{jk} X_k^{(1)} ds \quad (i=1, 2, \cdots, n), \tag{1.2.8}$$

where $\|\Phi_{ji}\| = \|\varphi_{ji}\|^{-1}$.

In order that the solution $\varphi_i(t, x, \lambda)$ may be periodic with period $\omega > 0$, it is necessary and sufficient that

$$\varphi_i(\omega/2, x, \lambda) = \varphi_i(-\omega/2, x, \lambda) \quad (i=1, 2, \cdots, n)$$

namely, from (1.2.3), that

$$\begin{aligned} &\varphi_i(\omega/2, x, a_0) + \varepsilon\varphi_i^{(1)}(\omega/2, x, a_0) + \cdots \\ &\quad = \varphi_i(-\omega/2, x, a_0) + \varepsilon\varphi_i^{(1)}(-\omega/2, x, a_0) + \cdots \\ &\qquad\qquad (i=1, 2, \cdots, n). \end{aligned} \tag{1.2.9}$$

By our assumption on $\varphi_i(t, x^{(0)}, a_0)$, (1.2.9) are evidently valid when $\varepsilon=0$, i.e. $\lambda=a_0$ and $x_i=x_i^{(0)}$ $(i=1, 2, \cdots, n)$. Therefore, for $\lambda=a_0+\varepsilon$, if we put $x_i=x_i^{(0)}+\delta x_i^{(0)}$, from (1.2.9) follows

$$\begin{aligned} &\sum_{j=1}^{n} \{\varphi_{ij}(\omega/2, x^{(0)}, a_0) - \varphi_{ij}(-\omega/2, x^{(0)}, a_0)\}\, \delta x_j^{(0)} \\ &\quad + \varepsilon\{\varphi_i^{(1)}(\omega/2, x^{(0)}, a_0) - \varphi_i^{(1)}(-\omega/2, x^{(0)}, a_0)\} + \cdots = 0\,, \end{aligned}$$

consequently, provided that

$$\det. |\varphi_{ij}(\omega/2, x^{(0)}, a_0) - \varphi_{ij}(-\omega/2, x^{(0)}, a_0)| \neq 0\,,$$

we have:

$$\delta x_j^{(0)} = -\varepsilon \sum_{i=1}^{n} \Psi_{ji} \Delta\varphi_i^{(1)} + O(\varepsilon^2)\,, \tag{1.2.10}$$

where

$$\| \Psi_{ji} \| = \| \varphi_{ji}(\omega/2, x^{(0)}, a_0) - \varphi_{ji}(-\omega/2, x^{(0)}, a_0) \|^{-1},$$

$$\Delta\varphi_i^{(1)} = \varphi_i^{(1)}(\omega/2, x^{(0)}, a_0) - \varphi_i^{(1)}(-\omega/2, x^{(0)}, a_0).$$

Thus, from (1.2.8) and (1.2.10), for $\lambda = a_0 + \varepsilon$, we obtain an approximately periodic solution such that

$$\varphi_i(0, x^{(0)} + \delta x^{(0)}, \lambda) = x_i^{(0)} + \delta x_i^{(0)} \qquad (i = 1, 2, \cdots, n)$$

where

$$\delta x_j^{(0)} = -\varepsilon \sum_{i=1}^{n} \Psi_{ji} \Delta\varphi_i^{(1)}. \tag{1.2.11}$$

Starting from this solution, by the iteration method of the preceding paragraph, we can find a periodic solution for $\lambda = a_0 + \varepsilon$. But, in this iteration process, as is remarked at the end of the preceding paragraph, we may use Ψ_{ji} of this paragraph as Ψ_{ji} in (1.1.13). This preferance would save much labor in actual computation.

Computation of a periodic solution for $\lambda = a_p$ from one for $\lambda = a_{p-1}$ is quite analogous.

Thus, from a given periodic solution for $\lambda = a_0$, we can compute seccessively periodic solutions for $\lambda = a_0, a_1, a_2, \cdots, a_N$ provided that, at each step

$$\det. | \varphi_{ij}(\omega/2, x, \lambda) - \varphi_{ij}(-\omega/2, x, \lambda) | \neq 0. \tag{1.2.12}$$

Chapter II Autonomous system

2.1 Fundamental formulas

Given an autonomous system

$$\frac{dx_i}{dt} = X_i(x_1, x_2, \cdots, x_n) \qquad (i = 1, 2, \cdots, n) \tag{2.1.1}$$

where $X_i(x)$'s are analytic in a certain domain Ω of n-dimensional Euclidean space. Also given a solution $\varphi_i(t)$ which is approximately periodic with the approximate period $\omega_0 > 0$, and of which the characteristic C_0 lies in Ω.

Let $\xi_i^{(\alpha)}(x)$ $(\alpha = 1, 2, \cdots, n-1)$ be the direction cosines of the normals of the characteristics of (2.1.1) which are orthogonal to each other[1)], then, from the definition follows

$$\begin{cases} \sum_{i=1}^{n} \xi_i^{(\alpha)} X_i = 0, \\ \sum_{i=1}^{n} \xi_i^{(\alpha)} \xi_i^{(\beta)} = \delta_{\alpha\beta}, \end{cases} \tag{2.1.2}$$

where $\delta_{\alpha\beta}$ is a Kronecker's delta. Since the characteristic C lying in the neighborhood of C_0 is expressed by

1) Existence of such normals is evident from analyticity of the characteristics.

$$x_i = \varphi_i + \sum_{\alpha=1}^{n-1} \rho_\alpha \xi_i^{(\alpha)}(\varphi) \qquad (i=1, 2, \cdots, n) \tag{2.1.3}$$

and, for C, from (2.1.1), there hold

$$\frac{dx_i}{d\tau} = X_i(x) \qquad (i=1, 2, \cdots, n), \tag{2.1.4}$$

substituting (2.1.3) into (2.1.4), we have:

$$\sum_{\alpha=1}^{n-1} \frac{d\rho_\alpha}{dt} \xi_i^{(\alpha)} + \sum_{\alpha=1}^{n-1} \rho_\alpha \frac{d\xi_i^{(\alpha)}}{dt} = X_i' \frac{d\tau}{dt} - X_i(\varphi) \qquad (i=1, 2, \cdots, n), \tag{2.1.5}$$

where

$$X_i' = X_i\Big(\varphi + \sum_\alpha \rho_\alpha \xi^{(\alpha)}\Big) \qquad (i=1, 2, \cdots, n) \tag{2.1.6}$$

and $\xi_i^{(\alpha)}$ denote $\xi_i^{(\alpha)}\{\varphi(t)\}$. Multiplying $\xi_i^{(\beta)}$ on both sides of (2.1.5) and adding on i, on account of (2.1.2), we obtain:

$$\frac{d\rho_\beta}{dt} = \sum_{j=1}^{n} X_j' \frac{d\tau}{dt} \xi_j^{(\beta)} - \sum_{\alpha=1}^{n-1} \rho_\alpha \Big(\sum_{j=1}^{n} \frac{d\xi_j^{(\alpha)}}{dt} \xi_j^{(\beta)} \Big) \qquad (\beta=1, 2, \cdots, n-1). \tag{2.1.7}$$

