JOURNAL OF SCIENCE OF THE HIROSHIMA UNIVERSITY, SER. A, VOL. 20, NO. 2, OCTOBER, 1956

On the Path in Matrix Space

By

Kakutaro MORINAGA and Fuzio MITSUDO

(Received March 25, 1956)

§ 1. The exponential expression of path.

In this paper we shall discuss the path in matrix space over the complex field. As usual literature we will define Me^{tA} and $e^{tA}M$ $(-\infty<t<\infty)$ as the *path*. Then, the path passing through two regular points M_1 and M_2 may be written as $M_1(M_1^{-1}M_2)^t$ [1] i.e.

$$e^{A_1}(e^{-A_1}e^{A_2})^t=e^{A_1}e^{tA_r} \quad (-\infty<t<\infty)$$

where $A_a=\log M_a$ $(a=1,2)$ and $A_r=\log M_1^{-1}M_2$.[2] Therefore, we have in general infinite numbers of path passing through two points M_1 and M_2.

By using C. H. Series (Campbell-Hausdorff Series): $H\{A_1, A_2\}\equiv\operatorname{Log} e^{A_1}e^{A_2})$ repeatedly for the case $\|M_a-E\|<\varepsilon$, ε being sufficiently small, we have

$$e^{A_1}(e^{-A_1}e^{A_2})^t=e^{(1-t)A_1+tA_2+\frac{t^2-t}{12}\{[A_1[A_1,A_2]]-[A_2[A_1,A_2]]\}+\cdots}$$
$$=e^{P\{A_1,A_2,t\}} \qquad (1)$$

where $P\{A_1, A_2, t\}\equiv H\{A_1, tH\{-A_1, A_2\}\}$.

Theorem 1. *$P\{A_1, A_2, t\}$ in (1) is the sum of alternants with respect to A_1 and A_2 of odd degree.*

For this purpose, at first we shall prove lemmas:

Lemma 1. *For corresponding branches, we always have*

$$M_1(M_1^{-1}M_2)^t\equiv(M_2M_1^{-1})^tM_1 \text{ i.e. } M_1e^{tA_r}=e^{tA_l}M_1.$$

Proof. If we take arbitrary branch A_l such that $M_2M_1^{-1}=e^{A_l}$, it follows:

$$(M_2M_1^{-1})^tM_1=(e^{A_l})^tM_1=e^{tA_l}M_1=M_1M_1^{-1}e^{tA_l}M_1=M_1e^{tM_1^{-1}A_lM_1}$$
$$=M_1(e^{M_1^{-1}A_lM_1})^t=M_1(M_1^{-1}e^{A_l}M_1)^t$$
$$=M_1(M_1^{-1}M_2M_1^{-1}M_1)^t=M_1(M_1^{-1}M_2)^t.$$

So we have $e^{tA_l}M_1=M_1e^{tM_1^{-1}A_lM_1}=M_1e^{tA_r}$.

And as above we have for arbitrary A_r satisfying $M_1^{-1}M_2=e^{A_r}$

$$M_1e^{tA_r}=e^{t(M_1A_rM_1^{-1})}M_1.$$

Hence, for arbitrary branch A_l or A_r it corresponds to $M_1^{-1}A_lM_1$ or to

1) See [3], p. 53, Theorem 1. Numbers in brackets refer to the references at the end of the paper.

2) Suffix r in A_r indicates the right, and we shall write a branch of $\log M_2M_1^{-1}$ by A_l.

$M_1A_rM_1^{-1}$ respectively and this correspondence is unique.[1)]

Lemma 2. *In this correspondence, principal branches* $\mathrm{Log}\, M_1^{-1}M_2$ $(\equiv A_r(P))$ *and* $\mathrm{Log}\, M_2M_1^{-1}$ $(\equiv A_l(P))$ *correspond to each other.*[2)]

Proof. $M_1e^{tA_r(P)}$ and $e^{tA_l(P)}M_1$ $(0\leq t\leq 1)$ pass through M_1 and M_2 and are contained in $M_1\mathfrak{M}_0=M_1\mathfrak{M}_0M_1^{-1}M_1=\mathfrak{M}_0M_1$.[3)] Since such path is unique, the above two paths coincide i.e. we have by Lemma 1

$$M_1e^{t\,\mathrm{Log}(M_1^{-1}M_2)}=e^{t\,\mathrm{Log}(M_2M_1^{-1})}M_1\,.$$

Now, we proceed to prove the theorem. In the case $\|M_1-E\|<\varepsilon$, and $\|M_2-E\|<\varepsilon$ (ε is sufficiently small), if we take the principal branches $\mathrm{Log}\, M_1$ $(\equiv A_1)$ and $\mathrm{Log}\, M_2$ $(\equiv A_2)$ then we have

$$e^{-A_1}e^{A_2}=e^{H\{-A_1,A_2\}}$$

where $\mathrm{Log}\, M_1^{-1}M_2=H\{-A_1, A_2\}$.

