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m-Connectedness and Polyhedral Inner Approximations of Plane Peano Continua

By

Akira TOMINAGA

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1. Introduction.

A plane peano continuum, M, is a set which is homeomorphic with a locally connected, connected compact (nonvoid) set in the plane π. Though its topological structure is generally clear, we shall somewhat notice the topological type in the large. The principal apparatus of investigation for M is cyclic element theory. We arrange plane peano continua in accordance with the notion "m-connectedness". Furthermore, for M we prove the equivalency of "simply connected", being equal to "1-connected", and some other notions, notably the equivalency of the topological notion "shrinking" and the metrical one "existence of the unique geodesic segment for each pair of points". The latter subject was proposed by Bing[1].

Next we intend to realise M in π as a mosaic consisting of segments and 2-simplices, i.e. to show that M is homeomorphic with a figure $\overline{\sum P_i}$ where P_i is an Euclidean polyheder[2], or, in other words, M is a *polyhedral inner approximation*.

Some of the definitions adopted in this paper are due to the Whyburn's book[3] and the Newman's one[4].

Notations for metric. Let R be a metric space with an associated distance-function $\rho(x, y)$ for $x, y \in R$. If $A \subseteq R$, $\rho(A)$ is the diameter of A for ρ, i.e. $\rho(A) = \sup\{\rho(x, y)/x, y \in A\}$. If $p \in R$, $\rho_p(A) = \sup\{\rho(p, x)/x \in A\}$. Let R be a metric space, having metric ρ, such that a pair of points, x, y, has a geodesic segment joining them. We shall denote the geodesic segment by $\langle x, y\rangle_\rho$, and denote $\langle x, y\rangle_\rho - y = \langle x, y)_\rho$ etc. d is the Euclidean metric in π. If $x, y \in \pi$, we abbreviate $\langle x, y\rangle_d$ by $\langle x, y\rangle$.

Notations for mapping. Let R, S be spaces and A a subset of R. If

(1) R. H. Bing: Partitioning continuous curves, Bull. Amer. Math. Soc. vol. **58** (1952), pp. 536–556.

(2) An Euclidean polyheder is the union of a finite set of simplices.

(3) G. T. Whyburn: Analytic topology, Amer. Math. Soc. Colloquium Publications, vol. **28** (1942), New York.

(4) M. H. A. Newman: Elements of the topology of plane sets of points, Cambridge at the University Press (1954).

f is a mapping of R to S, f/A is the restriction of f to A. "$R\cong S$" means "R is homeomorphic with S". Let $f_1, f_2, \cdots, f_k$ be a finite sequence of mappings of R into itself. We denote the composed mapping $f_k\cdots f_2f_1$ by $f_{(k\cdots 21)}$. If $f_1, f_2, \cdots, f_k, \cdots$ is an infinite sequence of mappings of R into itself and the composed mapping $\cdots f_k\cdots f_2f_1$ is nontrivial, we shall denote it by $f_{(\cdots k\cdots 21)}$.

Notation for domain. If ω is a simple closed curve in π, $[\omega]$ is the inner domain of ω.

2. *m*-Connected plane peano continua.

1) $\mathcal{C}_n$, $\mathcal{G}_n$, $\mathcal{K}_n$, $\mathcal{T}_n$ and $\mathcal{Q}_n$. Let M be a plane peano continuum. The collection, $\mathcal{C}$, of true cyclic elements of M is a null-sequence, i.e. a sequence of sets such that for any $\varepsilon>0$ at most a finite number of elements are of diameter greater than ε. If $C_\iota\in\mathcal{C}$, $C_\kappa\in\mathcal{C}$ and $C_\iota\neq C_\kappa$, then $C_\iota\cdot C_\kappa$ contains at most only one point in common. If α is a simple arc in M, $\alpha\cdot C_\iota$ is a simple arc or a point or the vacuous set, ϕ. Each complementary domain of C_ι has a simple closed curve as its frontier. Let ω_ι be the frontier of the unbounded complementary domain of C_ι, and let D_ι be the inner domain of ω_ι, i.e. $D_\iota=[\omega_\iota]$. C_ι will be said a *nodal-element* if $M-\overline{D_\iota}$ is connected. If C_ι is a nodal-element, $\overline{M-\overline{D_\iota}}\cdot C_\iota$ consists of only one point, $\boldsymbol{e}_\iota$. A *cut-element*, C_ι, of M is a true cyclic element such that $M-\overline{D_\iota}$ is not connected.

It is known that for C_ι and C_κ, either $D_\iota\cdot D_\kappa=\phi$ or $D_\iota\subseteq D_\kappa$ or $D_\iota\supseteq D_\kappa$. It will be said that C_ι *precedes* C_κ or C_κ *follows* C_ι, if $D_\iota\supset D_\kappa$ and $D_\iota\neq D_\kappa$. By "C_κ *is covered by* C_ι", it is meant that C_κ follows C_ι and there exists no true cyclic element which precedes C_κ and follows C_ι. C_ι will be called a *maximal* true cyclic element or a true cyclic element of the *first order* provided for each $C_\kappa\in\mathcal{C}$, $D_\iota\supseteq D_\kappa$ or $D_\iota\cdot D_\kappa=\phi$.

Let $\mathcal{K}_0$ be the collection consisting of only M and let $\mathcal{C}_1$ be the collection of maximal true cyclic elements, $C_{i_1}(i_1=1,2,\cdots)$. $\mathcal{D}_1$ is the collection D_{i_1}'s. The closure of each nonvoid component, T, of $M-\overline{\sum D_{i_1}}$ is a non-degenerate dendrite. We shall call $\overline{T}$ a *dendritic element of the first order* and denote by $\mathcal{T}_0=\{\overline{T^u}/u=1,2,\cdots\}$ the collection of $\overline{T}$'s. Now let $\mathcal{G}_1$ be the collection, $\{G_{i_1j_1}/i_1, j_1=1,2,\cdots\}$, of bounded complementary domains, $G_{i_1j_1}$, of maximal elements, C_{i_1}, and let $\mathcal{K}_1$ be the collection, $\{\overline{K}_{i_1j_1k_1}/i_1, j_1, k_1=1,2,\cdots\}$, of the closures of components, $K_{i_1j_1k_1}$, of $M-C_{i_1}$, contained in $G_{i_1j_1}$ for some i_1 and j_1. Then $\overline{K}_{i_1j_1k_1}\cdot C_{i_1}$ consists of only one point.