Then, since

$$\sum_{\alpha=1}^{n-1} \xi_i^{(\alpha)} \xi_j^{(\alpha)} = \delta_{ij} - \frac{X_i X_j}{\sum_{k=1}^{n} X_k^2} \qquad (i, j=1, 2, \cdots, n)$$

because of orthogonality of $\xi_i^{(\alpha)}$ and X_i, substituting (2.1.7) into (2.1.5), we have:

$$\sum_{j=1}^{n} X_j' \frac{d\tau}{dt} \Big(\delta_{ji} - \frac{X_i X_j}{\sum_k X_k^2} \Big) - \sum_{j=1}^{n} \Big(\sum_{\beta=1}^{n-1} \rho_\beta \frac{d\xi_j^{(\beta)}}{dt} \Big) \Big(\delta_{ji} - \frac{X_j X_i}{\sum_k X_k^2} \Big)$$
$$+ \sum_{\alpha=1}^{n-1} \rho_\alpha \frac{d\xi_i^{(\alpha)}}{dt} = X_i' \frac{d\tau}{dt} - X_i,$$

from which follows

$$\frac{d\tau}{dt} = \frac{\sum_{k=1}^{n} X_k^2 + \sum_{\alpha=1}^{n-1} \rho_\alpha \Big(\sum_{k=1}^{n} \frac{d\xi_k^{(\alpha)}}{dt} X_k \Big)}{\sum_{k=1}^{n} X_k' \cdot X_k}, \tag{2.1.8}$$

because at least one of $X_1, X_2, \cdots, X_n$ does not vanish. Then substitution of (2.1.8) into (2.1.7) entails

$$\frac{d\rho_\alpha}{dt} = R_\alpha(\rho, t) \qquad (\alpha=1, 2, \cdots, n-1), \tag{2.1.9}$$

where

(2.1.10) $R_\alpha(\rho, t)$

$$\equiv \sum_{j=1}^{n} X_j\Big(\varphi+\sum_{\beta}\rho_\beta\xi^{(\beta)}\Big)\xi_j^{(\alpha)}\frac{\sum_{k=1}^{n}X_k^2+\sum_{\beta=1}^{n-1}\rho_\beta\Big(\sum_{k=1}^{n}\frac{d\xi_k^{(\beta)}}{dt}X_k\Big)}{\sum_{k=1}^{n}X_k\cdot X_k\Big(\varphi+\sum_{\beta}\rho_\beta\xi^{(\beta)}\Big)}-\sum_{\beta=1}^{n-1}\rho_\beta\Big(\sum_{j=1}^{n}\frac{d\xi_j^{(\beta)}}{dt}\xi_j^{(\alpha)}\Big)$$

$$(\alpha=1, 2, \cdots, n-1).$$

2.2 Interation process for computing a periodic solution

For $\tilde{\omega}\doteqdot\omega_0$, the characteristic C_0 crosses again the normal hyperplane π of C_0 at $A\{\varphi_i(-\omega_0/2)\}$ at the time $\tilde{\omega}/2$ in B. Here $\tilde{\omega}$ is a root of the equation

(2.2.1) $$F(t)\equiv\sum_{i=1}^{n}\Big\{\varphi_i\Big(\frac{\tilde{\omega}}{2}\Big)-\varphi_i\Big(-\frac{\omega_0}{2}\Big)\Big\}X_i\Big\{\varphi\Big(-\frac{\omega_0}{2}\Big)\Big\}=0\,.$$

Consequently, following Newton's method, the value of $\tilde{\omega}$ can be found numerically by computing successively corrections $\delta\tilde{\omega}^{(p)}=\tilde{\omega}^{(p+1)}-\tilde{\omega}^{(p)}$ $(\tilde{\omega}^{(0)}=\omega_0)$ as follows:

(2.2.2) $$\delta\tilde{\omega}^{(p)}=-\frac{2\sum_{i=1}^{n}\Big\{\varphi_i\Big(\frac{\tilde{\omega}^{(p)}}{2}\Big)-\varphi_i\Big(-\frac{\omega_0}{2}\Big)\Big\}X_i\Big\{\varphi\Big(-\frac{\omega_0}{2}\Big)\Big\}}{\sum_{i=1}^{n}X_i\Big\{\varphi\Big(-\frac{\omega_0}{2}\Big)\Big\}X_i\Big\{\varphi\Big(\frac{\tilde{\omega}^{(p)}}{2}\Big)\Big\}}\,.$$

As is remarked at the end of 1.1, the denominator of the right-hand side can be replaced by

$$\sum_{i=1}^{n}X_i\Big\{\varphi\Big(-\frac{\omega_0}{2}\Big)\Big\}X_i\Big\{\varphi\Big(\frac{\omega_0}{2}\Big)\Big\}$$

or further by

$$\sum_{i=1}^{n}\Big[X_i\Big\{\varphi\Big(-\frac{\omega_0}{2}\Big)\Big\}\Big]^2,$$

because

$$X_i\Big\{\varphi\Big(\frac{\tilde{\omega}^{(p)}}{2}\Big)\Big\}\doteqdot X_i\Big\{\varphi\Big(\frac{\omega_0}{2}\Big)\Big\}\doteqdot X_i\Big\{\varphi\Big(-\frac{\omega_0}{2}\Big)\Big\}.$$

Since $\tilde{\omega}$ is a root of (2.2.1), $\varphi_i(\tilde{\omega}/2)-\varphi_i(-\omega_0/2)$ are expressed as follows:

(2.2.3) $$\varphi_i\Big(\frac{\tilde{\omega}}{2}\Big)-\varphi_i\Big(-\frac{\omega_0}{2}\Big)=\sum_{\alpha=1}^{n-1}\kappa_\alpha\xi_i^{(\alpha)}\Big(-\frac{\omega_0}{2}\Big)\quad(i=1, 2, \cdots, n)^{1)},$$

where

(2.2.4) $$\kappa_\alpha=\sum_{i=1}^{n}\Big\{\varphi_i\Big(\frac{\tilde{\omega}}{2}\Big)-\varphi_i\Big(-\frac{\omega_0}{2}\Big)\Big\}\xi_i^{(\alpha)}\Big(-\frac{\omega_0}{2}\Big)\quad(\alpha=1, 2, \cdots, n-1).$$

In order to seek for a periodic solution, putting

1) $\xi_i^{(\alpha)}(t)$ denote $\xi_i^{(\alpha)}\{\varphi(t)\}$.

(2.2.5)
$$l_{\alpha\beta}=\sum_{i=1}^{n}\xi_i^{(\alpha)}\left(\frac{\tilde{\omega}}{2}\right)\xi_i^{(\beta)}\left(-\frac{\omega_0}{2}\right) \qquad (\alpha, \beta=1, 2, \cdots, n-1),$$

let us consider the characteristic C satisfying the condition as follows:

(2.2.6)
$$\rho_\alpha\left(\frac{\tilde{\omega}}{2}\right)=\sum_{\beta=1}^{n-1}l_{\alpha\beta}\left\{\rho_\beta\left(-\frac{\omega_0}{2}\right)-\kappa_\beta\right\} \qquad (\alpha=1, 2, \cdots, n-1).$$

This condition expresses that the point Q of C corresponding to the point B by (2.1.3) is an orthogonal projection of the point P of C corresponding to the point A by (2.1.3) on the normal hyperplane π' of C_0 at B.

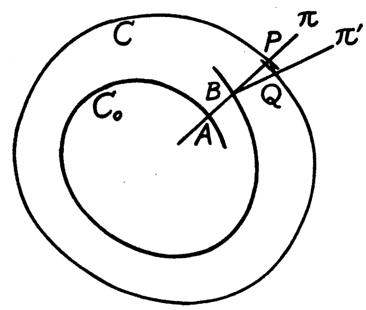

Fig. 1

Let $\rho_\alpha(t, c)$ $(\alpha=1, 2, \cdots, n-1)$ be a solution of (2.1.9) such that $\rho_\alpha(0, c)=c_\alpha$. Since $R_\alpha(\rho, t)$'s are analytic with respect to ρ_β for $|\rho_\beta|\ll 1$ and $R_\alpha(0, t)=0$, $\rho_\alpha(t, c)$'s are expanded with respect to c_β's as follows:

(2.2.7)
$$\rho_\alpha(t, c)=\sum_{\beta=1}^{n-1}\rho_{\alpha\beta}(t)c_\beta+[c]_2 \text{ [1]},$$

where

(2.2.8)
$$\rho_{\alpha\beta}(0)=\delta_{\alpha\beta}.$$

Substituting (2.2.7) into (2.1.9) and comparing the coefficients of c_β, we see that

(2.2.9)
$$\frac{d\rho_{\alpha\beta}}{dt}=\sum_{\gamma=1}^{n-1}R_{\alpha\gamma}(t)\rho_{\gamma\beta},$$

where

(2.2.10)
$$R_{\alpha\gamma}(t)\equiv\sum_{j,k=1}^{n}\xi_j^{(\alpha)}X_{jk}(\varphi)\xi_k^{(\gamma)}-\sum_{j=1}^{n}\xi_j^{(\alpha)}\frac{d\xi_j^{(\gamma)}}{dt},$$

$X_{jk}(x)$ being $\partial X_j/\partial x_k$.

