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Jordan and Jordan Triple Isomorphisms of Rings

By

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A *Jordan homomorphism* or semi-homomorphism of an associative ring $\mathfrak{A}$ into an associative ring $\mathfrak{B}$ is defined as a mapping $a \rightarrow a'$ such that

(I) $(a+b)'=a'+b'$,

(II) $(ab)'+(ba)'=a'b'+b'a'$.

In the ring $\mathfrak{B}$, if $2x'=0$ implies $x'=0$, it is called $\mathfrak{B}$ *has not additive order* 2. It is well known that, on the assumption that the additive order of $\mathfrak{B}$ is not 2, the additive mapping (II) is equivalent to the following:

(II)′ $(a^2)'=(a')^2$

and implies (III):

(III) $(aba)'=a'b'a'$.

In this paper we will consider the meanning of the mapping (III) (Theorem 1) and for the prime ring prove the generalization of G. Ancochea's theorem [1][1] (Theorem 2). Also, we will show a result similar to Jacobson-Rickart's theorem [3] for the one-to-one mapping (I), (III) (Theorem 3). Our principal result (Theorem 2) is based on the identities in Lemma 1 and 2. Recently, I. N. Herstein has proved some theorems for the Jordan homomorphisms [2]. His Theorem H is similar to our Theorem 2. The difference between his result and ours is that we do not require that the additive order of the image ring is not 3.

1. We may get the Jordan ring $\mathfrak{A}_j$ from the associative ring $\mathfrak{A}$ by introducing Jordan product $\{a, b\}=ab+ba$ for any pair of elements a, b in $\mathfrak{A}$. Then we can regard the Jordan homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$ as the homomorphism of $\mathfrak{A}_j$ into $\mathfrak{B}_j$. Such relation holds for the mapping (I), (III).

Theorem 1. *Let $a \rightarrow a'$ be an additive mapping which satisfies* (III) *of a ring $\mathfrak{A}$ into a ring $\mathfrak{B}$ of additive order different from* 2, *then it is a Jordan triple homomorphism, that is $\{\{a, b\}c\}'=\{\{a', b'\}c'\}$ for any $a, b, c \in \mathfrak{A}$. And conversely.*

Proof. For arbitrary elements $a, b, c \in \mathfrak{A}$

$$(abc+cba)'=((a+c)b(a+c)-aba-cbc)'=a'b'c'+c'b'a'.$$

Hence, $\{\{a, b\}c\}'=(abc+cba)'+(bac+cab)'=\{\{a', b'\}c'\}$. Conversely, $2\{a^2, b\}'=\{\{a, a\}b\}'=2\{(a')^2, b'\}$. Since the additive order of $\mathfrak{B}$ is not 2, we have $\{a^2, b\}'=\{(a')^2, b'\}$. Therefore, $2(aba)'=\{\{a, b\}a\}'-\{a^2, b\}'=2a'b'a'$. Thus, this theorem is proved.

1) Numbers in brackets refer to the references at the end of the paper.

In a similar fashion we can see the following:

PROPOSITION. *Let $\widetilde{\mathfrak{A}}$ be the additive subgroup of an associative ring $\mathfrak{A}$, and $\mathfrak{A}$ admits the operator $\frac{1}{2}$.*[2] *If $\widetilde{\mathfrak{A}}$ is closed under the multiplication aba, $\widetilde{\mathfrak{A}}$ is a Jordan triple subsystem of $\mathfrak{A}$.*[3] *And conversely.*

Hence we can regard the additive mapping which satisfies (III) of a ring $\mathfrak{A}$ into a ring $\mathfrak{B}$ as the homomorphism of $\mathfrak{A}_{j_3}$ into $\mathfrak{B}_{j_3}$, where $\mathfrak{A}_{j_3}$, $\mathfrak{B}_{j_3}$ are Jordan triple systems determined by $\mathfrak{A}$, $\mathfrak{B}$ respectively. We call therefore the additive mapping which satisfies (III) the *Jordan triple homomorphism* of $\mathfrak{A}$ into $\mathfrak{B}$.