By induction, we can readily define $\mathcal{C}_n$, $\mathcal{D}_n$, $\mathcal{G}_n$, $\mathcal{K}_n$ and $\mathcal{T}_n$ as follows: $\mathcal{C}_n=\{C_{i_1j_1k_1,\cdots,i_{n-1}j_{n-1}k_{n-1},i_n}\}$ where $C_{i_1j_1k_1,\cdots,i_{n-1}j_{n-1}k_{n-1},i_n}$ is covered by $C_{i_1j_1k_1,\cdots,i_{n-1}}$ and is contained in $K_{i_1j_1k_1,\cdots,i_{n-1}j_{n-1}k_{n-1}}$ and is called a true cyclic element of the n-th order, $\mathcal{D}_n=\{D_{i_1j_1k_1,\cdots,i_n}\}$ where $D_{i_1j_1k_1,\cdots,i_n}$ is the inner domain of the frontier, which is a simple closed curve, of the unbounded complementary domain of $C_{i_1j_1k_1,\cdots,i_nj_n}$, $\mathcal{G}_n=\{G_{i_1j_1k_1,\cdots,i_nj_n}\}$ where $G_{i_1j_1k_1,\cdots,i_nj_n}$ is a bounded complementary domain of $C_{i_1j_1k_1,\cdots,i_n}$, $\mathcal{K}_n=\{K_{i_1j_1k_1,\cdots,i_nj_nk_n}\}$ where $K_{i_1j_1k_1,\cdots,i_nj_nk_n}$ is a component of $M-C_{i_1j_1k_1,\cdots,i_n}$ which is contained in $G_{i_1j_1k_1,\cdots,i_nj_n}$, and $\mathcal{T}_n=\{\overline{T^u_{i_1j_1k_1,\cdots,i_nj_nk_n}}\}$ where $T^u_{i_1j_1k_1,\cdots,i_nj_nk_n}$ is a component of $K_{i_1j_1k_1,\cdots,i_nj_nk_n}-\overline{\sum_{i_{n+1}}D_{i_1j_1k_1,\cdots,i_nj_nk_n,i_{n+1}}}$.

Furthermore each component of $\pi-\overline{\pi-M}$ is contained in a true cyclic element of M, and we denote the collection of such components by $\mathcal{Q}_n=\{Q^v_{i_1j_1k_1,\cdots,i_n}\}$, where $Q^v_{i_1j_1k_1,\cdots,i_n}$ is contained in $C_{i_1j_1k_1,\cdots,i_n}$. Hence we have

$$(*)\quad\begin{cases} D_{i_1j_1k_1,\cdots,i_n}\supseteq G_{i_1j_1k_1,\cdots,i_nj_n},\\ \overline{G}_{i_1j_1k_1,\cdots,i_nj_n}\supseteq \overline{K}_{i_1j_1k_1,\cdots,i_nj_nk_n}\supseteq C_{i_1j_1k_1,\cdots,i_nj_nk_n,i_{n+1}},\\ \overline{K}_{i_1j_1k_1,\cdots,i_nj_nk_n}\supseteq \overline{T^u_{i_1j_1k_1,\cdots,i_nj_nk_n}}\\ \text{and } C_{i_1j_1k_1,\cdots,i_nj_nk_n,i_{n+1}}\supseteq Q^v_{i_1j_1k_1,\cdots,i_nj_nk_n,i_{n+1}}.\end{cases}$$

(Suffix notations, i_l, j_l, k_l, u, v etc., run independently on the positive integers.)

2) m-connectedness. It will be called that a plane peano continuum, M, is *m-connected* provided that $\mathcal{G}_m=\phi$. If M is m-connected, then $\mathcal{G}_n$, $\mathcal{K}_n$, $\mathcal{C}_{n+1}$, $\mathcal{T}_n$ and $\mathcal{Q}_{n+1}$ are all empty for $n\geqq m$ (by (*)). If $\mathcal{G}_n\neq\phi$ for any n, then M is ∞-connected. M will be said to be *simply connected* provided that if a simple closed curve, ω, is in M, so is the inner domain of ω. It is clear that "simply connected" is equivalent to "1-connected". If M is simply connected, each true cyclic element is a 2-cell, that is, a set which is homeomorphic with 2-simplex. If $\sum_n Q_n=\phi$, M is at most 1-dimensional. M will be called to be *simple* provided that M is simply connected and provided but at most two (non-degenerate or true) cyclic elements are cut-elements. If M is simply connected and is the union of a finite set of simple peano continua, it is called to be *regular*.

We have readily

THEOREM 1. *If M is m (or ∞)-connected, then for each $n\leqq m$ (or $n\leqq\infty$) there exists uniquely the minimal n-connected plane peano continuum, M_n, containing M. In particular, for each plane peano continuum M there exists uniquely the minimal simply connected set, M_1, containing M. Hence we can evidently arrange M_i's in order so that $M_1\supseteq M_2\supseteq\cdots\supseteq M$, and we have* $M=\prod_{n\leq m\,(or\,\infty)} M_n$.

Furthermore, we remark that if M is m-connected, each element of $\mathcal{K}_{m-k}(m>k)$ is k-connected, particularly each element of $\mathcal{K}_{m-1}$ is simply connected.

