JOURNAL OF SCIENCE OF THE HIROSHIMA UNIVERSITY, SER. A, VOL. 20, NO. 2, OCTOBER, 1956

# *Forms of Relativistic Dynamics Referred to the New Fundamental Group of Transformations*

By

Takashi SHIBATA and Toshiei KIMURA

(Received April 4, 1956)

## § 1. Introduction

In atomic theory, in order that a transition to the quantum theory may be possible, it is required that the equation of motion shall be expressible in the Hamiltonian form. And it seems that imperfections in atomic theory may arise from the use of inadequate Hamiltonians. Hence, for the improvement of atomic theory, it is necessary to set up new dynamical systems to represent atomic phenomena. This problem was first discussed by P. A. M. Dirac [1].* He obtained three forms of relativistic dynamics and investigated them in his papers [1, 2]. On the other hand, also our theory in the foregoing papers seems to lead to certain forms of relativistic dynamics [3]. We intend to develope relativistic dynamics referred to our fundamental group of transformations. In this paper, we shall introduce the dynamical systems based on the fundamental group of transformations, and compare them with that of Dirac's theory.

## § 2. Forms of relativistic dynamics deduced from the new group.

In the foregoing papers, we have proposed to replace the special Lorentz transformations (Lorentz transformations without rotation in general) by the new fundamental group of transformations, $G_3$, as representing the relations between the coordinates in two inertial systems one of which moves with uniform velocity to the other. Expressing the interval in Minkowski space as $-(dx^1)^2-(dx^2)^2-(dx^3)^2+(dx^4)^2=g_{\mu\nu}dx^\mu dx^\nu$, the actual form of the transformations of the group $G_3$, is given by [4]

$$x'^\lambda=A^\lambda{}_\mu x^\mu \qquad (\lambda, \mu=1,\cdots,4) \tag{2.1}$$

with

$$A^\lambda{}_\mu=\delta^\lambda{}_\mu+k^\lambda\frac{U_\mu-U'_\mu}{(kU')}+\frac{U'^\lambda-U^\lambda}{(kU)}k_\mu+\frac{(UU')-(UU)}{(kU)(kU')}k^\lambda k_\mu \tag{2.2}$$

where $k^\lambda$ is a fixed null-vector defined by

$$k^\lambda=(d^1, d^2, d^3, 1), \quad g_{\mu\nu}k^\mu k^\nu=0 \tag{2.3}$$

$U^\lambda$ is four-velocity corresponding to ordinary three-dimensional velocity

*) Numbers in brackets refer to the references at the end of the paper.

$u^h$ $(h=1,2,3)$, the relative velocity of the system to the other, i.e.

$$U^\lambda=(u^h/\sqrt{1-u^2/c^2},\quad c/\sqrt{1-u^2/c^2}),\quad \text{and}\quad U'^\lambda=(0,0,0,c) \tag{2.4}$$

The notation $(kU)$ means $g_{\mu\nu}k^\mu U^\nu$, similarly $(UU)=g_{\mu\nu}U^\mu U^\nu=c^2$.

In non-relativistic dynamics, the physical laws are invariant under the following transformations:

$$x'^h=a^h+b^{hi}x_i+u^h t\quad (h,i=1,2,3),\quad t'=t+a^0 \tag{2.5}$$

with $b^{ih}b_{ij}=\delta^h_j$, the $a$'s, $u$'s and $b^{hi}$ being constants. These transformations contain the Galilean transformations in two inertial systems one of which moves with uniform velocity $u^h$ to the other:

$$x'^h=x^h+u^h t\quad (h=1,2,3)\quad t'=t$$

as a special case. In the usual form of dynamics, say "instant form", the dynamical theory has been built up in terms of dynamical variables that refer to physical conditions at one instant of time $t=$constant. Among the transformations (2.5), *only* the translation of time changes the instant. Then, the change of dynamical system can be described by *one* Hamiltonian (it's strict definition will be given later). Further, owing to the fact that the Galilean transformations relating the equivalent observers leave the instant invariant, it is conceivable that the description of dynamical system by the "instant form" does not involve the internal inconsistency in the non-relativistic dynamics.