If we substitute (2.2.7) into (2.2.6), the condition (2.2.6) is written as follows:

(2.2.11)
$$\begin{aligned}\Phi_\alpha(c)&\equiv\rho_\alpha\left(\frac{\tilde{\omega}}{2}, c\right)-\sum_{\beta=1}^{n-1}l_{\alpha\beta}\left\{\rho_\beta\left(-\frac{\omega_0}{2}, c\right)-\kappa_\beta\right\}\\ &=\sum_{\beta=1}^{n-1}l_{\alpha\beta}\kappa_\beta+\sum_{\beta=1}^{n-1}\left\{\rho_{\alpha\beta}\left(\frac{\tilde{\omega}}{2}\right)-\sum_{\gamma=1}^{n-1}l_{\alpha\gamma}\rho_{\gamma\beta}\left(-\frac{\omega_0}{2}\right)\right\}c_\beta+[c]_2\\ &=0.\end{aligned}$$

1) [$]_m$ denotes the sum of the terms of the m-th and higher orders with respect to the arguments in the crotchets.

Since $l_{\alpha\beta} \doteqdot \delta_{\alpha\beta}$ and $\tilde{\omega} \doteqdot \omega_0$, instead of (2.2.11), let us consider the simpler condition

$$\sum_{\beta=1}^{n-1}\left\{\rho_{\alpha\beta}\left(\frac{\omega_0}{2}\right)-\rho_{\alpha\beta}\left(-\frac{\omega_0}{2}\right)\right\}c_\beta=-\kappa_\alpha \qquad (\alpha=1, 2,\cdots, n-1). \tag{2.2.12}$$

If

$$\det.\left|\rho_{\alpha\beta}\left(\frac{\omega_0}{2}\right)-\rho_{\alpha\beta}\left(-\frac{\omega_0}{2}\right)\right| \neq 0, \tag{2.2.13}$$

then (2.2.12) can be easily solved with respect to c_α.

If c_α's are so determined, we repeat the above process replacing the characteristic C_0 by the characteristic

$$C_1:\ x_i=\varphi_i^{(1)}(t),$$

where $\varphi_i^{(1)}(t)$ is a solution of (2.1.1) such that

$$\varphi_i^{(1)}(0)=\varphi_i(0)+\sum_{\alpha=1}^{n-1}c_\alpha\xi_i^{(\alpha)}\{\varphi(0)\}.$$

Again, starting from C_1, we obtain C_2 and so on. In this iteration process, the quantities

$$\rho_{\alpha\beta}\left(\frac{\omega_0}{2}\right)-\rho_{\alpha\beta}\left(-\frac{\omega_0}{2}\right)$$

change their values at each step, but their differences are so small that, at every step, we can make use of their values at the first step constantly. This modification facilitates actual computation in great deal.

If the above process converges, then, in the limit $c_\alpha=0$, namely $\kappa_\alpha=0$, consequently, (2.2.3) follows

$$\varphi_i\left(\frac{\tilde{\omega}}{2}\right)=\varphi_i\left(-\frac{\omega_0}{2}\right) \qquad (i=1, 2,\cdots, n).$$

This is to say that the limit solution $\varphi_i(t)$ is periodic with period $(\tilde{\omega}/2)+(\omega_0/2)$.

Summarizing the above discussions, we get the following method of computation for a periodic solution:

From approximately periodic solution $\varphi_i(t)$, we

1° *compute* $\tilde{\omega}$ *by* (2.2.2),

2° *find* $\rho_{\alpha\beta}(\omega_0/2)-\rho_{\alpha\beta}(-\omega_0/2)$ *by numerical integration of* (2.2.9),

3° *compute* κ_α *by* (2.2.4),

4° *compute* c_α *by* (2.2.12),

5° *find the solution* $\varphi_i^{(1)}(t)$ *for above* c_α *by numerical integration of* (2.1.1),

6° *compute* $\tilde{\omega}$ *for* $\varphi_i^{(1)}(t)$ *by* (2.2.2),

7° *compute* κ_α *for* $\varphi_i^{(1)}(t)$ *by* (2.2.4), 8° *compute* c_α *for above* κ_α *by* (2.2.12),

9° *find the solution* $\varphi_i^{(2)}(t)$ *for above* c_α *by numerical integration of* (2.2.1),

....

The solution $\varphi_i^{(m)}(t)$ *for which* $\kappa_\alpha=0$ *is a periodic solution whose period is* $(\tilde{\omega}/2)+(\omega_0/2)$.

Remark. The reason why we considered the normal hyperplane π of C_0 at $A\{\varphi_i(-\omega_0/2)\}$ lies in making the errors caused by numerical integration of differential equations as small as possible as in **1.1.**

2.3 Convergence of the iteration process

In this paragraph, we consider convergence of the iteration process found in the preceding paragraph.

Let us take C_0 in the neighborhood $U\subset\Omega$ of the characteristic $\bar{C}$ corresponding to the periodic solution with period $\bar{\omega}>0$.

Since $\rho_{\alpha\beta}(t)$'s are bounded for $|t|\leqq T\ (T>\bar{\omega})$ in U, there exists a positive number K_1 such that, for any characteristic in U,

$$|\rho_{\alpha\beta}(t)|\leqq K_1 \qquad \text{for } |t|\leqq T.$$

Put $\max_\alpha|\kappa_\alpha|=\delta$ and assume that, for ω_0 such that $|\omega_0-\bar{\omega}|$ is sufficiently small,

$$\det.\left|\rho_{\alpha\beta}\left(\frac{\omega_0}{2}\right)-\rho_{\alpha\beta}\left(-\frac{\omega_0}{2}\right)\right|>\eta>0,$$

then, from (2.2.12) follows

$$|c_\beta|<\delta\cdot\frac{(n-1)!\,2K_1}{\eta}, \tag{2.3.1}$$

consequently, from (2.2.7), there exists a positive number K_2 such that

$$|\rho_\alpha(t,c)|\leqq\delta K_2 \qquad (|t|\leqq T), \tag{2.3.2}$$

provided that δ is sufficiently small, namely C_0 lies sufficiently near $\bar{C}$. Also, from (2.2.3) follows

$$|\varphi_i(\tilde{\omega}/2)-\varphi_i(-\omega_0/2)|\leqq(n-1)\delta, \tag{2.3.3}$$

consequently there exists a positive number L such that

$$|\xi_i^{(\alpha)}(\tilde{\omega}/2)-\xi_i^{(\alpha)}(-\omega_0/2)|\leqq\delta L,$$

so that, from (2.2.5) follows

$$|l_{\alpha\beta}-\delta_{\alpha\beta}|\leqq\delta nL.$$

If $|\tilde{\omega}-\omega_0|\leqq\delta'$, there exists a positive number M such that

$$|\rho_{\alpha\beta}(\tilde{\omega}/2)-\rho_{\alpha\beta}(\omega_0/2)|\leqq\delta' M,$$

consequently, for c_α satisfying (2.2.12) approximately in such a way that

$$\sum_{\beta=1}^{n-1}\{\rho_{\alpha\beta}(\omega_0/2)-\rho_{\alpha\beta}(-\omega_0/2)+\sigma_{\alpha\beta}\}c_\beta=-\kappa_\alpha \qquad (\alpha=1, 2,\cdots, n-1)$$

where

$$|\sigma_{\alpha\beta}|\leqq\delta'' \qquad (\delta'': \text{sufficiently small}),$$

it is valid that

$$|\Phi_\alpha(c)|\leqq n(n-1)L\delta^2+(M\delta'+\delta''+n(n-1)LK_1\delta)\cdot\delta\,\frac{(n-1)!\,2K_1}{\eta}(n-1)+O(\delta^2),$$

therefore there exist positive numbers N_1, N_2 and N such that

$$|\Phi_\alpha(c)|\leqq(N_1\delta'+N_2\delta''+N\delta)\delta.$$

Then, from the geometrical meaning of $\Phi_\alpha(c)$, it is valid that

$$\overline{PQ}\leqq\overline{BP}\sin\theta+\sqrt{n-1}\,(N_1\delta'+N_2\delta''+N\delta)\delta \tag{2.3.4}$$

where θ is an angle between the hyperplanes π and π'. Now, from (2.3.3) follows

$$|\sin\theta|\leqq S\delta$$

for a certain positive constant S and, from (2.3.2) follows

$$\overline{BP}\leqq\sqrt{n-1}\,\delta(K_2+1).$$

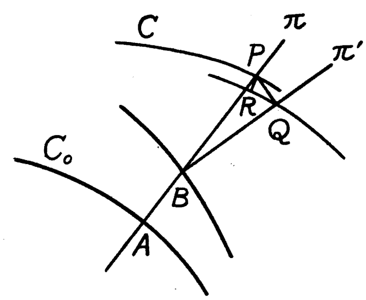

Fig. 2

Therefore, from (2.3.4), we see that

$$\overline{PQ}\leqq\delta\sqrt{n-1}\,\{\delta S(K_2+1)+(N_1\delta'+N_2\delta''+N\delta)\}. \tag{2.3.5}$$