Among plane peano continua, that is, continuous images of the closed interval $\langle 0,1\rangle$ in π, the simplest figures are dendrites (for example points, simple arcs, etc.), 2-cells and simple closed curves, and we can characterize them by $\mathcal{C}_n$, $\mathcal{G}_n$, $\mathcal{K}_n$, $\mathcal{T}_n$ and $\mathcal{Q}_n$, as follows: in order that

M is a $\begin{cases}\text{(i) dendrite,} \\ \text{(ii) 2-cell,} \\ \text{(iii) simple closed curve,}\end{cases}$ it is necessary and sufficient that

$\begin{cases}\text{(i) } C_1=\phi, \\ \text{(ii) } \phi=\mathcal{T}_0=\mathcal{G}_1 \text{ and } C_1 \text{ consists of only one element,} \\ \text{(iii) } \phi=\mathcal{T}_0=\mathcal{K}_1=\mathcal{Q}_1 \text{ and } \mathcal{G}_1 \text{ consists of only one element.}\end{cases}$

3. Simply connected.

In this paragraph we shall consider relations of several topological properties for simply connected continua, though some of them are known.

LEMMA 1. *Let Q be a square, $\square q_1q_2q_3q_4$, plus its interior, and H a disconnected closed set such that $H\cdot(q_1q_2+q_3q_4)=\phi$, $H\cdot q_1q_4\neq\phi$ and $H\cdot q_2q_3\neq\phi$, and such that no component of H meets both q_1q_4 and q_2q_3. Then there exists a cross-cut, α, in the interior of Q such that $\alpha\cdot H=\phi$, $a_1\in q_1q_2$, $a_2\in q_3q_4$, where $a_i(i=1,2)$ are end-points of α. Thus α separates $H\cdot Q$ in Q.*

PROOF. Let N be the sum of components of H meeting q_1q_4 and let $N_1=N+q_1q_4$, $N_2=(H-N)+q_2q_3$. Then we have $N_1\cdot N_2=\phi$. Let G be a refinement of the grating $\square q_1q_2q_3q_4$ such that no cell meets both $N_1\cdot Q$ and $N_2\cdot Q$. Let K be the sum of the elements, contained in Q, of G meeting $N_1\cdot Q$, and let K be thickened into K_1, by means of a refinement, G_1, of G, of which no element meets both K and $N_2\cdot Q$. Then K_1 is a true cyclic element, and the frontier of the unbounded complementary domain of K_1 is a simple closed curve, ω, such that $\omega\supset q_1q_4$, $(\omega-q_1q_4)\cdot(N_1+N_2)=\phi$. We can find the required cross-cut, α, as a subarc of ω.

THEOREM 2. *If M is any plane peano continuum, then the following properties are equivalent:*

(i) *that M is simply connected,*

(ii) *that M has a convex metric such that for each pair of points x, y, there exists the unique geodesic segment joining them,*

(iii) *that M is shrinkable,*

(iv) *that M does not separate π,*

(v) *unicoherence,*

(vi) *Brouwer property,*

(vii) *Phragmen-Brouwer property,*

(viii) *fixed-point property.*

These properties are cyclicly extensible.

PROOF. (i)$\rightarrow$(ii). Let M be a simply connected peano continuum, then each true cyclic element, C, of M is homeomorphic with 2-simplex, so that we can introduce a convex metric, having property (ii), on C. This property can be extended to the whole, and we denote the metric by ρ.

(ii)$\rightarrow$(iii). Let p be an arbitrary point of M. We define a continuous mapping, f, of $M\times\langle 0,1\rangle$ into M as follows:

$$f(x,r)=\begin{cases}\text{the point, on the geodesic segment joining } p \text{ to } x, \text{ such that}\\ \quad \rho(p,f(x,r))/\rho(p,x)=r, \text{ if } x\neq p,\\ p \text{ if } x=p.\end{cases}$$

(iii)$\to$(i). If M is not simply connected, there exists a simple closed curve, ω, in M which $[\omega]\not\subset M$. Let D_1 be a bounded complementary domain of M which meets $[\omega]$. Since M is shrinkable, so is ω in M, and therefore in $\pi-D_1$. However, this is impossible.

(i)$\to$(iv). If M separates π, that is, there exist points x, y, separated by M, then a simple closed curve, ω, in M separates x and y. Hence $[\omega]$ contains x or y, say x. Since $x\notin M$, M is not simply connected by definition.

(iv)$\to$(i). If M is not simply connected, there exists a simple closed curve, ω, contained in M, such that $[\omega]\not\subset M$. Since a bounded complementary domain of M is contained in $[\omega]$, M separates π.

(i)$\to$(v). For suppose on the contrary that M is not unicoherent. There exist two connected closed sets, A and B, such that $A\cdot B$ is disconnected and $M=A+B$. Then there exists a separation of $A\cdot B$, $A\cdot B=H_1+H_2$. We may suppose that A and B are locally connected. For since H_1 and H_2 are mutually exclusive compact sets, $d(H_1, H_2)=\eta>0$. On the other hand since M is a peano continuum, each point, x, of M has a neighborhood, U_x, such that $\overline{U_x}$ is locally connected and its diameter is not greater than $\eta/3$. Since H_i is compact, we can select a finite number of points, $x_i^k (k=1,\cdots,n_i)$, in H_i such that $H_i\subseteq\sum_1^{n_i}\overline{U_{x_i^k}}=K_i$. Obviously $K_1\cdot K_2=\phi$. Now let $A_1=A+K_1+K_2$, $B_1=B+K_1+K_2$, then $A_1\cdot B_1=K_1+K_2$. It is clear that the components of K_i are finite in number.

Let C_i be a component of K_i, and p_i a point in C_i. Draw an arc, α(or β), in A_1(or B_1), joining p_1 to p_2. Let a_1 be the last intersection of α with C_1, a_2 the next intersection with $(K_1+K_2)-C_1$ and a_3 the first intersection with C_2, for the direction from p_1 to p_2. b_i has the analogous meaning as a_i if we put β in place of α.