In the relativistic dynamics, the physical laws should be invariant under the inhomogeneous Lorentz transformations generated by the infinitesimal transformations of the form

$$x'^\mu=x^\mu+a^\mu+a^{\mu\nu}x_\nu\quad (a^{\mu\nu}=-a^{\nu\mu})\quad (\mu,\nu=1,\cdots,4) \tag{2.6}$$

$a^\mu$ and $a^{\mu\nu}$ being infinitesimal quantities. As a special case, the special Lorentz transformations, relating two coordinate systems one of which moves with uniform velocity to the other, are contained in (2.6). However, contrary to the case of non-relativistic dynamics, the instant $t=$constant is not invariant under the special Lorentz transformation. That is, the normal direction of the instant is altered by the special Lorentz transformation (which means the velocity of observer alter). Corresponding to this, it will be necessary to introduce extra Hamiltonians other than that of non-relativistic dynamics. From the analogy of non-relativistic dynamics, it is natural to consider that *the difference of the descriptions of a dynamical variable by two observers, one of which moves with uniform velocity to the other, is trivial and the two observers are equivalent to describe the dynamical system.* However, we are obliged to consider that the difference of the descriptions of a dynamical variable in two coordinate systems related by the special Lorentz transformations is caused by Hamiltonians. Therefore, we can say that the relativistic instant form has not a parallelism with non-relativistic dynamics. Hence, we are inclined to consider that this dynamical form is not a reasonable one, so far as we take the special Lorentz transformation as the transformation relating the two equivalent observers one of which moves with uniform velocity to the other. In the following paragraphs we shall show that, *by taking the new group $G_3$ of transformations in place of the special Lorentz transformations* and adopting the other dynamical forms than instant form, we can set up a relativistic dynamical theory having the parallelism with the non-relativistic dynamics. Accordingly, the difference of the descriptions of a dynamical variable in two coordinate systems one of which moves with uniform velocity to the other becoms to be trivial, and then we have not need of Hamiltonians to connect the two equivalent descriptions.

Corresponding to the hypersurface $t=$const. in the instant form, now we consider hypersurface in space-time which is left invariant under the group $G_3$. Such hypersurface is given by

$$\text{(I)}\quad k_\mu x^\mu=0 \qquad \text{or} \qquad \text{(II)}\quad x_\mu x^\mu=\text{const.} \tag{2.7}$$

Suppose now we have a state defined on one of the surfaces $k_\mu x^\mu=0$ and $x_\mu x^\mu=$const., instead of at one instant of time. From any state of motion of the dynamical system which is defined on the surface $k_\mu x^\mu=0$ or $x_\mu x^\mu=$const., we get another state of motion by displacing it under the inhomogeneous Lorentz transformation generated by (2.6). If we change the state by displacing it in such a way that (2.7) remains invariant, the corresponding change in the dynamical variables will be a trivial one. If we displace the state in a more general way, we make a non-trivial change in the dynamical variables. Thus, *from the supposition that the transformation of the group $G_3$ shall correspond to only a trivial change in dynamical variables, we have two forms of relativistic dynamics in which a state is defined on the surface $k_\mu x^\mu=0$ or $x_\mu x^\mu=$const. named by Dirac as "front form" or "point form" respectively* [1]. Dirac has obtained these forms by modifying the "instant form", whereas we have the two forms of relativistic dynamics by considering invariant hypersurface under the group $G_3$. We shall reduce the problem of constructing two dynamical forms based on the group $G_3$ to the mathematical one.

Take the four coordinates $q^\lambda$ of a point in space-time as dynamical coordinates and let their conjugate momenta be $p_\lambda$, so that

$$[q_\mu, q_\nu]=0, \quad [p_\mu, p_\nu]=0, \quad [p_\mu, q_\nu]=g_{\mu\nu} \tag{2.8}$$

where the bracket [ , ] denotes Poisson bracket notation. We suppose that Poisson brackets are subject to the following laws:

$$[\xi, \eta]=-[\eta, \xi] \text{ and } [f(\xi_1, \xi_2, \cdots), \eta]=\frac{\partial f}{\partial \xi_1}[\xi_1, \eta]+\frac{\partial f}{\partial \xi_2}[\xi_2, \eta]+\cdots \tag{2.9}$$

where $\xi, \eta$ are functions of the $q$'s and $p$'s, and $f$ is any function of various quantities $\xi_1, \xi_2, \cdots$, each of which is a function of the $q$'s and $p$'s, unless ambiguity occurs for passing from Hamilton's dynamics to quantum dynamics.