Then we may write

$$e^{A_1}(e^{-A_1}e^{A_2})^t=e^{A_1}e^{tH\{-A_1,A_2\}}=e^{H\{A_1,tH\{-A_1,A_2\}\}}$$
$$\equiv e^{P_r\{A_1,A_2,t\}}$$

and

$$(e^{A_2}e^{-A_1})^te^{A_1}=e^{tH\{A_2,-A_1\}}e^{A_1}=e^{H\{tH\{A_2,-A_1\},A_1\}}$$
$$=e^{P_l\{A_1,A_2,t\}}$$

where $\mathrm{Log}\, M_2M_1^{-1}=H\{A_2, -A_1\}$.
And it is clear that P_r and P_l are sum of alternants with respect to A_1 and A_2. By Lemma 1 and 2 it follows

$$e^{P_l\{A_1,A_2,t\}}=e^{P_r\{A_1,A_2,t\}}$$

and
$$\mathrm{Log}\, e^{P_l\{A_1,A_2,t\}}=P_l\{A_1, A_2, t\}=P_r\{A_1, A_2, t\}. \qquad (2)$$

In the above equation, as the orders of operation in either side are inverse to each other, if P_l contains

$$a^{i_1\cdots i_m}[A_{i_1}[A_{i_2}\cdots[A_{i_{m-1}}, A_{i_m}]]\cdots] \qquad (i_1,\cdots,i_m=1,2)$$

then P_r contains $a^{i_1\cdots i_m}[\cdots[[A_{i_m}, A_{i_{m-1}}]\cdots A_{i_2}]A_{i_1}]$.
And these two terms are transformable to each other by $(m-1)$-permutation, i.e.

$$[A_{i_1}[A_{i_2}\cdots[A_{i_{m-1}}, A_{i_m}]\cdots]=(-1)^{m-1}[\cdots[A_{i_m}, A_{i_{m-1}}]\cdots A_{i_2}]A_{i_1}].$$

So, we have:

The sum of terms with odd degree in $P_l\{A_1, A_2, t\}$ = the sum of terms with odd degree in $P_r\{A_1, A_2, t\}$, the sum of terms with even degree in $P_l\{A_1, A_2, t\}$ = − the sum of terms with even degree in $P_r\{A_1, A_2, t\}$. Therefore, by (2) $P_l\{A_1, A_2, t\}\equiv P\{A_1, A_2, t\}$ consists of the terms of odd degree only. So we completed the proof of Theorem 1.

---

1) Expression of a path is unique (left and right respectively).
2) A branch transformed the principal branch by some matrix is principal.
3) $\mathfrak{M}_0$ is the set of all the regular matrices without negative characteristic root.

§ 2. Convex set S.

We shall arrange the definitions with respect to the notions of "convex" and "tangent". In a set S of regular matrices, if there is a closed path $[C(\overline{M_1, M_2})]\subset S$ for any two points M_1 and $M_2 \in S$ where $[C(\overline{M_1, M_2})]\equiv\{M_1(M_1^{-1}M_2)^t\}$ $(0\leq t\leq 1)$, then we shall say S being *convex*; when $M_1e^{tA_1}, M_1e^{tA_2}\subset S$ $(0\leq t\leq 1)$, if we can choose small number ε such that for any $t_1, t_2, 0\leq t_1, t_2\leq \varepsilon$ $[C(\overline{M_1e^{t_1A_1}, M_1e^{t_2A_2}})]\subset(S\cap M_1\mathfrak{M}_0)$ (principal closed path $\subset S$) then S is defined as a *conditionally locally convex*. (In the case ε depends only on M_1, S becomes a *uniformly conditionally locally convex set*.), and if path $C(\overline{M_1, M_2})\subset S$ for any $[C(\overline{M_1, M_2})]\subset S$ then S is an *extensive set*. The set S, which is convex and uniformly conditionally more, if convex and extensive, will be said a *general path-plane*. Further-locally a general path plane S is locally convex, S is called a *path plane*.