On the other hand, joining a_1(or a_3) to b_1(or b_3) by an arc in C_1(or C_2), we have a simple closed curve, ω, which is contained in $a_1a_2a_3+a_3b_3+b_3b_2b_1+b_1a_1$ and $a_1a_2+b_1b_2+a_1b_1\subseteq\omega$. Let $a_1'a_2'$ and $b_1'b_2'$ be proper subarcs of a_1a_2 and b_1b_2 respectively, such that a_i'(or b_i') is near to a_i(or b_i) and $a_i'\neq a_i$(or $b_i'\neq b_i$), for $i=1,2$. In Lemma 1, put $Q=[\omega]$, $H=K_1+K_2$, $a_1'=q_1$, $a_2'=q_2$, $b_1'=q_4$ and $b_2'=q_3$. Then there exists a simple arc, e_1e_2, which does not meet K_1+K_2 and whose end-points, e_1, e_2, are in $A_1-(K_1+K_2)$ and in $B_1-(K_1+K_2)$ respectively. Since M is simply connected, $[\omega]\subseteq M$ and therefore $e_1e_2\subset M$. Then we have a separation $e_1e_2=e_1e_2\cdot A_1+e_1e_2\cdot B_1$, which is contrary to the connectedness of e_1e_2.

(v)$\to$(i). Suppose M is not simply connected, then there exists a simple closed curve, ω, such that $[\omega]\not\subset M$. Let $p\in[\omega]-M$, and let α be a cross-cut in $[\omega]$, containing p and having q_1 and q_2 as its end-points. q_1 and q_2

divide ω into the open arcs γ and γ'. Since ω is accessible from each complementary domain, there exist two disjoint unbounded topological rays, $\beta_i(i=1,2)$, contained in q_i+(the outer domain of ω) and having q_i's as their end-points. $\pi-(\beta_1+\alpha+\beta_2)=\Delta+\Delta'$, where Δ and Δ' are mutually exclusive domains and $\Delta'\supset\gamma'$.

Let C' be the component of $\Delta'\cdot M$ containing γ' and let C be the component of $M-C'$ containing γ. Then C and $\overline{M-C}$ are connected closed sets such that $C\cdot\overline{M-C}$ is a disconnected subset of $\beta_1+\alpha+\beta_2$. For if there exists a separation of $M-C$ such that $M-C=N_1+N_2$, C' is contained in N_1 or N_2, say in N_1. Since C is a closed subset of the locally connected, connected set M, every component of $M-C$ has limit points in C. Therefore if K is a component of N_2, then $\overline{K}\cdot C\neq\phi$. On the other hand, since $N_1\cdot\overline{N_2}=\phi$, we have $N_1\cdot\overline{K}=\phi$ and therefore $C'\cdot\overline{K}=\phi$, i.e. $\overline{K}\subseteq M-C'$. Since C is a component of $M-C'$ and $\overline{K}\cdot C\neq\phi$, we conclude $K\subseteq\overline{K}\subseteq C$. This is impossible.

(v), (vi) and (vii) are all equivalent[5].

(i)$\rightarrow$(viii). Let M be a simply connected plane peano continuum. Since fixed-point property is cyclicly extensible and each true cyclic element of M is a 2-cell, M has fixed-point property by Brouwer's theorem for fixed-point.

(viii)$\rightarrow$(i). For suppose M is not simply connected, then it contains a simple closed curve, ω, such that $[\omega]\not\subset M$. Let C be the true cyclic element containing ω. If $\mathfrak{N}$ is the family of (nonvoid) components of $M-C$, then for each element, N, of $\mathfrak{N}$, $\overline{N}\cdot C$ contains only one point, p_N. Let h be a continuous mapping of M onto C such that $h(x)=\begin{cases}p_N & \text{if } x\in N\in\mathfrak{N},\\ x & \text{if } x\in C.\end{cases}$ A bounded complementary domain of C has a simple closed curve, ω_1, as its frontier. Let f be a homeomorphism of the whole plane π such that $f(\omega_1)=\omega_0$, where ω_0 is the unit circle in π with center at the origin $(0,0)$. If λ is a ray issuing from $(0,0)$, $\lambda\cdot\omega_0$ consists of a single point, q_λ. Let g be the continuous mapping of $f(C)$ onto ω_0 such that, if $x\in\lambda\cdot f(C)$, $g(x)=q_\lambda$. Let r be the rotation of ω_0 through 90 degrees. Define a continuous mapping, φ, of M into itself by $\varphi=f^{-1}rgfh$. Obviously φ has no fixed-point.

4. Polyhedral inner approximations.

LEMMA 2. *Let F be a segmental set which is a dendrite and let $\alpha_i(i=1,2)$ be simple arcs which have a common end-point, p, and $\alpha_i\cdot F=p$. A spherical neighborhood, U_p, of p is divided by F into a finite number of open components, $\{D_j\}$. If for an arbitrary small U_p, subarcs of α_i's containing p are in the*

(5) Cf. R. L. Wilder: Topology of manifolds, Amer. Math. Soc. Colloquium Publications, vol. **32** (1949), New York.

same set, D_j+p, and if f is homeomorphism of α_1 onto α_2 under which p is fixed, then f can be extended to an automorphism, φ, of π such that each point of F is fixed under φ.

PROOF. We may assume that U_p is so small that $(\alpha_1+\alpha_2)\cdot U_p=(\alpha_1+\alpha_2)\cdot D_j+p$.