From (2.8) and (2.9), we can see that Poisson's identity $[\xi,[\eta,\zeta]]+[\eta,[\zeta,\xi]]+[\zeta,[\xi,\eta]]=0$ holds good for any three functions $\xi,\eta,\zeta$ of the $q$'s and $p$'s.

It is considered that each dynamical variable $\xi$ shall change to $\xi'$ in such a way that Poisson bracket $[\xi,\eta]$ of any two dynamical variables $\xi,\eta$, also changes to the Poisson bracket $[\xi',\eta']$ of the changed variables $\xi',\eta'$. From this consideration we may assume that, corresponding to the transformation of (2.6), each dynamical variable $\xi$ shall change according to the law

$$\xi'=\xi+[\xi,F] \tag{2.10}$$

where $F$ is a certain dynamical variable independent of each $\xi$. Since $F$ is infinitesimal, we can put

$$F=-a^{\mu}P_{\mu}+\tfrac{1}{2}a^{\mu\nu}M_{\mu\nu} \quad (M_{\mu\nu}=-M_{\nu\mu}) \tag{2.11}$$

$P_{\mu}$, $M_{\mu\nu}$ being finite variables, independent of the transformation of coordinates. In this way $F$ associates with each transformation of (2.6). If we suppose that the correspondence between (2.6) and (2.10) is isomorphic, we see that $P_{\rho}$, $M_{\mu\nu}$ must satisfy the following relations

$$\begin{aligned}&[P_{\mu},P_{\nu}]=0,\quad [M_{\mu\nu},P_{\rho}]=-g_{\mu\rho}P_{\nu}+g_{\nu\rho}P_{\mu},\\&[M_{\mu\nu},M_{\rho\sigma}]=-g_{\mu\rho}M_{\nu\sigma}+g_{\nu\rho}M_{\mu\sigma}-g_{\mu\sigma}M_{\rho\nu}+g_{\nu\sigma}M_{\rho\mu}\end{aligned} \tag{2.12}$$

which correspond to the commutation relations among the operators $T_{\rho}\equiv\partial/\partial x^{\rho}$, $T_{\mu\nu}\equiv x_{\mu}\partial/\partial x^{\nu}-x_{\nu}\partial/\partial x^{\mu}$ of the infinitesimal transformations (2.6). The $P_{\rho}$ and $M_{\mu\nu}$ are called the *fundamental quantities* corresponding to momentum energy and angular momentum. Among them, the $P_{\rho}$, $M_{\mu\nu}$ associated with the transformations of coordinates which leave the surface invariant are specially simple, and the other $P_{\rho}$, $M_{\mu\nu}$, being complicated, are called *Hamiltonians* since they play jointly the role of the single Hamiltonian in non-relativistic dynamics.

To construct a theory of a dynamical system it is necessary to find the expressions for $P_{\rho}$ and $M_{\mu\nu}$ satisfying the relations (2.12). In the following sections, this problem will be treated in a different way from Dirac's method.

## § 3. Expressions for the fundamental quantities $P_{\rho}$ and $M_{\mu\nu}$ in the "front form".

We work with dynamical variables referring to physical conditions on the surface $k_{\nu}x^{\nu}=0$. Here, besides the null-vector $k_{\mu}$, we introduce new vectors $\lambda_{\mu}$, $\alpha_{\mu}$, $\beta_{\mu}$, such that

$$\begin{aligned}&k^{\mu}\lambda_{\mu}=2,\ k^{\mu}\alpha_{\mu}=k^{\mu}\beta_{\mu}=0, && \lambda^{\mu}=(-d^1,-d^2,-d^3,1)\\&\lambda^{\mu}\lambda_{\mu}=0,\ \lambda^{\mu}\alpha_{\mu}=\lambda^{\mu}\beta_{\mu}=0, && \alpha_{\mu}=(\alpha^1,\alpha^2,\alpha^3,0)\\&\alpha^{\mu}\alpha_{\mu}=\beta_{\mu}\beta^{\mu}=1,\ \alpha^{\mu}\beta_{\mu}=0, && \beta^{\mu}=(\beta^1,\beta^2,\beta^3,0)\end{aligned} \tag{3.1}$$

and regard $k_\mu x^\mu$, $\lambda_\mu x^\mu$, $\alpha_\mu x^\mu$, $\beta_\mu x^\mu$ as four independent variables.