For a set S, and a point $M_0\in S$, we consider the following two sets:

$$\mathfrak{A}_{M_0}(S)=\{A; M_0e^A\in S\} \quad \text{and}$$

$$\mathfrak{A}_{M_0}(S, t)=\{uA; ||A||<t, A\in\mathfrak{A}_{M_0}(S), u \text{ being real}\},$$

then it is clear for $t_1<t_2$

$$\mathfrak{A}_{M_0}(S, t_1)\subset\mathfrak{A}_{M_0}(S, t_2)$$

So, we may define

$$\bigcap_t \overline{\mathfrak{A}_{M_0}(S, t)}\equiv\mathfrak{A}(S, M_0) \text{ where } \overline{\mathfrak{A}_{M_0}(S, t)} \text{ is closure of the set } \mathfrak{A}_{M_0}(S, t)$$

and we say $\mathfrak{A}(S, M_0)$ as the tangential set at M_0. Moreover we deduce a set

$$\mathfrak{A}(S, M_0)\text{-linear}=\overline{\bigcup_\varepsilon \mathfrak{A}_{M_0}(S, \varepsilon)}\text{-linear where}$$

$$\mathfrak{A}_{M_0}(S, \varepsilon)\text{-linear}=\{A; M_0e^{tA}\in S,\ 0\leq t<\varepsilon\},$$

which is called *linear tangential set* and it satisfies

$$\mathfrak{A}(S, M_0)\text{-linear}\subset\mathfrak{A}(S, M_0).$$

If $\mathfrak{A}(S, M_0)$-liner $\neq\mathfrak{A}(S, M_0)$, then M_0 will be said *singular* point and the other point will be said *ordinary*.

Ex. 1. If the set S consists of points $M_0e^{t(t_iA_0+\mathfrak{P}_i)}$ where $t_i=\dfrac{1}{2^i}$, $[A_0, \mathfrak{P}_i]=0$, $e^{\mathfrak{P}_i}=E$ $(i=1, 2,\cdots)$ and $A_0\neq kt_i\mathfrak{P}_i$, then we have $\mathfrak{A}(S, M_0)=\{tA_0$ and $t(t_iA_0+\mathfrak{P}_i)\}$ $(i=1, 2,\cdots)$ and $\mathfrak{A}(S, M_0)$-linear$=\{s(t_iA_0+\mathfrak{P}_i)\}$ $(i=1, 2,\cdots)$.

Ex. 2. When we consider the differentiable curve $C(t)$ $(\equiv S)$ which does not contain a part of path at M_0, it is clear

$$\mathfrak{A}(S, M_0)\neq\mathfrak{A}(S, M_0)\text{-linear}=\Phi.$$

Now, we consider about the properties of convexity of a set S. We will say *a set $\mathfrak{A}$ being a Lie triple system*[1] (this will be denoted as *Lie t.s.*)

1) See [2], pp. 150–153.

when $\mathfrak{A}$ satisfies the following conditions:

If $A_1, A_2, A_3 \in \mathfrak{A}$, then $t_1A_1+t_2A_2 \in \mathfrak{A}$ $(-\infty<t_1, t_2<\infty)$ and $[A_1[A_2, A_3]] \in \mathfrak{A}$. We will prove:[1]

Theorem 2. *When S is a path plane, the tangential set $\mathfrak{A}(S, M_0)$ at each point forms a Lie triple system.*

Proof. If $A_1, A_2 \in (\mathfrak{A}(S, M_0) \cap U(0))$ and $P_i=M_0e^{A_i}$ $(i=1, 2)$ (2)′

then it must be principal paths $C(\overline{M_0, P_i})$ and $C(\overline{P_1, P_2})$ are contained in S. So, we have $C(M_0, P_3) \subset S$ for any point $P_3 \in [C(P_1, P_2)]$ and from extensivity of path it holds by Theorem 1

$$M_0e^{sP\{A_1, A_2, t\}}(\equiv M_0e^{s\{(1-t)A_1+tA_2+\cdots\}}) \in S \quad (s \text{ is arbitrary constant}) \qquad (2)''$$

and

$$P\{A_1, A_2, t\} \in \mathfrak{A}(S, M_0).$$

Now, we take uvA_1 and uwA_2 instead of A_1 and A_2 respectively (u and v are arbitrary constants) and by consideration the case $s\to\infty$, $u\to 0$ holding $us=$constant we have from above relation

$$M_0e^{k\{(1-t)vA_1+twA_2\}} \in S \quad (k \text{ is arbtrary constant}),^{2)}$$

hence $$k\{(1-t)vA_1+twA_2\} \in \mathfrak{A}(S, M_0)$$

namely, $$t_1A_1+t_2A_2 \in \mathfrak{A}(S, M_0) \quad (-\infty<t_1, t_2<\infty).^{3)} \qquad (3)$$

