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Some Remarks on Zariski Rings

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Introduction. Given a Noetherian ring A with identity and an ideal $\mathfrak{m}$ of A such that $\bigcap_{n=1}^{\infty} \mathfrak{m}^n=(0)$, we may topologize A by adopting $\{\mathfrak{m}^n;\ n=1, 2,\cdots\}$ as a fundamental system of neighbourhoods of zero. This topologized ring is usually referred to as an $\mathfrak{m}$-adic ring, and is called a Zariski ring if its ideals are all closed. An $\mathfrak{m}$-adic ring is a Zariski ring if and only if $\mathfrak{m}$ is contained in its Jacobson radical, that is to say, the intersection of all its maximal ideals. In this note, unless otherwise stated, A will denote an $\mathfrak{m}$-adic Zariski ring and $\mathfrak{a}, \mathfrak{b}, \mathfrak{p}, \mathfrak{q}$ ideals of A; $\hat{A}$ will denote the completion of A and $\hat{\mathfrak{p}}, \hat{\mathfrak{q}}$ ideals of $\hat{A}$.

Now the following properties (α), (β) and (γ) are usually derived from the property $(\mathfrak{a}:cA)\hat{A}=\mathfrak{a}\hat{A}:c\hat{A}\ (c\in A)$ ([11], p. 353, Lemma 1; [9], p. 9, Proposition 1).

(α) $(\mathfrak{a}\cap\mathfrak{b})\hat{A}=\mathfrak{a}\hat{A}\cap\mathfrak{b}\hat{A}$ ([4], p. 54, Theorem 1).

(β) $(\mathfrak{a}:\mathfrak{b})\hat{A}=\mathfrak{a}\hat{A}:\mathfrak{b}\hat{A}$.

(γ) Let $\mathfrak{q}$ be $\mathfrak{p}$-primary and $\hat{\mathfrak{p}}$ be any prime divisor of $\mathfrak{q}\hat{A}$, then $\hat{\mathfrak{p}}\cap A=\mathfrak{p}$ ([1], p. 699, Proposition 6; [9], p. 9, Corollary 2).

(β) is proved as follows: $\mathfrak{a}:\mathfrak{b}=\mathfrak{a}:(b_1,\cdots,b_n)=(\mathfrak{a}:b_1A)\cap\cdots\cap(\mathfrak{a}:b_nA)$, so $(\mathfrak{a}:\mathfrak{b})\hat{A}=(\mathfrak{a}\hat{A}:b_1\hat{A})\cap\cdots\cap(\mathfrak{a}\hat{A}:b_n\hat{A})=\mathfrak{a}\hat{A}:\mathfrak{b}\hat{A}$.

Hence, from (α) and (β), we see that the mapping $\mathfrak{a}\rightarrow\mathfrak{a}\hat{A}$ is an isomorphism with respect to all the ideal-operations $(+,\ \cdot,\ :,\ \cap)$.

In § 1 we shall consider some relations between the prime divisors of $\mathfrak{a}$ and those of $\mathfrak{a}\hat{A}$. For proof, in addition to the above-mentioned properties, the following fact will be used: In a Noetherian ring with identity, a prime ideal $\mathfrak{p}$ is a prime divisor of $\mathfrak{a}$ if and only if $\mathfrak{p}=\mathfrak{a}:(p)$ for some $p\notin\mathfrak{a}$.

In § 2, as an application of the results obtained in § 1, the so-called transition theorem on lengths of primary ideals will be given for Zariski rings.

In § 3, by making use of Krull's Primidealkettensatz, some relations between maximal chains of prime ideals in A and those in $\hat{A}$ will be considered, and factorability will also be considered.

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§ 1. Prime divisors.

The following lemma is not new. But it plays an important role in this note, so we shall begin with it.

Lemma 1. *Let R be a Noetherian ring with identity and $\mathfrak{a}$ be an ideal of R. Then a prime ideal $\mathfrak{p}$ of R is a prime divisor of $\mathfrak{a}$ if and only if $\mathfrak{p}=\mathfrak{a}:(p)$ for some $p \notin \mathfrak{a}$.*