First we consider the case of a single particle. We take as dynamical coordinates the coordinates of the particle on the surface $k_\nu x^\nu=0$. Let these coordinates be $\lambda_\mu q^\mu$, $\alpha_\mu q^\mu$, $\beta_\mu q^\mu$ with $k_\mu q^\mu=0$. With this equation $\lambda^\mu p_\mu$ no longer has a meaning since in general the conjugate momenta of $k_\mu q^\mu$, $\lambda_\mu q^\mu$, $\alpha_\mu q^\mu$, $\beta_\mu q^\mu$ are $\frac{1}{2}\lambda^\mu p_\mu$, $\frac{1}{2}k^\mu p_\mu$, $\alpha^\mu p_\mu$, $\beta^\mu p_\mu$. Hence we must define $P_\rho$, $M_{\mu\nu}$, as functions of the $q$'s and $p$'s, such that they do not involve $\lambda^\mu p_\mu$, without changing the relations (2.12).

We shall solve the equations (2.12) under the supposition that

$$M_{\mu\nu}=q_\mu P_\nu-q_\nu P_\mu \tag{3.2}$$

Under this supposition, using the first relations of (2.12), the second relations of (2.12) are reduced to

$$[q_\mu, P_\rho]P_\nu-[q_\nu, P_\rho]P_\mu=-g_{\mu\rho}P_\nu+g_{\nu\rho}P_\mu$$

From the above, $(\partial P_\rho/\partial p^\mu-g_{\mu\rho})$ must be proportional to $P_\mu$, i.e.

$$\partial P_\rho/\partial p^\mu-g_{\mu\rho}=N_\rho P_\mu \tag{3.3}$$

since $[q_\mu, P_\rho]=-\partial P_\rho/\partial p^\mu$. The third relations of (2.12) are also reduced to the same equations as (3.3) by using the first relations of (2.12).

Further, from the condition that $P_\rho$ do not involve $\lambda^\mu p_\mu$, we have

$$k^\nu\frac{\partial P_\rho}{\partial p^\nu}=0 \tag{3.4}$$

since $P_\rho$ are regarded as functions of $k^\mu p_\mu$, $\alpha^\mu p_\mu$, $\beta^\mu p_\mu$ and $q$'s. The above condition is equivalent to the condition $[k_\nu q^\nu, P_\rho]=0$.

To solve the equations (3.3) and (3.4), we multiply (3.3) by $k^\mu$ and sum for $\mu=1,\cdots,4$. Then, by using (3.4), we have $-k_\rho=N_\rho k^\mu P_\mu$. Hence (3.3) are rewritten as

$$\frac{\partial}{\partial p^\mu}(P_\rho-p_\rho)=-\frac{k_\rho P_\mu}{k^\alpha P_\alpha} \tag{3.5}$$

from which $P_\rho-p_\rho$ are expressed in the form

$$P_\rho-p_\rho=-k_\rho B+f_\rho \tag{3.6}$$

where $f_\rho$ are independent of $p$'s, and $B$ must be the solution of

$$\frac{\partial B}{\partial p^\mu}=\frac{p_\mu+f_\mu-k_\mu B}{k^\alpha(p_\alpha+f_\alpha)} \tag{3.7}$$

in order that (3.6) satisfy (3.5). The equations (3.7) are rewritten as $\frac{\partial}{\partial p^\mu}\{k^\alpha(p_\alpha+f_\alpha)B\}=p_\mu+f_\mu$. Therefore, $B$ is determined by

$$k^\alpha(p_\alpha+f_\alpha)B=\tfrac{1}{2}(p_\mu+f_\mu)(p^\mu+f^\mu)-\tfrac{1}{2}m^2 \tag{3.8}$$