If p is a cut-point of F, there exists a cross-cut, $bpa=\langle b,p\rangle+\langle p,a\rangle$, in U_p such that bpa is a subarc of F containing p and has a, b as end-points, and such that D_j is bounded by the simple closed curve $\widehat{ab}+bpa$ where $\widehat{ab}$ is a subarc of the frontier, $\mathfrak{F}(U_p)$, of U_p. Since $(\alpha_1+\alpha_2)\cdot F=p$, for each point, x, of $F-p$, there exists a spherical neighborhood, V_x, such that $\overline{V}_x\cdot(\alpha_1+\alpha_2)=\phi$. Let a', b' be points in $\widehat{ab}\cdot V_a$, $\widehat{ab}\cdot V_b$, respectively. a'(or b') and p can be joined by an arc, $a'p$(or $b'p$), in $D_j\cdot\sum_{x\in\langle ap\rangle}V_x$(or $D_j\cdot\sum_{x\in\langle bp\rangle}V_x$). Since $F-U_p=K$ is compact, there exist a finite number of points, $x_i(i=1,\cdots,m)$, in K such that $\sum_1^m V_i=V\supset K$ where $V_{x_i}=V_i$. Now let $W=[a'p+pb'+(\mathfrak{F}(U_p)-\widehat{a'b'})]$ where $\widehat{a'b'}\subset\widehat{ab}$, then $\overline{W}+\overline{V}=N$ is a peano continuum containing F and $N\cdot(\alpha_1+\alpha_2)=p$. If N is not simply connected, the bounded complementary domain, $Q_i(i=1,\cdots,n)$, of N are finite in number. Let q_i be a point of Q_i and R the unbounded complementary domain of N. Since F is a dendrite, there exists an arc, α_i, joining q_i to a point of R and contained in $\pi-F$. α_i can be covered by a region, Δ_i, being a sum of finite spherical neighborhoods, contained in $\pi-F$, whose centers are on α_i. Let P be the component of $N-\sum_1^n\Delta_i$ such that $P\supseteq F$. Then P is a simply connected plane peano continuum and its frontier, ω, is a simple closed curve such that the inner (or outer) domain of ω contains $F-p$ (or $(\alpha_1+\alpha_2)-p$) and $p\in\omega$.

If p is an end-point of F, $\{D_j\}$ contains only one element, D. In this case there exists an end-cut, $\langle p,a\rangle$, in U_p, which is a subarc of F and has p as its inner extremity, a as its end-point. Let V'_a be a spherical neighborhood of a such that $\overline{V'_a}\subset V_a$. $\mathfrak{F}(U_p)\cdot\mathfrak{F}(V'_a)$ consists of two points a', b'. The following process exactly reproduces that in case where p is a cut-point. Hence we can find a simple closed curve, ω, such that $\omega\cdot F=p$ and the inner (or outer) domain contains $F-p$ (or $(\alpha+\alpha')-p$).

By a preliminary automorphism of π the point at infinity can be placed in $[\omega]$. Thus Lemma 1 is reduced to the following version: *let $\alpha_i(i=1,2)$ be end-cuts in $[\omega]$, having p as a common end-point and a_i as the inner extremity of α_i. Let f be a homeomorphism of α_1 onto α_2 under which p is fixed. Then there exists an automorphism, φ, of π, such that φ carries $[\omega]$ onto itself and agrees with f on α_1, and such that φ leaves invariant each point of $\pi-[\omega]$.*

To prove this, let q be a point of ω different from p. a_i and q can be joined by an arc, a_iq, in $\overline{D}$ such that pa_iq is a cross-cut in D. p and q divide ω into the arcs β and β'. Let f' be a homeomorphism of a_1q

onto a_2q such that $f'(q)=q$. Let $D_i=[\beta+pa_iq]$ and $D'_i=[\beta'+pa_iq]$. Then there exists a homeomorphism ψ(or ψ') of $\overline{D}_1$(or $\overline{D}'_1$) onto $\overline{D}_2$(or $\overline{D}'_2$) such that

$$\psi(x)=\begin{cases} x & \text{if } x\in\beta, \\ f(x) & \text{if } x\in pa_1, \\ f'(x) & \text{if } x\in qa_1 \end{cases} \left(\text{or } \psi'(x)=\begin{cases} x & \text{if } x\in\beta', \\ f(x) & \text{if } x\in pa_1, \\ f'(x) & \text{if } x\in qa_1 \end{cases}\right).$$

Finally we define

$$\varphi(x)=\begin{cases} \psi(x) & \text{if } x\in\overline{D}_1, \\ \psi'(x) & \text{if } x\in\overline{D}'_1, \\ x & \text{if } x\in\notin[\omega], \end{cases}$$

which is the required homeomorphism.

Q.E.D.

(A) **Dendrite.** Let T be a dendrite, and ρ a convex metric on T. We may assume, without loss of generality, that $\rho(T)=1$.

1) There exists a pair of points, a_1, b_1, such that $\rho(a_1, b_1)=\rho(T)$. Let $\langle a_1, b_1\rangle_\rho=\kappa_1$ and $\mathscr{B}_1=\{\kappa_1\}$. Now let $\mathscr{B}_{m+1}$ be the collection of components, N's, of $T-\sum_1^{n_m}\kappa_i$, such that $\rho_a(N)\geqq 1/2^{m+2}$ where $a=\overline{N}\cdot\sum_1^{n_m}\kappa_i$. Since T is locally connected, $\mathscr{B}_{m+1}$ contains at most a finite number of elements and $\mathscr{B}_{m+1}=\{N_{n_m+1}, N_{n_m+2},\cdots, N_{n_{n+1}}\}$. For each $i(n_m<i\leqq n_{m+1})$, we can find a point, $b_i\in N_i$, such that $\rho(a_i, b_i)=\rho_{a_i}(N_i)$. Let $\langle a_i, b_i\rangle_\rho=\kappa_i$. Hence we have a sequence of geodesic segments,

$$\kappa_1, \kappa_2\cdots.$$

It is readily seen that $T=\overline{\sum\kappa_i}$.