Let $\overline{PR}$ be the distance measured from P in a normal hyperplane of C at P to R of C lying near Q. Then, when δ is sufficiently small, neglecting the terms of higher orders, it is valid that

$$\overline{PQ}^2=\overline{PR}^2+\overline{RQ}^2 \;{}^{1)}, \tag{2.3.6}$$

consequently, because $\delta\leqq\overline{AB}$, from (2.3.5) follows

$$\frac{\overline{PR}}{\overline{AB}}\leqq\frac{\overline{PQ}}{\overline{AB}}\leqq K \tag{2.3.7}$$

1) Let $x_i=\varphi_i(t)$ be the equations of C and assume that $\{\varphi_i(0)\}$ represents P. Then, in terms of the same notations as in (2.1.3), C is represented by

$$x_i=\varphi_i(t)+\sum_{\alpha=1}^{n-1}\rho_\alpha\xi_i^{(\alpha)}(t) \qquad (i=1, 2,\cdots, n)$$

in the neighborhood of R. In the neighborhood of R, putting $\rho_\alpha(0)=c_\alpha$, x_i's are expanded as follows:

$$x_i=\varphi_i(0)+tX_i+\sum_{\alpha=1}^{n-1}c_\alpha\xi_i^{(\alpha)}(0)+[c, t]_2,$$

consequently, neglecting the terms of the second and higher orders, we have:

$$\sum_{i=1}^{n}[x_i-\varphi_i(0)]^2=t^2\sum_{i=1}^{n}X_i^2+\sum_{\alpha=1}^{n}c_\alpha^2,$$

which proves (2.3.6).

where

(2.3.8) $$K=\sqrt{n-1}\,\{S(K_2+1)\delta+(N_1\delta'+N_2\delta''+N\delta)\}\,.$$

The relation (2.3.7) expresses that

$$\sum_{\alpha=1}^{n-1}(\kappa_\alpha^{(m+1)})^2\leqq K^2\sum_{\alpha=1}^{n-1}(\kappa_\alpha^{(m)})^2,$$

where $\kappa_\alpha^{(m)}$'s are κ_α obtained at m-th step and $\kappa_\alpha^{(0)}=\kappa_\alpha$. Thus we see that, if C_0 is taken sufficiently near $\overline{C}$ and δ', δ'' are taken sufficiently small, then $K<1$, consequently, since the characteristic obtained at each step of our process lies always in U as is seen from (2.3.2), our iteration process can be continued indefinitely, and the sequences $\{\kappa_\alpha^{(m)}\}$ $(\alpha=1,2,\cdots,n-1)$ all converge to zero, namely our iteration process converges.

Since PQ is nearly perpendicular to the hyperplane π' when $\delta',\delta''\leqq O(\delta)$, the distance $\overline{PR}$ is equal to $\overline{PQ}\sin\theta'$ neglecting the terms of the higher orders, where θ' is an angle between the hyperplane π' and the normal hyperplane of C at P. Consequently, since $\sin\theta'=O(\delta)$, it becomes that $\overline{PR}=O(\delta^3)$. Then, since $\overline{AB}\geqq(\delta)$, we have

$$\overline{PR}=O(\overline{AB}^3)\,,$$

namely *our iteration process is of the third order* [2], in other words, *convergence of the process is more rapid by one order than that of Newton's iteration process*[1].

Chapter III One-parameter family of autonomous systems

In this chapter, as in Chap. I, we consider a one-parameter family of the systems. When the dimension of the systems is two, the discussions have already been carried on almost completely by the present writer [1]. Here we are going to develop the parallel theory for the case of n dimensions as far as possible. But our present aim is to compute periodic solutions minutely, consequently the discussions are carried on in connection with the iteration process of the preceding chapter.

3.1 Variation of a periodic solution

Given a family of autonomous systems

(3.1.1) $$\frac{dx_i}{dt}=X_i(x,\lambda)\qquad(i=1,2,\cdots,n)$$

like (2.1.1) depending analytically on a parameter λ for $a<\lambda<b$. Assuming that a periodic solution is known for $\lambda=a_0$ $(a<a_0<b)$, let us seek for a periodic solution of (3.1.1) for $\lambda=a_0+\varepsilon$, $|\varepsilon|$ being sufficiently small.

Let the periodic solution of (3.1.1) for $\lambda=a_0$ be $\varphi_i(t)$ and its period and characteristic be $\omega_0>0$ and C_0 respectively. Then, as in Chap. II, the

1) Newton's iteration process is of the second order.

characteristic C for $\lambda=a_0+\varepsilon$ lying in the neighborhood of C_0 is represented by (2.1.3) and, from (2.1.8) and (2.1.9), we have:

$$\frac{d\tau}{dt}=\frac{\sum_{k=1}^{n} X_k^2+\sum_{\alpha=1}^{n-1}\rho_\alpha\left(\sum_{k=1}^{n}\frac{d\xi_k^{(\alpha)}}{dt}X_k\right)}{\sum_{k=1}^{n} X_k\cdot X_k(\varphi+\sum_\alpha \rho_\alpha\xi^{(\alpha)},\lambda)}, \tag{3.1.2}$$

$$\frac{d\rho_\alpha}{dt}=R_\alpha(\rho,t,\lambda) \tag{3.1.3}$$

$$\equiv\sum_{j=1}^{n} X_j(\varphi+\sum_\beta \rho_\beta\xi^{(\beta)},\lambda)\xi_j^{(\alpha)}\frac{d\tau}{dt}-\sum_{\beta=1}^{n-1}\rho_\beta\left(\sum_{j=1}^{n}\frac{d\xi_j^{(\beta)}}{dt}\xi_j^{(\alpha)}\right)$$

$$(\alpha=1,2,\cdots,n-1),$$

where

$$X_k=X_k(\varphi,a_0)\qquad (k=1,2,\cdots,n).$$

Since $R_\alpha(\rho,t,\lambda)$'s are analytic with respect to ρ_β and ε $(=\lambda-a_0)$ for $|\rho_\beta|$, $|\varepsilon|\ll 1$, the solution $\rho_\alpha=\rho_\alpha(t,c,\lambda)$ $(\alpha=1,2,\cdots,n-1)$ of (3.1.3) such that $\rho_\alpha(0,c,\lambda)=c_\alpha$ is expanded with respect to c_β and ε as follows:

$$\rho_\alpha(t,c,\lambda)=\rho_\alpha^{(0)}(t,c,\varepsilon)+\rho_\alpha^{(1)}(t,c,\varepsilon)+\rho_\alpha^{(2)}(t,c,\varepsilon)+\cdots, \tag{3.1.4}$$

where $\rho_\alpha^{(k)}(t,c,\varepsilon)$'s $(k=0,1,2,\cdots)$ are the sums of the terms of k-th order with respect to c_β and ε. From $\rho_\alpha(0,c,\lambda)=c_\alpha$, it is evident that

$$\begin{cases}\rho_\alpha^{(0)}(0,c,\varepsilon)=\rho_\alpha^{(2)}(0,c,\varepsilon)=\cdots=0,\\ \rho_\alpha^{(1)}(0,c,\varepsilon)=c_\alpha\end{cases} \tag{3.1.5}$$