where $m^2$ is independent of $p$'s. From (3.6), by using (3.8), we have

$$P_\rho=p_\rho+f_\rho-\tfrac{1}{2}k_\rho\{(p_\mu+f_\mu)(p^\mu+f^\mu)-m^2\}\{k^\alpha(p_\alpha+f_\alpha)\}^{-1} \tag{3.9}$$

which are the solutions of (3.3) and (3.4). Next we shall determine $m^2$

and $f_\rho$ such that $P_\rho$ satisfy the first relations of (2.12). Since $P^\mu P_\mu = m^2$, we have $[m^2, P_\nu]=0$, i.e. by using (3.5),

$$\frac{\partial m^2}{\partial q^\nu} - k_\nu \frac{1}{k^\sigma P_\sigma} P^\mu \frac{\partial m^2}{\partial q^\mu} = 0$$

These equations show that $m^2$ is independent of $\lambda_\mu q^\mu, \alpha_\mu q^\mu, \beta_\mu q^\mu$. Hence $m$ must be constant since we are dealing with $k_\mu q^\mu = 0$. Further, inserting (3.9) into $[P_\mu, P_\nu]=0$, we have, after some calculation,

$$\frac{\partial f_\mu}{\partial q^\nu} - \frac{\partial f_\nu}{\partial q^\mu} = 0 \qquad \text{viz.}\ \ f_\mu = \frac{\partial f}{\partial q^\mu}$$

$f$ being arbitrary function of $q$'s. So we have the result: *For the "front form", under the supposition that* $M_{\mu\nu} = q_\mu P_\nu - q_\nu P_\mu$, *the expressions for* $P$ *satisfying the relations (2.12) are given by*

$$P_\rho = p_\rho + f_\rho - \tfrac{1}{2} k_\rho \{(p_\mu + f_\mu)(p^\mu + f^\mu) - m^2\} \{k^\alpha (p_\alpha + f_\alpha)\}^{-1} \tag{3.10}$$

*where* $m$ *is constant and* $f_\sigma = \partial f / \partial q^\sigma$, $f$ *being arbitrary function of* $q$'s.

We can see that the above-mentioned expressions for $P_\rho$ coincide with that of $P_\rho$ in the "front from" obtained by Dirac [1], if we put $f_\rho = 0$ in (3.10) and transform the coordinates such as in Dirac's theory. Dirac has obtained the expressions assuming the following forms:

$$P_\mu = p_\mu + \lambda_\mu (p^\sigma p_\sigma - m^2), \quad M_{\mu\nu} = q_\mu p_\nu - q_\nu p_\mu + \lambda_{\mu\nu} (p^\sigma p_\sigma - m^2) \tag{3.11}$$

whereas we have the result under the supposition $M_{\mu\nu} = q_\mu P_\nu - q_\nu P_\mu$.

We conclude the section by giving the Hamiltonians in the "front form". Since the transformations generated by the infinitesimal transformations with the symbols: $k^\mu T_\mu, \alpha^\mu T_\mu, \beta^\mu T_\mu, \alpha^\mu k^\nu T_{\mu\nu}, \beta^\mu k^\nu T_{\mu\nu}, \lambda^\mu k^\nu T_{\mu\nu}, \alpha^\mu \beta^\nu T_{\mu\nu}$ ($T_\mu \equiv \partial/\partial x^\mu$, $T_{\mu\nu} \equiv x_\mu \partial/\partial x^\nu - x_\nu \partial/\partial x^\mu$) leave $k_\nu x^\nu = 0$ invariant, the associated quantities: $k^\mu P_\mu, \alpha^\mu P_\mu, \beta^\mu P_\mu, \alpha^\mu k^\nu M_{\mu\nu}, \beta^\mu k^\nu M_{\mu\nu}, \lambda^\mu k^\nu M_{\mu\nu}, \alpha^\mu \beta^\nu M_{\mu\nu}$ are specially simple and the other three quantities: $\lambda^\mu P_\mu, \alpha^\mu \lambda^\nu M_{\mu\nu}, \beta^\mu \lambda^\nu M_{\mu\nu}$ give the Hamiltonians.