2) Let f_1 be an automorphism of π such that

(1) $f_1(\kappa_1)=\langle(0,0), (1,0)\rangle=\lambda_1$, so that $f_1(a_1)=(0,0)$ and $f_1(b_1)=(1,0)$,

(2) f_1/κ_1 is an isometric mapping for ρ, d.

A small subarc of $f_1(\kappa_2)$ containing $f_1(a_2)$ is in $H_1+f_1(a_2)$ or $H_2+f_1(a_2)$, say in $H_1+f_1(a_2)$, where H_1 (or H_2) is the open half-plane whose elements have positive (or negative) y-coordinates. We draw the perpendicular, having $f_1(a_2)$ as its foot, to the x-axis, which is contained in $H_1+f_1(a_2)$, and cut it at the point, b'_2, such that $d(f_1(a_2), b'_2)=\rho(\kappa_2)$. By Lemma 1, there exists an automorphism, f_2, of π such that

(1) $f_2(\kappa_2)=\langle f_1(a_2), b'_2\rangle=\lambda_2$ so that $f_2(b_2)$ and $f_2(x)=x$ for $x\in\kappa_1$,

(2) f_2/κ_2 is an isometric mapping for ρ, d.

Enclose λ_2 by a simple closed curve, ω_2, such that

(i) $\omega_2\cdot(\lambda_1+\lambda_2)=f_1(a_2)$,

(ii) λ_2 is an end-cut in $[\omega_2]$ and $\lambda_1-f_1(a_2)\subset\pi-\overline{[\omega_2]}$.

By induction, we shall define λ_i, f_i and $\omega_i(n_m<i\leqq n_{m+1})$ as follows. An open subarc of κ_i, having a_i as its end-point, is contained in one of the domains into which a small enough neighborhood of a_i is separated by $\sum_1^{m}\kappa_j$, say in D. There exists a segmental arc, λ_i, such that

(a) λ_i has $f_{(n_m\cdots 1)}(a_i)$ as its end-point,

(b) a small subsegment of λ_i containing $f_{(n_m\cdots 1)}(a_i)$ is in $f_{(n_m\cdots 1)}(D+a_i)$,

(c) (the length of λ_i for $d)=\rho(\kappa_i)$,

(d) $\lambda_i\subset[\omega_j]$ if $f_{(n_m\cdots 1)}(a_i)\in[\omega_j]$, $\lambda_i\cdot[\omega_j]=\phi$ if $f_{(n_m\cdots 1)}(a_i)\notin[\omega_j]$ $(1\leqq j<i)$.

Then there exists an automorphism, f_i, of π such that

(1) $f_{(i\cdots 1)}(\kappa_i)=\lambda_i$,

(2) $f_{(i\cdots 1)}/\kappa_i$ is an isometric mapping for ρ, d.

Next we enclose λ_i by a simple closed curve, ω_i, such that

(i) $\omega_i\cdot\sum_1^i\lambda_j=f_{(i\cdots 1)}(a_i)$,

(ii) λ_i is an end-cut in $[\omega_i]$ and $\omega_i\subset[\omega_j]$ if $\lambda_i\subset[\omega_j]$, $\omega_i\cdot[\omega_j]=\phi$ if $\lambda_i\cdot[\omega_j]=\phi$. $(1\leqq j<i)$.

We note here that if branch points of T, i.e. points having order >2, are finite in number, the procedure for ω_i is unnecessary.

3) $T\cong\overline{\sum\lambda_i}$. For we first notice that $T=\overline{\sum\kappa_i}$ and $\sum\kappa_i\cong\sum\lambda_i$. Let $p\in T-\sum\kappa_i$ and let $\{p_k\}$ be an arbitrary sequence in $\sum\kappa_i$ converging to p. Let $f_{(\cdots i\cdots 1)}(p)=\lim\limits_{k\to\infty}f_{(\cdots i\cdots 1)}(p_k)$. By the condition ($d$) for λ_i and (ii) for φ_i, $f_{(\cdots i\cdots 1)}$ is an $1-1$ continuous mapping of T onto $\overline{\sum\lambda_i}$. Since T is compact, $f_{(\cdots i\cdots 1)}$ is the required homeomorphism.

(B) **True cyclic elements.** 1) Let C be a true cyclic element. The frontier of each complementary domain of C is a simple closed curve. Let ω be the frontier of the unbounded complementary domain of C. The collection of the bounded complementary domains of C is a null-sequence, $\{D_i/i=1,2,\cdots\}$, each of whose elements is a simply connected domain. Let $\mathcal{A}$ be the collection of components of $\sum\limits_{k\neq l}\overline{D_k}\cdot\overline{D_l}$. Each element of $\mathcal{A}$ is a simple arc. Let S be the set of all points in $\sum\mathcal{F}(D_i)$ which have closed neighborhoods (relative to C) homeomorphic with 2-simplex. Let $\mathcal{B}$ be the collection of components of S. Each element of $\mathcal{B}$ is a simple closed curve or an open simple arc. Let $\mathcal{E}=\mathcal{A}+\mathcal{B}$. $\{\omega\}+\mathcal{E}$ is a countable collection, $\{\kappa_i/i=1,2,\cdots;\ \kappa_1=\omega\}$, of mutually exclusive sets.

2) Let λ_1 be the frontier of a convex 2-cell. There exists an automorphism, g_1, of π such that $g_1(\kappa_1)=\lambda_1$ and $g_1([\kappa_1])=[\lambda_1]$. By induction, we shall define a sequence of automorphisms, $g_2, g_3,\cdots$, of π as follows. Let $\kappa_i'=g_{(i-1\cdots 0)}(\kappa_i)$ and $\varepsilon_i=d(\kappa_i')$. If $\kappa_i\in\mathcal{A}$, there exist D_k, D_l such that $\kappa_i\subset\overline{D_k}\cdot\overline{D_l}$. Let a, b be end-points of κ_i'. In an $\varepsilon_i/2^i$-neighborhood of κ_i', we can draw a simple arc, λ_i, such that λ_i is segmentwise[(6)] and is an end-cut in D, where $D=g_{(i-1\cdots 0)}(D_k+D_l)+(\kappa_i'-(a+b))$ and such that λ_i has a, b as its end-points. Let g_i' be a homeomorphism of κ_i' onto λ_i such that $g_i'(a)=a$, $g_i'(b)=b$. Then we can define an automorphism, g_i, of π such that $g_i(x)=\begin{cases}g_i'(x) & \text{if } x\in\kappa_i',\\ x & \text{if } x\notin D.\end{cases}$

(6) A simple arc or simple closed curve, α, is segmentwise if $\alpha=\overline{\sum P_i}$ where P_i is a segmental set.