$$(\alpha=1,2,\cdots,n-1).$$

If we put

$$\rho_\alpha^{(1)}(t,c,\varepsilon)=\sum_{\beta=1}^{n-1}\rho_{\alpha\beta}(t)c_\beta+\rho_\alpha^{(1)}(t)\varepsilon, \tag{3.1.6}$$

then, as in Chap. II, substitution of (3.1.4) into (3.1.3) entails

$$\rho_\alpha^{(0)}(t,c,\varepsilon)=0,$$

$$\frac{d\rho_{\alpha\beta}}{dt}=\sum_{\gamma=1}^{n-1}R_{\alpha\gamma}(t)\rho_{\gamma\beta}, \tag{3.1.7}$$

$$\frac{d\rho_\alpha^{(1)}}{dt}=\sum_{\gamma=1}^{n-1}R_{\alpha\gamma}(t)\rho_\gamma^{(1)}+\sum_{i=1}^{n}X_i^{(1)}\xi_i^{(\alpha)}, \tag{3.1.8}$$

where $R_{\alpha\gamma}(t)$'s are the same as in (2.2.10) and $X_i^{(1)}=\partial X_i(\varphi,a_0)/\partial a_0$. From (3.1.5) and (3.1.6), it must be that

$$\rho_{\alpha\beta}(0)=\delta_{\alpha\beta},\qquad \rho_\alpha^{(1)}(0)=0.$$

Consequently $\{\|\rho_{\alpha\beta}(t)\|\}$ $(\beta=1,2,\cdots,n-1)$ is a fundamental system of solutions of

$$\frac{dr_\alpha}{dt}=\sum_{\gamma=1}^{n-1} R_{\alpha\gamma}(t) r_\gamma \qquad (\alpha=1, 2, \cdots, n-1),$$

therefore, from (3.1.8), it follows that

$$\rho_\alpha^{(1)}(t)=\sum_{\beta=1}^{n-1} \rho_{\alpha\beta}(t) \int_0^t \sum_{\gamma=1}^{n-1} \bar{\rho}_{\beta\gamma}\Big(\sum_{i=1}^{n} \xi_i^{(\gamma)} X_i^{(1)}\Big) dt, \tag{3.1.9}$$

where $\|\bar{\rho}_{\alpha\beta}\|=\|\rho_{\alpha\beta}\|^{-1}$.

Then, in order that the characteristic C may be closed with a period near ω_0, namely that the solution corresponding to c_β may be periodic with a period near ω_0, it is necessary and sufficient that

$$\rho_\alpha\Big(\frac{\omega_0}{2}, c, \lambda\Big)=\rho_\alpha\Big(-\frac{\omega_0}{2}, c, \lambda\Big) \qquad (\alpha=1, 2, \cdots, n-1), \tag{3.1.10}$$

namely, from (3.1.4) and (3.1.6), that

$$\sum_{\beta=1}^{n-1}\Big\{\rho_{\alpha\beta}\Big(\frac{\omega_0}{2}\Big)-\rho_{\alpha\beta}\Big(-\frac{\omega_0}{2}\Big)\Big\}c_\beta+\varepsilon\Big\{\rho_\alpha^{(1)}\Big(\frac{\omega_0}{2}\Big)-\rho_\alpha^{(1)}\Big(-\frac{\omega_0}{2}\Big)\Big\}+[c, \varepsilon]_2=0 \qquad (\alpha=1, 2, \cdots, n-1). \tag{3.1.11}$$

Consequently, if

$$\det.\Big|\rho_{\alpha\beta}\Big(\frac{\omega_0}{2}\Big)-\rho_{\alpha\beta}\Big(-\frac{\omega_0}{2}\Big)\Big| \neq 0, \tag{3.1.12}$$

the values of c_β are computed approximately by solving the linear equations

$$\sum_{\beta=1}^{n-1}\Big\{\rho_{\alpha\beta}\Big(\frac{\omega_0}{2}\Big)-\rho_{\alpha\beta}\Big(-\frac{\omega_0}{2}\Big)\Big\}c_\beta+\varepsilon\Big\{\rho_\alpha^{(1)}\Big(\frac{\omega_0}{2}\Big)-\rho_\alpha^{(1)}\Big(-\frac{\omega_0}{2}\Big)\Big\}=0 \qquad (\alpha=1, 2, \cdots, n-1). \tag{3.1.13}$$

The condition (3.1.10) may also be written as

$$\rho_\alpha(\omega_0, c, \lambda)=c_\alpha \qquad (\alpha=1, 2, \cdots, n-1), \tag{3.1.10$'$}$$

for which, corresponding to (3.1.12), the condition

$$\det.|\rho_{\alpha\beta}(\omega_0)-\delta_{\alpha\beta}| \neq 0 \tag{3.1.12$'$}$$

suffices. Equivalence of this condition to (3.1.12) is easily seen as in Remark 2 of 1.1.

From the the condition (3.1.11), we see also that the condition (3.1.12) or (3.1.12′) assures the unique existence of a periodic solution for $\lambda=a_0+\varepsilon$. In this case, from (3.1.2), it is readily seen that the period ω of this periodic solution becomes

$$\omega=\int_0^{\omega_0}\frac{\sum_{k=1}^{n}X_k^2+\sum_{\alpha=1}^{n-1}\rho_\alpha(t,c,\lambda)\Big(\sum_{k=1}^{n}\frac{d\xi_k^{(\alpha)}}{dt}X_k\Big)}{\sum_{k=1}^{n}X_k\cdot X_k(\varphi+\sum_\alpha\rho_\alpha(t,c,\lambda)\xi^{(\alpha)},\lambda)}dt, \tag{3.1.14}$$

from which, evidently, $\omega(a_0)=\omega_0$ and $\omega(\lambda)$ is expanded with respect to ε.

When once c_α's are computed approximately by (3.1.13), the periodic solution for $\lambda=a_0+\varepsilon$ and its period are computed starting from the solution corresponding to these c_α's by the method of the preceding chapter. In this computation, as is seen from 2.2, we can make use of $\rho_{\alpha\beta}$ and ω_0 of the periodic solution for $\lambda=a_0$ for those of the approximately periodic solution of the preceding chapter provided that $|\varepsilon|$ is sufficiently small.

So long as the condition (3.1.12) or (3.1.12′) holds, we can continue the above process, consequently we can find the periodic solution of (3.1.1) for any value of λ to which continuation of the above process is possible starting from $\lambda=a_0$.

When $\varepsilon=0$, from (3.1.4) and (3.1.6) follows

$$\rho_\alpha(\omega_0,c,a_0)=\sum\rho_{\alpha\beta}(\omega_0)c_\beta+[c]_2 \qquad (\alpha=1,2,\cdots,n-1).$$

Consequently we see that

1° *When the absolute values of the characteristic roots of* $\|\rho_{\alpha\beta}(\omega_0)\|$ *are all less than unity, the periodic solution* $x_i=\varphi_i(t)$ *is orbitally stable;*

2° *When at least one of the absolute values of the characteristic roots of* $\|\rho_{\alpha\beta}(\omega_0)\|$ *is greater than unity, the periodic solution* $x_i=\varphi_i(t)$ *is not orbitally stable;*

3° *When the absolute values of the characteristic roots of* $\|\rho_{\alpha\beta}(\omega_0)\|$ *are all equal or less than unity and at least one of them is equal to unity, the stability of the periodic solution* $x_i=\varphi_i(t)$ *depends upon the terms of the higher orders in the expansions of* $\rho_\alpha(\omega_0,c,a_0)$.

Since the functions $\rho_{\alpha\beta}(\omega_0)$ are continuous with the corresponding periodic solution as is seen from the above discussions, the orbital stability of the periodic solution is unaltered when λ varies continuously, so long as the stability can be discriminated according to the above criteria 1° or 2°.

3.2 Deformation of continuum of periodic solutions

Let us consider the case where the system (3.1.1) for $\lambda=a_0$ admits of a continuum of periodic solutions with continuous periods in the neighborhood of the periodic solution $x_i=\varphi_i(t)$.