**Note.** If we work with dynamical variables referring to physical conditions on the surface $k_\nu x^\nu = \text{const.} \neq 0$, the quantity $\lambda^\mu k^\nu M_{\mu\nu}$ does not become simple and belongs to Hamiltonians because it associates with the infinitesimal transformation with the symbol $\lambda^\mu k^\nu T_{\mu\nu}$ which gives $k_\nu x^\nu$ an infinitesimal change proportional to $-2k_\nu x^\nu$, not remaining the expression $k_\nu x^\nu$ itself invariant.

## § 4. Expressions for $\mathbf{P}_\rho$ and $\mathbf{M}_{\mu\nu}$ in the "point form".

In the "point form", the condition (3.4) should be replaced by

$$[q^\mu q_\mu, P_\rho] = 0 \qquad \text{i.e.}\ \ q^\mu \frac{\partial P_\rho}{\partial p^\mu} = 0 \tag{4.1}$$

By the same way as in § 3, we can solve (2.12) under the conditions (3.2) and (4.1), not using Dirac's assumption (3.11). The result is that

$$P_\rho = p_\rho + f_\rho + q_\rho B \tag{4.2}$$

where $B$ is determined by

$$q^\sigma q_\sigma B^2 + 2q^\alpha(p_\alpha + f_\alpha)B + (p_\mu + f_\mu)(p^\mu + f^\mu) = m^2 \tag{4.3}$$

in which $m$ is constant and $f_\alpha = \partial f/\partial q^\alpha$, $f$ being arbitrary function of $q$'s. This result coincides with that of Dirac's theory if $f_\rho = 0$.

## § 5. Interaction terms for a system composed of several particles.

We consider the case of "front form". For a dynamical system composed of several particles, we denote the ten fundamental quantities by $\bar{P}_\rho$, $\bar{M}_{\mu\nu}$. As in Dirac's theory, $k^\mu \bar{P}_\mu$, $\alpha^\mu \bar{P}_\mu$, $\beta^\mu \bar{P}_\mu$, $\alpha^\mu k^\nu \bar{M}_{\mu\nu}$, $\beta^\mu k^\nu \bar{M}_{\mu\nu}$, $\lambda^\mu k^\nu \bar{M}_{\mu\nu}$, $\alpha^\mu \beta^\nu \bar{M}_{\mu\nu}$ will be just the sum of their values for the particles separately. The Hamiltonians $\lambda^\mu \bar{P}_\mu$, $\alpha^\mu \lambda^\nu \bar{M}_{\mu\nu}$, $\beta^\mu \lambda^\nu \bar{M}_{\mu\nu}$ will be the sum of their values for the particles separately plus interaction terms. Therefore, we can put

$$\bar{P}_\rho = \Sigma P_\rho + k_\rho U, \quad \bar{M}_{\mu\nu} = \Sigma M_{\mu\nu} + 2\alpha_{[\mu}k_{\nu]}V + 2\beta_{[\mu}k_{\nu]}W \tag{5.1}$$

where the notation $\alpha_{[\mu}k_{\nu]}$ means $\frac{1}{2}(\alpha_\mu k_\nu - \alpha_\nu k_\mu)$. Here $P_\rho$ and $M_{\mu\nu}$ are given by (3.10) and (3.2) for each particle, and $U$, $V$, $W$ represent interaction terms. The $U$, $V$, $W$ must satisfy certain conditions to make the Hamiltonians satisfy the same relations as (2.12) in which $P_\rho$, $M_{\mu\nu}$ are replaced by $\bar{P}_\rho$, $\bar{M}_{\mu\nu}$. We shall obtain these conditions as follows. From the conditions $[\bar{P}_\mu, \bar{P}_\nu] = 0$, we have $k_\mu[U, \Sigma P_\nu] + k_\nu[\Sigma P_\mu, U] = 0$. Therefore, $[U, \Sigma P_\nu]$ must be proportional to $k_\nu$, i.e.

$$[U, \Sigma P_\nu] = k_\nu L \tag{5.2}$$

$L$ being defined by $\lambda^\nu[U, \Sigma P_\nu] = 2L$. From the second relations of (2.12) in which $P_\rho$, $M_{\mu\nu}$ are replaced by $\bar{P}_\rho$, $\bar{M}_{\mu\nu}$, we have

$$k_\rho[\Sigma M_{\mu\nu}, U] + 2\alpha_{[\mu}k_{\nu]}\{[V, \Sigma P_\rho] + k_\rho[V, U]\} + 2\beta_{[\mu}k_{\nu]}\{[W, \Sigma P_\rho] + k_\rho[W, U]\} = -g_{\mu\rho}k_\nu U + g_{\nu\rho}k_\mu U \tag{5.3}$$