Furthermore if $\kappa_i \in \mathscr{B}$, there exists D_k such that $\kappa_i \subset \mathscr{F}(D_k)$. Each point, p, of κ'_i has a neighborhood, U_p, in π such that $\overline{U_p} \cdot g_{(i-1\cdots 1)}((\mathscr{F}(D_k)-\kappa_i)+\sum_{j\neq k} D_j)=\phi$, $d(U_p)<\varepsilon_i/2^i$ and κ'_i is an end-cut in $[\omega]$ and such that $\overline{U_p \cdot g_{(i-1\cdots 1)}(C)}$ is a closed 2-cell. When κ'_i is an open arc, we can easily find a simple closed curve, ω, in $\sum_{p\in\kappa'_i} U_p+a+b$, such that a, b divide ω into two arcs α, β and such that $\alpha-(a+b)$ (or $\beta-(a+b)$) is contained in the interior of $g_{(i-1\cdots 1)}(C)$ (or in $g_{(i-1\cdots 1)}(D_k)$). There exists an end-cut, λ_i, being segmentwise, in $[\omega]$ such that λ_i has a, b as its end-points. Let g_i be a homeomorphism of $[\omega]$ onto itself such that $g_i([\alpha+\kappa'_i])=[\alpha+\lambda_i]$, $g_i(\beta+\kappa'_i])=[\beta+\lambda_i]$ and $g_i(x)=x$ for $x \notin [\omega]$.

When κ'_i is a simple closed curve, we can find two simple closed curves, ω, ω', in $\sum_{p\in\kappa'_i} U_p$, such that ω (or ω') is contained in the interior of $g_{(i-1\cdots 1)}(C)$ (or in $g_{(i-1\cdots 1)}(D_k)$). Thus κ'_i is in the interior of the annulus, $A(\omega, \omega')$, bounded by ω and ω'. There exists a simple closed curve, λ_i, containing in the interior of $A(\omega, \omega')$ such that λ_i is segmentwise and $[\lambda_i] \supset \omega'$. We can readily construct such an automorphism, g_i, of π as $g_i(A(\omega, \kappa'_i))=A(\omega, \lambda_i)$, $g_i(A(\omega', \kappa'_i))=A(\omega', \lambda_i)$ and $g_i(x)=x$ for $x \notin A(\omega, \omega')$. $C \cong g_{(\cdots i \cdots 1)}(C)$ shows M is a polyhedral inner approximation.

(C) **Simply connected continua.**

LEMMA 3. *Let C_0, C_1 be end-points or nodal-elements of a simple continuum, F, and let*

$$a_i=\begin{cases} C_i \text{ if } C_i \text{ is an end-point,} \\ \text{a point, belonging to the frontier of } C_i \text{ and being different from } \boldsymbol{e}_i \\ \qquad \text{(para. 2,1), if } C_i \text{ is a nodal-element.}\end{cases}$$

Then there exists a simple closed curve, ω, such that $\omega \cdot F=a_0+a_1$ and such that $F-(a_0+a_1)$ is contained in $[\omega]$.

PROOF.(7) Let ε be a positive number. If a_0 is an end-point of F, there exists a cut-point, q, of F in $U(a_0, \varepsilon)$ such that the component, N, of $F-q$ containing a_0 is in $U(a_0, \varepsilon)$. We denote $q+N$ by N_1. Otherwise, since a_0 belongs to a nodal-element (which is a 2-cell) and is different from $\boldsymbol{e}_0$, there exists a cut-arc, γ, of F in $U(a_0, \varepsilon)$ such that the component, N', of $F-\gamma$ containing a_0 is in $U(a_0, \varepsilon)$. In this case we shall also denote $\gamma+N'$ by N_1. In either case, N_1 and $\overline{F-N_1}$ are continua. If δ is the distance of a_0 from $F-N_1$, any two points x and y, belonging to $U(a_0, \delta)$, can be joined by an arc which meets neither $\overline{F-N_1}$ nor the frontier, B, of $U(a_0, \varepsilon)$.

Furthermore, since F is simply connected, x and y can not be separated by F in π (by Theorem 2). Since $F \cdot B \subseteq \overline{F-N_1}$, we have $F \cdot (B+\overline{F-N_1}) =\overline{F-N_1}$. Hence x and y are not separated by $F+B$ and there exists a

(7) See footnote (4).

continuum joining x to y in $U(a_0, \varepsilon)-F$. Thus $\pi-F$ is locally connected at a_0, similarly at a_1, so a_0 and a_1 are accessible from $\pi-F$. Thus it is easy to show that there exists the required simple closed curve. Q.E.D.

Lemma 2 can be deduced in more general form,

LEMMA 4. *Let F be regular (para. 2), and let $\{\alpha_i/i=1,2\}$ be simple continua such that both $F+\alpha_1$ and $F+\alpha_2$ are regular and such that $\alpha_1\cdot F=\alpha_2\cdot F=$only one point, p. A small enough neighborhood, U_p, of p is divided by F into a finite number of components, $\{D_j\}$. If small non-degenerate subcontinua of α_i's, containing p, are in the same set D_j+p and if f is an orientation-preserving homeomorphism of α_1 onto α_2 under which p is fixed, f can be extended to an automorphism, φ, of π such that each point of F is fixed under φ.*

Let ρ be a convex metric having the property (ii) in Theorem 2. We may assume that $\rho(M)=1$.