Then, from (3.1.10′), the relations

$$\rho_\alpha(\omega_0,c,a_0)=c_\alpha \qquad (a=1,2,\cdots,n-1) \tag{3.2.1}$$

hold for any c_α such that $|c_\alpha|\leqslant 1$. Consequently, writing c_α as u_α, any periodic solution belonging to the continuum is expressed as

$$x_i=\varphi_i(t,u_1,u_2,\cdots,u_{n-1}) \qquad (i=1,2,\cdots,n) \tag{3.2.2}$$

and, from (3.1.14), its period $\omega=\omega(u)$ is analytic with respect to u_α. Let the closed characteristic corresponding to the periodic solution (3.2.2) be C_u. Then, if there exists a closed characteristic C for $\lambda=a_0+\varepsilon$ lying in the neighborhood of C_0 with a period near ω_0, there exists a characteristic C_u passing through the intersection of C and the normal hyperplane of C_0 at $x_i=\varphi_i(0)$. For such characteristic C_u, from (3.1.10′), it is valid that

$$\rho_\alpha\{\omega(u), 0, a_0+\varepsilon\}=0 \qquad (\alpha=1, 2,\cdots, n-1).$$

From (3.1.4) and (3.1.6), these conditions are written as follows:

$$\rho_\alpha^{(1)}\{\omega(u)\}+O(\varepsilon)=0 \qquad (\alpha=1, 2,\cdots, n-1). \tag{3.2.3}$$

Now, since (3.2.1) hold also for C_u, it is valid that

$$\rho_{\alpha\beta}\{\omega(u)\}=\delta_{\alpha\beta} \qquad (\alpha, \beta=1, 2,\cdots, n-1),$$

therefore, from (3.1.9), it becomes that

$$\rho_\alpha^{(1)}\{\omega(u)\}=\int_0^{\omega(u)}\sum_{\beta=1}^{n-1}\bar{\rho}_{\alpha\beta}\left(\sum_{i=1}^{n}\xi_i^{(\beta)}X_i^{(1)}\right)dt \quad (\alpha=1, 2,\cdots, n-1). \tag{3.2.4}$$

From (3.2.3), there exists a periodic solution for $\lambda=a_0+\varepsilon$ only when there exists a solution $u_\alpha^{(0)}$ of the equations

$$\rho_\alpha^{(1)}\{\omega(u)\}=0 \qquad (\alpha=1, 2,\cdots, n-1). \tag{3.2.5}$$

Conversely, if there exists a solution $u_\alpha^{(0)}$ of the above equations and moreover

$$\det.|\partial\rho_\alpha^{(1)}\{\omega(u^{(0)})\}/\partial u_\beta^{(0)}|\neq 0, \tag{3.2.6}$$

then, evidently, there exists a unique solution $\tilde{u}_\alpha$ of the equations (3.2.3), namely, there exists a unique periodic solution for $\lambda=a_0+\varepsilon$. The period of this periodic solution is calculated from (3.1.2) as follows:

$$\begin{aligned}\tilde{\omega}&=\int_0^{\omega(\tilde{u})}\frac{d\tau}{dt} \\ &=\int_0^{\omega(\tilde{u})}\frac{\sum_{k=1}^{n}X_k^2+\sum_{\alpha=1}^{n-1}\rho_\alpha\left(\sum_{k=1}^{n}\frac{d\xi_k^{(\alpha)}}{dt}X_k\right)}{\sum_{k=1}^{n}X_k\left\{X_k+\sum_{\alpha=1}^{n-1}\rho_\alpha\left(\sum_{j=1}^{n}X_{kj}\xi_j^{(\alpha)}\right)+\varepsilon X_k^{(1)}+\cdots\right\}}dt \\ &=\omega(\tilde{u})+\int_0^{\omega(\tilde{u})}\left\{\frac{1}{\sum_{k=1}^{n}X_k^2}\cdot\sum_{\alpha=1}^{n-1}\rho_\alpha\left(\sum_{k=1}^{n}X_k\frac{d\xi_k^{(\alpha)}}{dt}-\sum_{k,j=1}^{n}X_kX_{kj}\xi_j^{(\alpha)}\right)\right. \\ &\qquad\left.-\frac{\varepsilon}{\sum_{k=1}^{n}X_k^2}\sum_{k=1}^{n}X_kX_k^{(1)}+[\rho, \varepsilon]_2\right\}dt,\end{aligned} \tag{3.2.7}$$

where $\rho_a=\rho_a(t, 0, \lambda)$ is a solution of (3.1.3) representing a periodic solution for $\lambda=a_0+\varepsilon$.

When the condition (3.2.6) is satisfied, the values $u_a^{(0)}$ are evidently computed by Newton's method if the approximate values of $u_a^{(0)}$ are known.

3.3 Stability of a periodic solution produced by deformation of continuum of periodic solutions

In this paragraph, let us investigate the stability of a periodic solution for $\lambda=a_0+\varepsilon$ obtained in the preceding paragraph. Putting $u_a=\tilde{u}_a+v_a$, we denote C_u by C_v, consequently $C_{\tilde{u}}$ by C_0. Let the equation of C_0 be

$$x_i = \varphi_i(t) \qquad (i=1, 2, \cdots, n)$$

and C_v' be the characteristic for $\lambda=a_0+\varepsilon$ through the intersection of C_v and the normal hyperplane

$$\pi:\ x_i = a_i+\sum_{a=1}^{n-1} \xi_i^{(a)} w_a$$

of C_0 at $A\{a_i=\varphi_i(0)\}$, where w_a's are the parameters.

Now, if the characteristic $C'_{v^{(0)}}$ crossing the normal hyperplane π in P crosses the normal hyperplane

$$\pi':\ x_i=a_i+\sum_{a=1}^{n-1}{}'\xi_i^{(a)}w_a$$

of C_0' in a point

$$Q:\ x_i=a_i+\sum_{a=1}^{n-1}{}'\xi_i^{(a)}c_a^{(0)},$$

then it is valid that

$$(3.3.1)\quad c_a^{(0)}=v_a^{(0)}+\varepsilon[v^{(0)}]_2 +O(\varepsilon^2),$$

provided that ${}'\xi_i^{(a)}=\xi_i^{(a)}+O(\varepsilon)$ are suitably chosen. For, let the equation of $C'_{v^{(0)}}$ be $x_i=\psi_i(t)$ and suppose that P corresponds to a point for $t=0$. Then, in Q, it holds that

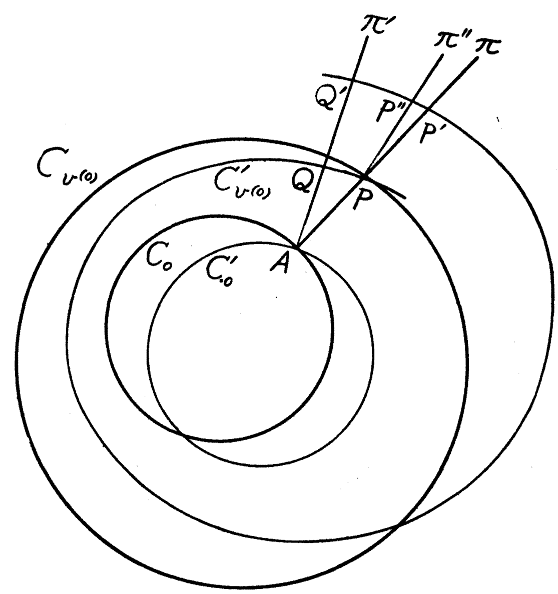

Fig. 3

$$\sum_{i=1}^{n} \{\psi_i(t)-a_i\}(X_i+\varepsilon X_i^{(1)}+\cdots)=0,$$

from which follows

$$\sum_{i=1}^{n} \{\psi_i(0)-a_i+t({}'X_i+\varepsilon\, {}'X_i^{(1)}+\cdots)+\cdots\}(X_i+\varepsilon X_i^{(1)}+\cdots)=0,$$

namely follows

$$t\sum_{i=1}^{n} X_i{}'X_i+\varepsilon\sum_{i=1}^{n}\{\psi_i(0)-a_i\}X_i^{(1)}+\cdots=0,$$

where $'X_i$, $'X_i^{(1)}$, $\cdots$ denote the values at P. From this follows that the time t required to reach Q from P is $\varepsilon[v^{(0)}]_1$. Now, from the assumption that $'\xi_i^{(\alpha)}=\xi_i^{(\alpha)}+O(\varepsilon)$, we can write $'\xi_i^{(\alpha)}$ in the forms:

$$'\xi_i^{(\alpha)}=\xi_i^{(\alpha)}+\varepsilon\sum_{\beta=1}^{n-1}\lambda_{\alpha\beta}\xi_i^{(\beta)}+\varepsilon\lambda_\alpha X_i. \tag{3.3.2}$$