Proof. Let $\mathfrak{a}=\mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n$ be a normal decomposition and $\mathfrak{q}_i$ be $\mathfrak{p}_i$-primary $(i=1,\cdots,n)$. Take an element r such that $r \in \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_{i-1} \cap \mathfrak{q}_{i+1} \cap \cdots \cap \mathfrak{q}_n$ and $r \notin \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n$, then $r \notin \mathfrak{q}_i$ and $\mathfrak{a}:(r)=\mathfrak{q}_i:(r)$. Since $\mathfrak{q}_i:(r)$ is $\mathfrak{p}_i$-primary we shall have $\mathfrak{p}_i=\mathfrak{q}_i:(rr')$ for a suitable element r'. This shows that $\mathfrak{p}_i$ is of the form: $\mathfrak{p}_i=\mathfrak{a}:(p)$ $(p \notin \mathfrak{a})$. Conversely, let $\mathfrak{p}=\mathfrak{a}:(p)$ $(p \notin \mathfrak{a})$, then $\mathfrak{p}=(\mathfrak{q}_1:(p)) \cap \cdots \cap (\mathfrak{q}_n:(p))$. Since $\mathfrak{q}_i:(p)$ is $\mathfrak{p}_i$-primary or unit ideal $(i=1,\cdots,n)$, we may conclude that $\mathfrak{p}=\mathfrak{q}_i:(p)$ and $\mathfrak{p}=\mathfrak{p}_i$ for some i.

From this lemma we may obtain the following:

Lemma 2. *Let $\mathfrak{p}$ be any prime divisor of $\mathfrak{a}$. Then every prime divisor of $\mathfrak{p}\hat{A}$ is also a prime divisor of $\mathfrak{a}\hat{A}$.*

Proof. Let $\mathfrak{p}=\mathfrak{a}:pA$, then $\mathfrak{p}\hat{A}=\mathfrak{a}\hat{A}:p\hat{A}$. If $\hat{\mathfrak{p}}$ is a prime divisor of $\mathfrak{p}\hat{A}$, then $\hat{\mathfrak{p}}=\mathfrak{p}\hat{A}:\hat{p}\hat{A}$. So $\hat{\mathfrak{p}}=\mathfrak{a}\hat{A}:p\hat{p}\hat{A}$; that is, $\hat{\mathfrak{p}}$ is a prime divisor of $\mathfrak{a}\hat{A}$.

We want to prove the converse of this lemma, and for this purpose it is enough to prove the following:

Lemma 3. *Let $\mathfrak{q}$ be a $\mathfrak{p}$-primary ideal. Then $\mathfrak{q}\hat{A}$ and $\mathfrak{p}\hat{A}$ have the same prime divisors.*

Proof. Denote by s the length of $\mathfrak{q}$. For $s=1$ $(\mathfrak{q}=\mathfrak{p})$ the lemma is trivial. Assume then that the lemma is proved for $\mathfrak{p}$-primary ideals of lengths less than s. We may further assume that $\mathfrak{q}=(0)$; then we have $\mathfrak{p}=(0):pA$, where $p \neq 0$, $p \in \mathfrak{p}$. Let r be an element of A such that $r \notin \mathfrak{p}$ and that $\mathfrak{q}_1=pA:rA$ is the isolated $\mathfrak{p}$-primary component of pA, and let $\hat{\mathfrak{p}}_1,\cdots,\hat{\mathfrak{p}}_n$ be the prime divisors of $\mathfrak{p}\hat{A}$. Then by our inductive hypothesis $\hat{\mathfrak{p}}_1,\cdots,\hat{\mathfrak{p}}_n$ are the prime divisors of $\mathfrak{q}_1\hat{A}$. Now let $(0)=\hat{\mathfrak{q}}_1 \cap \cdots \cap \hat{\mathfrak{q}}_n \cap \hat{\mathfrak{q}}_{n+1} \cap \cdots \cap \hat{\mathfrak{q}}_{n+m}$ be a normal decomposition of the zero ideal of $\hat{A}$ and $\hat{\mathfrak{q}}_i$ be $\hat{\mathfrak{p}}_i$-primary $(i=1, 2, \cdots, n)$ (Lemma 2), where it may be assumed that $\mathfrak{q}_1\hat{A}\supseteq\hat{\mathfrak{q}}\cap\cdots\cap\hat{\mathfrak{q}}_n$. To prove the lemma it is sufficient to show that $\hat{\mathfrak{q}}_1\cap\cdots\cap\hat{\mathfrak{q}}_n\subseteq(0)$. Let $\hat{a}$ be any element of $\hat{\mathfrak{q}}_1\cap\cdots\cap\hat{\mathfrak{q}}_n$. Then we have $\hat{a}\in\mathfrak{q}_1\hat{A}$, and hence $r\hat{a}=p\hat{a}_1$, because $\mathfrak{q}_1\hat{A}=p\hat{A}:r\hat{A}$. On the other hand, from the facts that $\hat{\mathfrak{p}}_1,\cdots,\hat{\mathfrak{p}}_n$ are the prime divisors of $\mathfrak{p}\hat{A}$ and $\mathfrak{p}\hat{A}=(0):p\hat{A}$, we may conclude that $\mathfrak{p}\hat{A}=(\hat{\mathfrak{q}}_1:p\hat{A})\cap\cdots\cap(\hat{\mathfrak{q}}_n:p\hat{A})$ is a normal decomposition of $\mathfrak{p}\hat{A}$ (here, $(\hat{\mathfrak{q}}_{n+1}:p\hat{A})\cap\cdots\cap(\hat{\mathfrak{q}}_{n+m}:p\hat{A})$ can be omitted). Since $\hat{a}\in\hat{\mathfrak{q}}_i$, $r\hat{a}=p\hat{a}_1\in\hat{\mathfrak{q}}_i$ $(i=1, 2,\cdots, n)$; so $\hat{a}_1\in\mathfrak{p}\hat{A}$, and $p\hat{a}_1=r\hat{a}=0$. From this we have $\hat{a}=0$, because $r\notin\mathfrak{p}$, and this completes the proof of this lemma.