2. Jacobson-Rickart showed some identities in the mapping such that holds (I), (II)′ and (III) [3]. For instance,

(1) $((ab)'-a'b')((ab)'-b'a')=0,$

(2) $((ab)'-b'a')((ab)'-a'b')=0,$

(3) $((abc)'-a'b'c')b'((abc)'-c'b'a')=0,$ *for all* $a, b, c \in \mathfrak{A}$.

We can prove the next identity which is useful in the following.

LEMMA 1. *Let $a \to a'$ be a Jordan homomorphism of a ring $\mathfrak{A}$ into a ring $\mathfrak{B}$ of additive order different from 2. Then for all $a, b, c \in \mathfrak{A}$*

(4) $((ab)'-a'b')c'((ab)'-b'a')+((ab)'-b'a')c'((ab)'-a'b')=0.$

PROOF. For simplicity, we write $(ab)'-a'b' \equiv x'$, $(ab)'-b'a' \equiv y'$. Then $x'c'y'+y'c'x'$

$$=2(abcab)'-(ab)'c'(ab+ba)'-(ab+ba)'c'(ab)'+(abcba)'+(bacab)'$$
$$=(2abcab-abc(ab+ba)-(ab+ba)cab+abcba+bacab)'$$
$$=0.$$

Using (1) and (2) we obtain immediately the following:

COROLLARY. *On the assumption of the Lemma 1*

(5) $\{\{c', (ab)'-a'b'\}(ab)'-b'a'\}=0,$

(6) $[[c', (ab)'-a'b'](ab)'-b'a']=0,$[4] *for all* $a, b, c \in \mathfrak{A}$.

LEMMA 2. *On the assumption of the above Lemma 1, holds the following:*

(7) $x'(x'c'+c'y')+(x'c'+c'y')y'=(c[a,b]^2)'$ *for every* $a, b, c \in \mathfrak{A}$. *Where* $x' \equiv (ab)'-a'b'$, $y' \equiv (ab)'-b'a'$.

PROOF. Since $x'c'+c'y'=(c[a,b])'$, the left side of (7) is equal to $((ab)'-a'b')(c[a,b])'+(c[a,b])'((ab)'-b'a')$

$$=(abc[a,b]+c[a,b]ab-abc[a,b]-c[a,b]ba)'$$
$$=(c[a,b]^2)'$$

The ring $\mathfrak{B}$ is called a *prime ring* if $a'\mathfrak{B}b'=(0)$ implies $a'=0$ or $b'=0$.[5]

LEMMA 3. *Let $a \to a'$ be the Jordan isomorphism of a ring $\mathfrak{A}$ onto a*

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2) This means that $2x=a$ has a unique solution $(\frac{1}{2})a$ for every a in $\mathfrak{A}$.

3) The additive subgroup $\widetilde{\mathfrak{A}}$ of $\mathfrak{A}$ is called the Jordan triple system when $\widetilde{\mathfrak{A}}$ is closed under the multiplication $\{\{a, b\}c\}$. Such a most simple example is the subspace of quaternion on the complex field, which has the basis i, j or i, j and k. They are not the Jordan algebras.

4) $[a, b]$ means the Lie product $ab-ba$.

5) If there exists a element $v \neq 0$ in the prime ring $\mathfrak{B}$ such that $mv=0$ (m: positive integer), then $mz=0$ for all z in $\mathfrak{B}$. Because, since $mv\mathfrak{B}z=(0)$ $v\mathfrak{B}mz=(0)$, hence we obtain $mz=0$. I owe this remark to Prof. K. Morinaga.

prime ring $\mathfrak{B}$ *of additive order different from* 2, *then this mapping is either an isomorphism or an anti-isomorphism.*