Substitution of these into the relations

$$\sum_{i=1}^{n}{}'\xi_i^{(\alpha)}(X_i+\varepsilon X_i^{(1)}+\cdots)=0,\quad \sum_{i=1}^{n}{}'\xi_i^{(\alpha)}\,{}'\xi_i^{(\beta)}=\delta_{\alpha\beta},$$

entails

$$\begin{cases}\lambda_\alpha\sum_{i=1}^{n}X_i^2+\sum_{i=1}^{n}\xi_i^{(\alpha)}X_i^{(1)}+O(\varepsilon)=0,\\[2mm] \lambda_{\alpha\beta}+\lambda_{\beta\alpha}+\varepsilon\sum_{\gamma=1}^{n-1}\lambda_{\alpha\gamma}\lambda_{\beta\gamma}+\varepsilon\lambda_\alpha\lambda_\beta\sum_{i=1}^{n}X_i^2=0.\end{cases}$$

Consequently, if we choose $\lambda_{\alpha\beta}$ so that $\lambda_{\alpha\beta}=\lambda_{\beta\alpha}$ (this is evidently possible), we see that

$$\begin{cases}\lambda_\alpha=-\dfrac{\sum_{i=1}^{n}\xi_i^{(\alpha)}X_i^{(1)}}{\sum_{i=1}^{n}X_i^2}+O(\varepsilon),\\[2mm] \lambda_{\alpha\beta}=O(\varepsilon).\end{cases} \tag{3.3.3}$$

Then

$$c_\alpha^{(0)}=\sum_{i=1}^{n}\{\psi_i(t)-a_i\}{}'\xi_i^{(\alpha)}$$

are calculated as follows:

$$\begin{aligned}c_\alpha^{(0)}&=\sum_{i=1}^{n}\{\psi_i(0)-a_i+t('X_i+\varepsilon\,'X_i^{(1)}+\cdots)+\cdots\}\{\xi_i^{(\alpha)}+\varepsilon\lambda_\alpha X_i+O(\varepsilon^2)\}\\ &=\sum_{i=1}^{n}\left\{\sum_{\beta=1}^{n-1}\xi_i^{(\beta)}v_\beta^{(0)}+\varepsilon[v^{(0)}]_1\cdot X_i+\varepsilon[v^{(0)}]_2+O(\varepsilon^2)\right\}\{\xi_i^{(\alpha)}+\varepsilon\lambda_\alpha X_i+O(\varepsilon^2)\}\\ &=v_\alpha^{(0)}+\varepsilon[v^{(0)}]_2+O(\varepsilon^2),\end{aligned}$$

namely we have (3.3.1).

If we choose $|v_\alpha^{(0)}|$ sufficiently small, $C'_{v^{(0)}}$ crosses again the hyperplane π after the time nearly equal to $\omega(v^{(0)})$ ($\fallingdotseq\omega(0)\fallingdotseq\tilde{\omega}$) in a point

$$P':\ x_i=a_i+\sum_{\alpha=1}^{n-1}\xi_i^{(\alpha)}v_\alpha^{(1)}$$

near P, consequently, in the neighborhood of P', it crosses again the hyperplane π' in a point

$$Q': \ x_i = a_i + \sum_{\alpha=1}^{n-1} {}'\xi_i^{(\alpha)} c_\alpha^{(1)} .$$

Then, by (3.3.1), it is also valid that

$$c_\alpha^{(1)} = v_\alpha^{(1)} + \varepsilon [v^{(1)}]_2 + O(\varepsilon^2) . \tag{3.3.4}$$

Now, with respect to C_0', from (3.1.4), it holds that

$$c_\alpha^{(1)} = \sum_{\beta=1}^{n-1} \rho_{\alpha\beta}(\tilde{\omega}) c_\beta^{(0)} + [c^{(0)}]_2 ,$$

consequently, from (3.3.1) and (3.3.4), we have:

$$v_\alpha^{(1)} = \sum_{\beta=1}^{n-1} \rho_{\alpha\beta}(\tilde{\omega}) v_\beta^{(0)} + [v^{(0)}]_2 + \varepsilon [v^{(0)}]_2 + O(\varepsilon^2) . \tag{3.3.5}$$

Now, when $|v_\alpha^{(0)}|$'s are sufficiently small, analogously to ${}'\xi_i^{(\alpha)}$, we can choose the direction cosines ${}''\xi_i^{(\alpha)}$ of the mutually orthogonal normals of $C_{v^{(0)}}$ at P so that they may be written as follows:

$${}''\xi_i^{(\alpha)} = \xi_i^{(\alpha)} + \lambda_\alpha X_i + [v^{(0)}]_2$$

where $\lambda_\alpha = [v^{(0)}]_1$. Then, in the same manner as (3.3.1) was proved, it is easily proved that, in the neighborhood of P', $C_{v^{(0)}}'$ crosses the normal hyperplane π'' of $C_{v^{(0)}}$ at P in a point

$$P'': \ x_i = a_i + \sum_{\alpha=1}^{n-1} \xi_i^{(\alpha)} v_\alpha^{(0)} + \sum_{\alpha=1}^{n-1} {}''\xi_i^{(\alpha)} v_\alpha^{(2)},$$

for which it holds that

$$v_\alpha^{(2)} = v_\alpha^{(1)} - v_\alpha^{(0)} + [v^{(0)}]_2 . \tag{3.3.6}$$

Then, with respect to $C_{v^{(0)}}$, from (3.1.4), it holds that

$$v_\alpha^{(2)} = \varepsilon \rho_\alpha^{(1)} \{\omega(v^{(0)})\} + O(\varepsilon^2) .$$

Since $\rho_\alpha^{(1)}\{\omega(0)\} = O(\varepsilon)$ from (3.2.3), for sufficiently small $|v_\alpha^{(0)}|$, the above relations are written as follows:

$$v_\alpha^{(2)} = \varepsilon \sum_{\beta=1}^{n-1} \sigma_{\alpha\beta} v_\beta^{(0)} + \varepsilon [v^{(0)}]_2 + O(\varepsilon^2) , \tag{3.3.7}$$

where

$$\sigma_{\alpha\beta} = \left. \frac{\partial \rho_\alpha^{(1)} \{\omega(v)\}}{\partial v_\beta} \right|_{v_\gamma = 0} . \tag{3.3.8}$$

Then, substituting (3.3.5) and (3.3.7) into (3.3.6), we get:

$$\sum_{\beta=1}^{n-1} \rho_{\alpha\beta}(\tilde{\omega})v_\beta^{(0)} - v_\alpha^{(0)} + [v^{(0)}]_2 + \varepsilon[v^{(0)}]_2 + O(\varepsilon^2) = \varepsilon \sum_{\beta=1}^{n-1} \sigma_{\alpha\beta} v_\beta^{(0)} + \varepsilon[v^{(0)}]_2 + O(\varepsilon^2),$$

consequently, for sufficiently small $|v_\alpha^{(0)}|$ and $|\varepsilon|$, neglecting the terms of the higher orders, we have:

$$\sum_{\beta=1}^{n-1} \{\rho_{\alpha\beta}(\tilde{\omega}) - \delta_{\alpha\beta} - \varepsilon\sigma_{\alpha\beta}\} v_\beta^{(0)} = 0.$$

Here $v_\alpha^{(0)}$ are arbitary values such that their absolute values are sufficiently small, consequently, within the first order of ε, we have:

$$\rho_{\alpha\beta}(\tilde{\omega}) = \delta_{\alpha\beta} + \varepsilon\sigma_{\alpha\beta} \tag{3.3.9}$$

or

$$\|\rho_{\alpha\beta}(\tilde{\omega})\| = \exp(\varepsilon\|\sigma_{\alpha\beta}\|). \tag{3.3.10}$$