We now have

Theorem 1. *Let $\mathfrak{p}$ be any prime divisor of $\mathfrak{a}$. Then every prime divisor of $\mathfrak{p}\hat{A}$ is also a prime divisor of $\mathfrak{a}\hat{A}$, and conversely, any prime divisor of $\mathfrak{a}\hat{A}$ is a prime divisor of $\mathfrak{p}\hat{A}$ for some prime divisor $\mathfrak{p}$ of $\mathfrak{a}$. Moreover, if $\mathfrak{p}$ is isolated, then every isolated prime divisor of $\mathfrak{p}\hat{A}$ is also an isolated prime divisor of $\mathfrak{a}\hat{A}$, and conversely, any isolated prime divisor of $\mathfrak{a}\hat{A}$ is an isolated prime divisor of $\mathfrak{p}\hat{A}$ for some isolated prime divisor $\mathfrak{p}$ of $\mathfrak{a}$.*

To prove the second part, let $\hat{\mathfrak{p}}$ be any isolated prime divisor of $\mathfrak{a}\hat{A}$, then $\mathfrak{a}=\mathfrak{a}\hat{A}\cap A\subseteq\hat{\mathfrak{p}}\cap A$. So $\hat{\mathfrak{p}}\cap A$ will contain an isolated prime divisor $\mathfrak{p}$ of $\mathfrak{a}$. Then $\mathfrak{p}\hat{A}\subseteq\hat{\mathfrak{p}}$, and $\hat{\mathfrak{p}}$ will coincide with an isolated prime divisor of $\mathfrak{p}\hat{A}$. Conversely, let $\mathfrak{p}$ be any isolated prime divisor of $\mathfrak{a}$, then it will be seen easily from (γ) that every isolated prime divisor of $\mathfrak{p}\hat{A}$ is also an isolated prime divisor of $\mathfrak{a}\hat{A}$.

Lemma 4. *Let $\mathfrak{q}$ be $\mathfrak{p}$-primary and $\hat{\mathfrak{p}}_1,\cdots,\hat{\mathfrak{p}}_n$ be the prime divisors of $\mathfrak{p}\hat{A}$. Let $\mathfrak{q}\hat{A}=\hat{\mathfrak{q}}_1\cap\cdots\cap\hat{\mathfrak{q}}_n$ be a normal decomposition of $\mathfrak{q}\hat{A}$ and $\hat{\mathfrak{q}}_i$ be $\hat{\mathfrak{p}}_i$-primary $(i=1,\cdots, n)$. Then we have $\hat{\mathfrak{q}}_i\cap A=\mathfrak{q}$ $(i=1,\cdots, n)$.*

Proof. If $\mathfrak{q}\subset\hat{\mathfrak{q}}_i\cap A$, then there exists an element p such that $p\notin\mathfrak{q}$, $p\in\hat{\mathfrak{q}}_i\cap A$. Since $\mathfrak{q}'=\mathfrak{q}:pA$ is $\mathfrak{p}$-primary, $\hat{\mathfrak{p}}_i$ must be a prime divisor of $\mathfrak{q}'\hat{A}$ by Lemma 3. On the other hand, $\mathfrak{q}'\hat{A}=\mathfrak{q}\hat{A}:p\hat{A}=(\hat{\mathfrak{q}}_1:p\hat{A})\cap\cdots\cap(\hat{\mathfrak{q}}_n:p\hat{A})$, so we have a normal decomposition of $\mathfrak{q}'\hat{A}$ by shortening this representation, and we see that $\hat{\mathfrak{p}}_i$-component has vanished in it. This is a contradiction.