PROOF. We shall suppose that $x' \neq 0$ and $y' \neq 0$, then using (1) and (2), from (4) by multiplying x', y' we obtain $x'x'c'y'=0$ and $y'y'c'x'=0$ for every $c \in \mathfrak{A}$. Since the ring $\mathfrak{B}$ is prime, the above relations implies $(x^2)'=(y^2)'=0$. It is clear that $x'+y'=[a, b]'$, hence $(x^2)'+(y^2)'=([a, b]^2)'$, also $[a, b]^2=0$. If we substitute the above values in (7), $2x'c'y'=0$ for every $c' \in \mathfrak{B}$. This contradicts our assumption. Therefore, it follows that either $(ab)'=a'b'$ or $(ab)'=b'a'$. Hence by the generalization of Hua's theorem ([2] Lemma 1), $'$ is either an isomorphism or an anti-isomorphism.

LEMMA 4. *Let* $a \rightarrow a'$ *be the Jordan homomorphism of a ring* $\mathfrak{A}$ *onto a prime ring* $\mathfrak{B}$ *of additive order different from* 2 *and* K *the kernel of this mapping. Then* K *is the two-sided ideal of* $\mathfrak{A}$.

PROOF. It is clear that K is an additive subgroup of $\mathfrak{A}$. In (4) let $b=k$, $k \in K$, then we obtain $2(ak)'c'(ak)'=0$ for all $c' \in \mathfrak{B}$, that is $(ak)'c'(ak)'=0$. Since $\mathfrak{B}$ is the prime ring, it follows $(ak)'=0$. Hence $ak \in K$ for all $a \in \mathfrak{A}$. Similarly, if put $a=k$, $k \in K$, then $kb \in K$ for all $b \in \mathfrak{A}$.

THEOREM 2. *Let* $a \rightarrow a'$ *be the Jordan homomorphism of a ring* $\mathfrak{A}$ *onto a prime ring* $\mathfrak{B}$ *of additive order different from* 2, *then this mapping is either a homomorphism or an anti-homomorphism.*

PROOF. Let K be the kernel of the Jordan homomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$, then K is the two-sided ideal of $\mathfrak{A}$ from Lemma 4. Hence, the factor group $\mathfrak{A}/K$ becomes the quotient ring modulo K by the definition that the product of two cosets of ideal K is $(a+K)(b+K)=ab+K$. And the Jordan homomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$ induces the Jordan isomorphism of a ring $\mathfrak{A}/K$ onto $\mathfrak{B}$. Therefore, from Lemma 3 this mapping is an isomorphism or an anti-isomorphism and $'$ is a homomorphism or an anti-homomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$. q.e.d.

3. Next we will prove the Jacobson-Rickart's theorem ([2] Theorem 2) for Jordan triple isomorphism.

THEOREM 3. *Let* $a \rightarrow a'$ *be the Jordan triple isomorphism of a ring* $\mathfrak{A}$ *into a ring* $\mathfrak{B}$ *which has not a divisor of zero. Then, this mapping is a ternary isomorphism or a ternary anti-isomorphism, that is,*

$$(abc)'=a'b'c' \text{ for all } a, b, c \in \mathfrak{A} \quad \text{or} \quad (abc)'=c'b'a' \text{ for all } a, b, c \in \mathfrak{A}.$$

PROOF. From (3), we have for any a, b, c in $\mathfrak{A}$ (A): $b'=0$ or (B): $(abc)'=a'b'c'$ or (C): $(abc)'=c'b'a'$. If every $b'=0$ our result is obvious. Therefore, it is sufficient to prove in the case some $b' \neq 0$. In this case, if both (B) and (C) hold for all $a, c \in \mathfrak{A}$ then the theorem is correct. Hence for some a, c let (α): $(abc)'=a'b'c' \neq c'b'a'$ $(b' \neq 0)$, then (i) $(abe)'=a'b'e'$ for every $e \in \mathfrak{A}$, because if suppose $(abe)'=e'b'a' \neq a'b'e'$, then $(ab(c+e))'=a'b'(c'+e')$ or $=(c'+e')b'a'$, but since $(ab(c+e))'=a'b'c'+e'b'a'$, this leads to a contradiction. Similarly, we can obtain on the same assumption, for every $e \in \mathfrak{A}$ (ii) $(ebc)'=e'b'c'$ and (iii) $(aec)'=a'e'c'$, since in the case $e'+b'=0$ this expression is also correct. Next, if we suppose that (β): $(abc)'=c'b'a'\neq a'b'c'$, then by the same way for every $e\in\mathfrak{A}$ (iv) $(abe)'=e'b'a'$, (v) $(ebc)'=c'b'e'$, (vi) $(aec)'=c'e'a'$. Moreover, from (α) (vii) $(ebf)'=e'b'f'$ for all $e, f\in\mathfrak{A}$. Because, let $(ebf)'=f'b'e'\neq e'b'f'$ then