Next, we take $P\{vA_1, wA_2, t\}$ and $(1-t)vA_1+twA_2(\in \mathfrak{A}(S, M_0))$ for A_1 and A_2 in (2)′ and (2)″ and by the same reasoning as above we have

$$v^2w[A_1[A_1, A_2]]-vw^2[A_2[A_1, A_2]] \in \mathfrak{A}(S, M_0)$$

and v, w being arbitrary it follows from (3)

$$[A_1[A_1, A_2]] \in \mathfrak{A}(S, M_0).$$

Therefore[4] $$[A_1, [A_2, A_3]] \in \mathfrak{A}(S, M_0) \qquad (4)$$

for $A_1, A_2, A_3 \in \mathfrak{A}(S, M_0)$.

Hence S must be a Lie *t.s.*

If we use $\mathfrak{A}(S, M_0)$-linear instead of $\mathfrak{A}(S, M_0)$ in the process proving theorem 2, similarly we have:

Theorem 3. *When S is a general path plane, the tangential linear set i.e. $\mathfrak{A}(S, M_0)$-linear at each point forms a Lie t.s.*

Remark 1. Conversely, when (3) and (4) hold good, we have by some permutations of order for alternants and from Theorem 1 for A_1 and $A_2 \in (\mathfrak{A}(S, M_0) \cap U(0))$

$$\begin{aligned} P\{A_1, A_2, t\} &= \sum b^{i_1\cdots i_{2m+1}}[A_{i_1}\cdots[A_{i_{2m-2}}[A_{2m-1}[A_{i_{2m}}, A_{i_{2m+1}}]]\cdots] \\ &= \sum b^{i_1\cdots i_{2m+1}}[A_{i_1}\cdots[A_{i_{2m-2}}, A'_{(i_{2m-1}\, i_{2m}\, i_{2m+1})}]\cdots] \\ &= \sum b^{i_1\cdots i_{2m+1}}A'_{(i_1,\cdots,i_{2m+1})}=A' \end{aligned}$$

---

1) See [1].

2) We have this result from local property of S i. e. local path plane because we use only $us=k$ (k may be small) and property of $\mathfrak{A}$ (S, M_0).

3) The fact that the linearity (3) of $\mathfrak{A}(S, M_0)$ has been deduced is a characteristic result of this method.

4) See [4], p. 39, Theorem 2.

where all terms $A'_{(2m-1\ i_{2m}\ i_{2m+1})}, \cdots$ and $A' \in \mathfrak{A}(S, M_0)$ by (3) and (4).

So, it follows

$$[C(\overline{M_0 e^{A_1}, M_0 e^{A_2}})] = M_0 e^{P\{A_1, A_2, t\}} \subset M_0 e^{\mathfrak{A}(S, M_0)}$$

namely

$$[C(\overline{M_0 e^{A_1}, M_0 e^{A_2}})] \subset S.$$

Hence, when the tangential set $\mathfrak{A}(S, M)$ forms a Lie *t.s.*, $Me^{\mathfrak{A}(S, M_0)}$ is a local path plane at M.

And similarly we have the result:

When the linear tangential set $\mathfrak{A}(S, M)$*-linear forms a Lie t.s., S is a local general path plane at M.*

Remark 2. In a path-plane it is clear that $\mathfrak{A}(S, M_0) = \mathfrak{A}(S, M_0)$-linear, so there is no singular point.

Remark 3. Since analytical continuation of Lie triple system is available over $M_0\mathfrak{M}_0$ as it will be explained in Theorem 4, if $\mathfrak{A}(S, M_0)$ forms a Lie triple system then $\mathfrak{A}(S_0, M_1)$ ($S_0 \equiv M_0 e^{\mathfrak{A}_0(S, M_0)}$) does so for $M_1 \in M_0 e^{\mathfrak{A}_0(S, M_0)}$.[1] Therefore $M_0 e^{\mathfrak{A}(S, M_0)} \cap M_0\mathfrak{M}_0$ is a local path-plane when $\mathfrak{A}(S, M_0)$ forms a Lie triple system.

Now, we shall discuss the continuation of Lie triple system.