§ 2. The transition theorem.

Let $\mathfrak{q}_1$ and $\mathfrak{q}_2 (\mathfrak{q}_2 \subset \mathfrak{q}_1)$ be $\mathfrak{p}$-primary such that no further $\mathfrak{p}$-primary ideal can be inserted between $\mathfrak{q}_1$ and $\mathfrak{q}_2$. Let $\hat{\mathfrak{p}}$ be an isolated prime divisor of $\mathfrak{p}\hat{A}$, and let $\hat{\mathfrak{q}}_0$, $\hat{\mathfrak{q}}_1$ and $\hat{\mathfrak{q}}_2$ be the isolated $\hat{\mathfrak{p}}$-primary components of $\mathfrak{p}\hat{A}$, $\mathfrak{q}_1\hat{A}$ and $\mathfrak{q}_2\hat{A}$ respectively. Then we shall show that $L(\hat{\mathfrak{q}}_2) - L(\hat{\mathfrak{q}}_1) = L(\hat{\mathfrak{q}}_0)$.[1]

By Lemma 4, $\hat{\mathfrak{q}}_1 \cap A = \mathfrak{q}_1$, $\hat{\mathfrak{q}}_2 \cap A = \mathfrak{q}_2$. Let $\hat{\mathfrak{q}}$ be $\hat{\mathfrak{p}}$-primary such that $\hat{\mathfrak{q}}_2 \subseteq \hat{\mathfrak{q}} \subset \hat{\mathfrak{q}}_1$. Then it will be easily seen that $\hat{\mathfrak{q}} \cap A = \mathfrak{q}_2$. On the other hand, let q be an element of $\mathfrak{q}_1$ which does not belong to $\mathfrak{q}_2$. As $\mathfrak{q}_2 : qA$ is $\mathfrak{p}$-primary, we have $\mathfrak{p} = \mathfrak{q}_2 : qq'A$ for a suitable element q'. This shows that, for a suitable element p of $\mathfrak{q}_1$, $\mathfrak{p} = \mathfrak{q}_2 : pA$. From this $\mathfrak{p}\hat{A} = \mathfrak{q}_2\hat{A} : p\hat{A}$, and consequently $\hat{\mathfrak{q}}_0 = \hat{\mathfrak{q}}_2 : p\hat{A}$.

Now, since $p \notin \mathfrak{q}_2$ and $p \in \mathfrak{q}_1$, $\hat{\mathfrak{q}}_2 + p\hat{A}$ has $\hat{\mathfrak{q}}_1$ as the $\hat{\mathfrak{p}}$-primary component. Passing to $(\hat{A}/\hat{\mathfrak{q}}_2)_{\hat{\mathfrak{p}}/\hat{\mathfrak{q}}_2}$, our assertion is reduced to the following:

Lemma 5. *Let R be a Noetherian primary ring with identity, and let $\mathfrak{q}_0$, $\mathfrak{q}_1 (\mathfrak{q}_1 \subset \mathfrak{q}_0)$ be ideals of R such that $\mathfrak{q}_1 = (r)$ and $\mathfrak{q}_0 = (0) : (r)$ with a non-zero element r. Then $L(\mathfrak{q}_0) = L(0) - L(\mathfrak{q}_1)$.*

Proof. Consider $\mathfrak{q}_1$ as R-module. Then the annihilator of $\mathfrak{q}_1$ is $\mathfrak{q}_0$. So $\mathfrak{q}_1$ may be regarded as a free $R/\mathfrak{q}_0$-module, and from this our assertion follows.

The result obtained above now enables us to give readily the transition theorem on the lengths of primary ideals of a Zariski ring, which we shall state in the following form.