In order to simplify the above equations, we multiply the above by $k^\nu$ and sum for $\nu = 1, \cdots, 4$. Then we have

$$k^\nu[\Sigma M_{\mu\nu}, U] = k_\mu U \tag{5.4}$$

Similarly, multiplying (5.3) by $\alpha^\nu$ or $\beta^\nu$ and summing for $\nu = 1, \cdots, 4$, respectively, we have

$$k_\rho \alpha^\nu[\Sigma M_{\mu\nu}, U] - k_\mu\{[V, \Sigma P_\rho] + k_\rho[V, U]\} = \alpha_\rho k_\mu U \tag{5.5}$$

$$k_\rho \beta^\nu[\Sigma M_{\mu\nu}, U] - k_\mu\{[W, \Sigma P_\rho] + k_\rho[W, U]\} = \beta_\rho k_\mu U \tag{5.6}$$

From (5.5) and (5.6), multiplying by $\lambda^\rho$ respectively, the following relations

$$\alpha^\nu[\Sigma M_{\mu\nu}, U] = k_\mu A, \qquad \beta^\nu[\Sigma M_{\mu\nu}, U] = k_\mu B \tag{5.7}$$

with $2A = \lambda^\rho[V, \Sigma P_\rho] + 2[V, U]$, $2B = \lambda^\rho[W, \Sigma P_\rho] + 2[W, U]$ are obtained. Hence (5.5) and (5.6) are rewritten as

$$[V, \Sigma P_\rho] + k_\rho[V, U] = k_\rho A - \alpha_\rho U \tag{5.8}$$

$$[W, \Sigma P_\rho] + k_\rho[W, U] = k_\rho B - \beta_\rho U \tag{5.9}$$

Further, by using (5.7), the equations (5.4) are solved for $[U, \Sigma M_{\mu\nu}]$, viz.

$$[U, \Sigma M_{\mu\nu}] = 2\alpha_{[\mu}k_{\nu]}A + 2\beta_{[\mu}k_{\nu]}B + \lambda_{[\mu}k_{\nu]}U \tag{5.10}$$

The equations (5.8), (5.9), (5.10) together are equivalent to (5.3). By the same procedure, from the third relations of (2.12) in which $P_\rho$, $M_{\mu\nu}$ are replaced by $\bar{P}_\rho$, $\bar{M}_{\mu\nu}$, we can obtain

$$[V, \Sigma M_{\rho\sigma}] = 2\alpha_{[\rho}k_{\sigma]}D + 2\beta_{[\rho}k_{\sigma]}E + \lambda_{[\rho}k_{\sigma]}V + 2\alpha_{[\rho}\beta_{\sigma]}W \tag{5.11}$$

$$[W, \Sigma M_{\rho\sigma}] = 2\alpha_{[\rho}k_{\sigma]}F + 2\beta_{[\rho}k_{\sigma]}G + \lambda_{[\rho}k_{\sigma]}W + 2\beta_{[\rho}\alpha_{\sigma]}V \tag{5.12}$$

$$[V, W] = F - E \tag{5.13}$$

$D, E, F, G$ being defined by means of (5.11) and (5.12), e.g. $\alpha^\rho\lambda^\sigma[V, \Sigma M_{\rho\sigma}] = 2D$, etc. So we have the result: the interaction terms $U, V, W$ must satisfy the relations (5.2) and (5.8)—(5.13).

## References


[1] P. A. M. Dirac: Forms of relativistic dynamics. Rev. Mod. Phys., Vol. **21** (1949), 392.

[2] ———: Generalized Hamiltonian dynamics. Canadian Journ. Math. Vol. **2** (1950), 129.

[3] T. Shibata and T. Kimura: Spin-orbit interaction energy of an electron based on the new fundamental group of transformations. This journal Vol. **20** (1956), 37.

[4] T. Shibata: The "Lorentz transformations without rotation" and the new fundamental group of transformations in special relativity and quantum mechanics. This journal Vol. **19** (1955), 101.