Theorem 4. *When $\mathfrak{A}(S, M)$ forms a Lie t.s. and $S_0 = Me^{\mathfrak{A}_0(S, M)}$, $\mathfrak{A}(S_0, R)$ forms a Lie t.s. for $R \in Me^{\mathfrak{A}_0(S, M)}$.*[2]

Proof. We take the small open set $V(0)$ such that $P\{A_1, A_2, t\}$ may be defined for any $A_1, A_2 \in V(0)$, and define

$$D_M = Me^{V(0)} \cap Me^{\mathfrak{A}_0(S, M)}.$$

Then, as $\mathfrak{A}(S, M)$ ($\equiv \mathfrak{A}(S_0, M)$) is a Lie triple system,

$$\text{principal-path} \quad [C(\overline{P_3, P_4})] \subset S_0 \tag{5}$$

for $P_3, P_4 \in D_M$ and the path has property of local extensivity in D_M.

And, moreover we define open set $D_{M_1}(D_M)$ such that

$$D_{M_1}(D_M) = M_1 e^{V(0)} \cap D_M \quad \text{for } M_1 \in D_M.$$

So, from (5) we have

$$\text{principal-path } [C(M_1, P_2)] \subset (M_1 e^{\mathfrak{A}_0(S_0, M_1)} \cap Me^{\mathfrak{A}_0(S_0, M)})$$

for M_1 and P_2 satisfying

$$P_2 \in D_{M_1}(D_M).$$

From (5),[3] $\mathfrak{A}(S_0, M_1)$ also forms a Lie *t.s.*. (6)

(If S is extensive, then $M_1 e^{\mathfrak{A}(S, M_1)} \subset S$).

As we can assert the homeomorphism[4] between $\mathfrak{A}_0$ and $M\mathfrak{M}_0$, the neighbourhood of M_1 in S_0 for $M_1 \in M\mathfrak{M}_0$ corresponds topologically to the

---

1), 2) See [3], p. 54. In this case $\mathfrak{A}_0$ means the part of $\mathfrak{A}$ in which absolute value of imaginary parts of characteristic roots are smaller than π and $e^{\mathfrak{A}_0} = \mathfrak{M}_0$.

3) See foot-note 2). p. 82.

4) See [5], p. 111, Theorem III.

neighbourhood of $\mathfrak{A}(S_0, M_1)$. And from the definition of S_0, dimension of $\mathfrak{A}(S_0, M_1)$ is equal to that of $\mathfrak{A}(S_0, M)$. So, it follows that in the (small) neighbourhood $U(M_1)$, $Me^{\mathfrak{A}(S_0,M)}$ coincides with $M_1e^{\mathfrak{A}(S,M_1)}$. And moreover a path $C(M, P_2, P_5)$ passing through $M, P_2, P_5(P_2, P_5 \in D_{M_1}(D_M))$ is a path in $M_1e^{\mathfrak{A}(S_0,M_1)}$ and it is clear that such paths cover the neighbourhood $U_1(C(\overline{M,M_1}))\cap S_0$ of the principal-path $C(\overline{M,M_1})$.

From the fact that $V(0)$ is definite and Me^{tA} $(0 \leq t \leq 1)$ is continuum, for any point $R(=Me^A)$ of S_0 we can take points M_i $(i=1,\cdots l)$ on Me^{tA} $(0 \leq t \leq 1)$ such that

$$M_\lambda \in M_{\lambda-1}e^{V(0)}$$

where $M_0=M$, $M_{l+1}=R$, $\lambda=1,2,\cdots,l+1$. By using the above process $(l+1)$-times repeatedly for the paths contained in

$$\bigcap_j U_j(C(\overline{M,M_1})) \qquad (j=1,2,\cdots,\lambda),$$

we have $Re^{\mathfrak{A}(S_0,R)}$ such that

$$U(R)\cap Re^{\mathfrak{A}(S_0,R)} \subset S_0$$

and $\mathfrak{A}(S_0, R)$ forms a Lie *t.s.*.

References

[1] E. Cartan, *La Géometrie des groupes de transformations*, J. Math. pures et appliqués, Tome **6** (1927), pp. 1–119.

[2] N. Jacobson, *Lie and Jordan Triple Systems*, A. J. Math. Vol. **71**, No. 1 (1949), pp. 149–170.

[3] K. Morinaga and T. Nono, *On the Matrix Space*, J. Sci. Hiroshima Univ., (A), Vol. **19**, No. 1 (1955), pp. 51–69.

[4] G. D. Mostow, *Some New Decomposition Theorems for Semi-Simple Lie Groups*, Memoirs of A. M. S., No. 14 (1955).

[5] K. Morinaga and T. Nono, *On the logarithmic functions of matrices* I, J. Sci. Hiroshima Univ., (A), Vol. **14**, No. 2 (1955), pp. 107–114.

Mathematical Institute,
Hiroshima University