Theorem 2. *Let $\mathfrak{q}$ be $\mathfrak{p}$-primary and $\hat{\mathfrak{p}}$ be an isolated prime divisor of $\mathfrak{p}\hat{A}$. Then $L(\mathfrak{q})L(\mathfrak{p}\hat{A}_{\hat{\mathfrak{p}}}) = L(\mathfrak{q}\hat{A}_{\hat{\mathfrak{p}}})$.*[2]

We proceed with the same notations as in Theorem 2. Then the $\mathfrak{p}$-primary component of $\mathfrak{q}^n$ is $\mathfrak{q}^{(n)}$, and $\mathfrak{q}^n\hat{A} = \mathfrak{q}^{(n)}\hat{A} \cap \cdots$. Since, by Theorem 1, the $\hat{\mathfrak{p}}$-primary component of $\mathfrak{q}^{(n)}\hat{A}$ is also the $\hat{\mathfrak{p}}$-primary component of $\mathfrak{q}^n\hat{A}$ and since, if $\hat{\mathfrak{q}}$ is the $\hat{\mathfrak{p}}$-primary component of $\mathfrak{q}\hat{A}$, $\hat{\mathfrak{q}}^{(n)}$ is the $\hat{\mathfrak{p}}$-primary component of $\mathfrak{q}^n\hat{A}$, so $\hat{\mathfrak{q}}^{(n)}$ is also the $\hat{\mathfrak{p}}$-primary component of $\mathfrak{q}^{(n)}\hat{A}$. Hence, from Theorem 2, we have $L(\mathfrak{q}^{(n)})L(\mathfrak{p}\hat{A}_{\hat{\mathfrak{p}}}) = L(\hat{\mathfrak{q}}^{(n)})$. Then an important theorem of P. Samuel ([8], Chap. 1, Theorem 11, Chap. 2, Theorem 1,3; [9], Chap. 2,

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1) By $L(\mathfrak{q})$ is meant the length of a primary ideal $\mathfrak{q}$.

2) M. Nagata reported without proof in ([5], pp. 237–238) that the transition theorem holds good in a general local ring.

Sec. 3) shows that, for sufficiently large values of n, $L(\mathfrak{q}^{(n)})(L(\hat{\mathfrak{q}}^{(n)}))$ is equal to a polynomial in n. The degree of this polynomial is equal to the rank r of $\mathfrak{p}$ (rank r' of $\hat{\mathfrak{p}}$) and its leading term has the form $en^r/r!(e'n^{r'}/r'!)$, where $e(e')$ is a non-zero integer. If we write $e=e(\mathfrak{q})(e'=e(\hat{\mathfrak{q}}))$, then $e(\mathfrak{q})(e(\hat{\mathfrak{q}}))$ is called the *multiplicity* of $\mathfrak{q}(\hat{\mathfrak{q}})$. We now have

Corollary. *Let $\mathfrak{q}$ be $\mathfrak{p}$-primary and $\hat{\mathfrak{q}}$ be the $\hat{\mathfrak{p}}$-primary component of $\mathfrak{q}\hat{A}$, where $\hat{\mathfrak{p}}$ is an isolated prime divisor of $\mathfrak{p}\hat{A}$. Then $e(\hat{\mathfrak{q}})=L(\mathfrak{p}\hat{A}_{\hat{\mathfrak{p}}})e(\mathfrak{q})$ and* rank $\mathfrak{p}$ = rank $\hat{\mathfrak{p}}$.

To our last assertion concerning on the rank we shall, in the next section, give another proof which depends on the Krull's Primidealkettensatz and on its converse.

§ 3. Maximal chains of prime ideals and the factorability.

From Krull's Primidealkettensatz ([7], p. 60, Theorem 7) and the results obtained in § 1, we derive the following:

Proposition 1. *Let $\mathfrak{p}$ be a prime ideal, then every isolated prime divisor of $\mathfrak{p}\hat{A}$ has the same rank as $\mathfrak{p}$.*[3)]

Proof. Denote by r the rank of $\mathfrak{p}$, then it is possible to find r elements $a_1, a_2, \cdots, a_r$ of A such that $\mathfrak{p}$ is an isolated prime divisor of $(a_1, a_2, \cdots, a_r)A$. Let $\hat{\mathfrak{p}}$ be any isolated prime divisor of $\mathfrak{p}\hat{A}$, then, by Theorem 1, $\hat{\mathfrak{p}}$ is also an isolated prime divisor of $(a_1, a_2, \cdots, a_r)\hat{A}$. Hence we have rank $\hat{\mathfrak{p}} \leqq r$. On the other hand, there exists a prime ideal chain in A such that $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_r = \mathfrak{p}$. Since $\mathfrak{p}_{r-1}\hat{A} \subset \hat{\mathfrak{p}}$, $\hat{\mathfrak{p}}$ will contain an isolated prime divisor $\hat{\mathfrak{p}}_{r-1}$ of $\mathfrak{p}_{r-1}\hat{A}$. Since $\hat{\mathfrak{p}} \cap A = \mathfrak{p}$ and $\hat{\mathfrak{p}}_{r-1} \cap A = \mathfrak{p}_{r-1}$, $\hat{\mathfrak{p}}$ contains $\hat{\mathfrak{p}}_{r-1}$ properly. If we proceed in this way, we shall see that $r \leqq$ rank $\hat{\mathfrak{p}}$.