Thus, when $|\varepsilon|$ is sufficiently small, from the end of **3.1**, we obtain the criteria on the orbital stability of the periodic solution corresponding to C_0' as follows:

1° *When the real parts of the characteristic roots of* $\varepsilon\|\sigma_{\alpha\beta}\|$ *are all negative, the absolute values of the characteristic roots of* $\|\rho_{\alpha\beta}(\tilde{\omega})\|$ *are all less than unity and the periodic solution is orbitally stable;*

2° *When at least one of the characteristic roots of* $\varepsilon\|\sigma_{\alpha\beta}\|$ *has a positive real part, the absolute value of at least one of the characteristic roots of* $\|\rho_{\alpha\beta}(\tilde{\omega})\|$ *is greater than unity and the periodic solution is not orbitally stable.*

Since, from (3.2.3), there holds

$$\tilde{u}_\alpha = u_\alpha^{(0)} + O(\varepsilon) \qquad (\alpha = 1, 2, \cdots, n-1),$$

from (3.3.8) follows

$$\frac{\partial\rho_\alpha^{(1)}\{\omega(u^{(0)})\}}{\partial u_\beta^{(0)}} = \sigma_{\alpha\beta} + O(\varepsilon),$$

consequently, when $|\varepsilon|$ is sufficiently small, the above criteria on the stability holds good also when $\sigma_{\alpha\beta}$'s are replaced by $\partial\rho_\alpha^{(1)}\{\omega(u^{(0)})\}/\partial u_\beta^{(0)}$.

Remark. When $\det.|\rho_{\alpha\beta}(\tilde{\omega}) - \delta_{\alpha\beta}| \neq 0$, namely when $\det.|\varepsilon\sigma_{\alpha\beta}| \neq 0$ for sufficiently small $|\varepsilon|$, the method of the present chapter, combined with that of **2.2**, is used in a following way:

1° we compute the solution $u_\alpha^{(0)}$ of (3.2.5) by Newton's method;

2° by the method of **2.2**, we compute the periodic solution for $\lambda = a_0 + \varepsilon$ starting from the periodic solution for $\lambda = a_0$ corresponding to $u_\alpha^{(0)}$ obtained in 1°;

3° by the method of **3.1**, we proceed to compute the periodic solution for further values of λ.

Appendix

Two dimensional autonomous system

If the normal of the characteristic of (2.1.1) is oriented so that its direction cosines may be

(1) $$\xi_1 = -\frac{X_2}{\sqrt{X_1^2+X_2^2}}, \quad \xi_2 = \frac{X_1}{\sqrt{X_1^2+X_2^2}},$$

then, the equation (2.2.9) becomes

(2) $$\frac{d\rho_1}{dt} = \left\{\left(\frac{\partial X_1}{\partial x_1}+\frac{\partial X_2}{\partial x_2}\right) - \frac{d}{dt}\log\sqrt{X_1^2+X_2^2}\right\}\rho_1,$$

therefore we have:

(3) $$\rho_1 = \frac{\sqrt{\mathring{X}_1^2+\mathring{X}_2^2}}{\sqrt{X_1^2+X_2^2}}e^{h(t)},$$

where $\mathring{X}_i = X_i\{\varphi(0)\}$ and

(4) $$h(t) = \int_0^t \left(\frac{\partial X_1}{\partial x_1}+\frac{\partial X_2}{\partial x_2}\right)dt.$$

Put

$$x_i' = \varphi_i(-\omega_0/2), \qquad X_i' = X_i(x'),$$

then, since

$$\rho_1\left(\frac{\omega_0}{2}\right) \doteqdot \frac{\sqrt{\mathring{X}_1^2+\mathring{X}_2^2}}{\sqrt{X_1'^2+X_2'^2}}e^{h(\omega_0/2)},$$

from (3), the equation (2.2.12) is solved as follows:

(5) $$c = \frac{\{\varphi_1(\tilde{\omega}/2)-x_1'\}X_2' - \{\varphi_2(\tilde{\omega}/2)-x_2'\}X_1'}{\sqrt{\mathring{X}_1^2+\mathring{X}_2^2}\,(e^{h(\omega_0/2)}-e^{h(-\omega_0/2)})},$$

provided that

(6) $$h\left(\frac{\omega_0}{2}\right) - h\left(-\frac{\omega_0}{2}\right) = \int_{-\omega_0/2}^{\omega_0/2}\left(\frac{\partial X_1}{\partial x_1}+\frac{\partial X_2}{\partial x_2}\right)dt \neq 0.$$

When the direction cosines of the normal of the characteristic of (3.1.1) for $\lambda = a_0$ are assumed to be (1), the equations (3.1.7) and (3.1.8) become

(7) $$\frac{d\rho_1}{dt} = \left\{\left(\frac{\partial X_1}{\partial x_1}+\frac{\partial X_2}{\partial x_2}\right) - \frac{d}{dt}\log\sqrt{X_1^2+X_2^2}\right\}\rho_1,$$

(8) $$\frac{d\rho^{(1)}}{dt} = \left\{\left(\frac{\partial X_1}{\partial x_1}+\frac{\partial X_2}{\partial x_2}\right) - \frac{d}{dt}\log\sqrt{X_1^2+X_2^2}\right\}\rho^{(1)} + \frac{1}{\sqrt{X_1^2+X_2^2}}(X_1X_2^{(1)} - X_2X_1^{(1)}),$$

therefore, from (3) and (3.1.9), we have:

$$(9)\qquad \begin{cases} \rho_1 = \dfrac{\sqrt{\overset{\circ}{X}{}_1^2 + \overset{\circ}{X}{}_2^2}}{\sqrt{X_1^2 + X_2^2}} e^{h(t)}, \\ \rho^{(1)} = \dfrac{1}{\sqrt{X_1^2 + X_2^2}} e^{h(t)} I(t), \end{cases}$$

where

$$(10)\qquad I(t) = \int_0^t e^{-h(t)} (X_1 X_2^{(1)} - X_2 X_1^{(1)})\, dt.$$

Then, when (6) is valid[1], the equation (3.1.13) is solved as follows:

$$(11)\qquad c = -\frac{e^{h(\omega_0/2)} I(\omega_0/2) - e^{h(-\omega_0/2)} I(-\omega_0/2)}{\sqrt{\overset{\circ}{X}{}_1^2 + \overset{\circ}{X}{}_2^2}\, (e^{h(\omega_0/2)} - e^{h(-\omega_0/2)})} \varepsilon.$$

When the system (3.1.1) for $\lambda = a_0$ admits of a continuum of periodic solutions[2], the equation (3.2.5) becomes

$$(12)\qquad I\{\omega(u)\} = 0$$

and the periodic solution for $\lambda = a_0 + \varepsilon$ is determined uniquely corrèsponding to a root $u^{(0)}$ of (12) provided that

$$(13)\qquad \left. \frac{dI\{\omega(u)\}}{du} \right|_{u=u^{(0)}} \neq 0.$$

When $|\varepsilon|$ is sufficiently small, the relation (3.3.10) expresses that

$$(14)\qquad \int_0^{\tilde{\omega}} \left(\frac{\partial X_1}{\partial x_1} + \frac{\partial X_2}{\partial x_2} \right) dt = \frac{\varepsilon}{\sqrt{\overset{\circ}{X}{}_1^2 + \overset{\circ}{X}{}_2^2}} \cdot \frac{dI\{\omega(u^{(0)})\}}{du^{(0)}},$$

where the left-hand side expresses the integral taken along the periodic solution with period $\tilde{\omega}$ for $\lambda = a_0 + \varepsilon$.

The formulas (9), (11) and (12) are those already obtained in the previous paper [1], but the relation (14) is a relation newly found in this paper.

References

1. M. Urabe, *Infinitesimal deformation of cycles*, J. Sci. Hiroshima Univ., Ser. A, **18**, 37–53 (1954).
2. D. R. Hartree, *Notes on iterative processes*, Proc. Cambridge Philos. Soc., **45**, 230–236 (1949).

Department of Mathematics,
Faculty of Science,
Hiroshima University

1) In the two dimensional case, the condition of continuity of periods is evidently unnecessary.
2) In the two dimensional case, the condition of continuity of periods is unnecessary, because (3.2.1) hold without it.