$$((a+e)b(c+f))' \quad =(a'+e')b'(c'+f')$$
$$\text{or } =(c'+f')b'(a'+e')$$

but the left side is equal to $a'b'c'+a'b'f'+e'b'c'+f'b'e'$, from (α), (i) and (ii), hence the former implies a contradiction. For the latter, the left side also can be written $a'b'c'+c'b'e'+f'b'a'+f'b'e'$, from (α), (iv) and (v), hence it is impossible. Similarly for all $e, f\in\mathfrak{A}$ (viii) $(aef)'=a'e'f'$, (ix) $(efc)'=e'f'c'$. And on the assumption (β) for all $e, f\in\mathfrak{A}$ (x) $(ebf)'=f'b'e'$, (xi) $(aef)'=f'e'a'$, (xii) $(efc)'=c'f'e'$. Hence we can conclude that (xiii): $(efg)'=e'f'g'$ for every $e, f, g\in\mathfrak{A}$. Because, let $(efg)'=g'f'e'\neq e'f'g'$ $(f'\neq 0)$, then

$$((a+e)(b+f)(c+g))' \quad =(a'+e')(b'+f')(c'+g'),$$

but the left side is $a'b'c'+a'b'g'+a'f'c'+a'f'g'+e'b'c'+e'b'g'+e'f'c'+g'f'e'$ by (α), (vii), (viii) and (ix), hence $e'f'g'=g'f'e'$, this is a contradiction,

$$\text{or } =(c'+g')(b'+f')(a'+e'),$$

but the left side can be written $a'b'c'+c'b'e'+c'f'a'+c'f'e'+g'b'a'+g'b'e'+g'f'a'+g'f'e'$ by (α), (x), (xi) and (xii), hence $c'b'a'=a'b'c'$, this is impossible. And in the case $f'+b'=0$, the relation (xiii) is also correct by (vii). Also we obtain on the assumption (β) for all $e, f, g\in\mathfrak{A}$ $(efg)'=g'f'e'$.

From the above proof, it is clear that

COROLLARY. *For Jordan triple homomorphism of a ring $\mathfrak{A}$ into a ring $\mathfrak{B}$ always if $(abc)'=a'b'c'$ or $(abc)'=c'b'a'$ then this mapping is either a ternary homomorphism or a ternary anti-homomorphism.*

PROPOSITION. *Let $a\to a'$ be the Jordan triple homomorphism of a ring $\mathfrak{A}$ into a ring $\mathfrak{B}$. Then for every $a, b, c, e\in\mathfrak{A}$.*

$$(8)\quad ((abc)'-a'b'c')e'((abc)'-c'b'a')+((abc)'-c'b'a')e'((abc)'-a'b'c')=0.$$

PROOF. The left side of (8) is equal to

$$\begin{aligned}&2(abceabc)'-(abc)'e'(abc+cba)'-(abc+cba)'e'(abc)'\\&\qquad+(abcecba)'+(cbaeabc)'\\&=0.\end{aligned}$$

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References

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[3] N. Jacobson and C. E. Rickart: *Jordan homomorphisms of rings.* Trans. Amer. Math. Soc. vol. 69 (1950) pp. 479–502.

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