Corollary. rank $\mathfrak{a}$ = rank $\mathfrak{a}\hat{A}$.

Lemma 6. *Let $\mathfrak{p}_1$ and $\mathfrak{p}_2 (\mathfrak{p}_1 \subset \mathfrak{p}_2)$ be prime such that* rank $\mathfrak{p}_2/\mathfrak{p}_1 = 1$, *and let $\hat{\mathfrak{p}}_2$ be any isolated prime divisor of $\mathfrak{p}_2\hat{A}$. Then $\hat{\mathfrak{p}}_2$ contains an isolated prime divisor $\hat{\mathfrak{p}}_1$ of $\mathfrak{p}_1\hat{A}$ such that* rank $\hat{\mathfrak{p}}_2/\hat{\mathfrak{p}}_1 = 1$.

3) In the latest issue of Nagoya Math. J., which was published in June 1956, M. Nagata ([6], p. 53, Lemma 2.1) gave our Proposition 1 in the case where A is semi-local. But his proof depends essentially on the result which is to be given in his forthcoming paper.

Proof. Since $\hat{A}/\mathfrak{p}_1\hat{A}$ is the completion of $(\mathfrak{m}+\mathfrak{p}_1)/\mathfrak{p}_1$-adic Zariski ring $A/\mathfrak{p}_1$, our assertion follows easily from Proposition 1.

Corollary. *Let* $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_r$ *be a chain of prime ideals such that* rank $\mathfrak{p}_i/\mathfrak{p}_{i-1}=1$ $(i=1,\cdots,r)$. *Then there exists a chain of prime ideals* $\hat{\mathfrak{p}}_0 \subset \hat{\mathfrak{p}}_1 \subset \cdots \subset \hat{\mathfrak{p}}_r$ *such that* $\hat{\mathfrak{p}} \cap A = \mathfrak{p}$ *and* rank $\hat{\mathfrak{p}}_i/\hat{\mathfrak{p}}_{i-1}=1$ $(i=1,\cdots,r)$.

We consider now a maximal chain of prime ideals in $A: \mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_d$, that is, $\mathfrak{p}_d$ is maximal, $\mathfrak{p}_0$ is of rank zero and rank $\mathfrak{p}_i/\mathfrak{p}_{i-1}=1$ $(i=1,\cdots,d)$. Then, by Lemma 6, there exists a maximal chain of prime ideals in $\hat{A}: \hat{\mathfrak{p}}_0 \subset \hat{\mathfrak{p}}_1 \subset \cdots \subset \hat{\mathfrak{p}}_d$ $(=\mathfrak{p}_d\hat{A})$, such that each $\hat{\mathfrak{p}}$ lies over $\mathfrak{p}$. Hence, if we denote by (M) the following property:

(M) *All the maximal chains of prime ideals have the same length,*

then we have

Theorem 3. *If* $\hat{A}$ *has* (M), *so also has* A.

Now, by a theorem due to I. S. Cohen ([2], Theorem 19), every complete local domain has the property (M). So, by virtue of Theorem 3, every maximal chain of prime ideals in an analytically irreducible local ring A, in particular, in a regular local ring, has the same length as the dimension of A.

Finally, as another application of Theorem 1, we shall prove the following:

Proposition 2. *If* $\hat{A}$ *is a unique factorization domain, so also is* A.[4)]

Proof. It is sufficient to prove that any minimal prime ideal of A is principal. Let $\mathfrak{p}$ be a minimal prime ideal, and take a non-zere element p of $\mathfrak{p}$. Since $\hat{A}$ is a unique factorization domain, every prime divisor of $p\hat{A}$ is a minimal prime ideal; hence, by Theorem 1, every prime divisor of $\mathfrak{p}\hat{A}$ is also a minimal prime ideal, and consequently $\mathfrak{p}\hat{A}$ is principal; so $\mathfrak{p}$ is principal ([10], p. 3, Theorem 1) and this completes the proof.

4) Proposition 2 was proved by Krull ([3], p. 236, Satz 8) in the case where A is a regular local ring.

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