1) There exists, a pair of points a_1, b_1, such that $\rho(a_1, b_1)=\rho(M)$. Let $\langle a_1, b_1\rangle_\rho=\mu_1$ and let C^1 be the collection of true cyclic elements, C, meeting μ_1 with simple arcs. We denote $\mu_1+\sum_{C\in C^1} C$ by κ_1 and put $\mathcal{B}_1=\{\kappa_1\}$. Now let $\mathcal{B}_{m+1}$ be the collection of components, N, of $M-\sum_1^{n_m}\kappa_i$ such that $\rho_a(N)\geqq 1/2^{m+2}$ where $a=\bar{N}\cdot\sum_1^{n_m}\kappa_i$. Then we have $\mathcal{B}_{m+1}=\{N_{n_m+1}, N_{n_m+2},\cdots, N_{n_{m+1}}\}$. For each $i(n_m<i\leqq n_{m+1})$, there exists a point, $b_i\in N_i$, such that $\rho(a_i, b_i)=\rho_{a_i}(N_i)$. Let $\langle a_i, b_i\rangle_\rho=\mu_i$ and let C^i be the collection of true cyclic elements meeting μ_i with simple arcs. We denote $\mu_i+\sum_{C\in C^i} C$ by κ_i. Hence we have a sequence of simple continua

$$\kappa_1, \kappa_2, \cdots$$

such that $M=\overline{\sum\kappa_i}$.

It is readily seen that $\sum_1^m\kappa_i$ is regular.

2) Let φ_1 be an automorphism of π such that

(1) $\varphi_1(\mu_1)=\langle(0,0), (1,0)\rangle=\nu_1$ so that $\varphi_1(a_1)=(0,0)$ and $\varphi_1(b_1)=(1,0)$,

(2) φ_1/μ_1 is an isometric mapping for p, d.

Let $C\in C^1$ and let x, y be the first intersection of ν_1 and the last one with $f_1(C)$ respectively, taking the direction of ν_1 from (0,0) to (1,0). If ζ is the frontier of C, then x, y divide $\varphi_1(\zeta)$ into two arcs α, α'. Two small subarcs of α(or α'), containing x and y respectively, are contained in $\bar{H}_1$(or $\bar{H}_2$), where H_i has been defined in case (A). Let γ(or γ') be the simple arc, contained in $\sum\alpha+\nu_1$(or $\sum\alpha'+\nu_1$), such that $\gamma\supset\sum\alpha$(or $\gamma'\supset\sum\alpha'$). Next we construct the figure consisting of ν_1 and equilateral triangles, Δxyz, contained in $\bar{H}_1$. Let $\beta_1=\sum(xz+zy)+\beta_1'$ where $\beta_1'=\nu_1-\sum xy$. Then β_1 is a simple arc having a_1, b_1 as its end-points.

In Lemma 3, put $F=\nu_1+\sum(xy+yz)+[\Delta xyz]=\lambda_1$. There exists a simple closed curve, ω, such that γ, γ', β_1 and ν_1 are all cross-cuts in $[\omega]$ and have a_1, b_1 as end-points in common. Let ω_1, ω_2 be two subarcs of ω divided by

a_1 and b_1. We may also assume that $[\omega_1+\gamma]\cdot\gamma'=\phi$ and $[\omega_1+\beta_1]\cdot\nu_1=\phi$. Then there exists a homeomorphism, φ_1', of $[\omega_1+\gamma]$ (or $[\omega_2+\gamma']$) onto $[\omega_1+\beta_1]$ (or $[\omega_2+\nu_1]$) such that $\varphi_1'(\alpha)=xz+zy$ (or $\varphi_1'(\alpha')=xy$) and such that each point of $\omega_1+\beta_1'$ (or $\omega_2+\beta_1'$) is fixed. It is easy to extend φ_1' to an automorphism, ψ_1, of π such that each point of $\pi-[\omega]$ is fixed under f_1.

We put $f_1=\psi_1\cdot\varphi_1$.

The following process is a slight modification of the methods in case (A) where M is a dendrite. We put only ν_i in place of λ_i in case (A) and construct λ_j by a similar way as above ((C), 2), using Lemma 4.

(D) Let M be a **plane peano continuum.** For N, being the minimal simply connected continuum containing M, we construct a sequence of simple continua, $\kappa_0^1, \kappa_0^2, \cdots$, by (C). Let $N_{i_1j_1k_1,\cdots,i_nj_nk_n}$ be the minimal simply connected continuum containing $K_{i_1j_1k_1,\cdots,i_nj_nk_n}$. In the same way as in case (C), we have a sequence of simple continua $\{\kappa^s_{i_1j_1k_1,\cdots,i_nj_nk_n}/s=1,2,\cdots\}$. On the other hand, for each $C_{i_1j_1k_1,\cdots,i_n}\in\mathcal{C}_n$, we have also $\mathcal{E}_{i_1j_1k_1,\cdots,i_n}=\{\mu^s_{i_1j_1k_1,\cdots,i_n}\}$ as in case (B). (If an element, N, of $\mathcal{E}_{i_1j_1k_1,\cdots,i_n}$ has a cut-point, p, of M, N is divided into the components of $N-p$). Thus we have a sequence $\mathcal{S}=\{\kappa_0^s/s=1,2,\cdots\}+\{\kappa^s_{i_1j_1k_1,\cdots,i_nj_nk_n}/s, i_1, j_1, k_1, \cdots, i_n, j_n, k_n, n=1,2,\cdots\}+\{\mu^s_{i_1j_1k_1,\cdots,i_n}/s, i_1, j_1, k_1, \cdots, i_1, n=1,2,\cdots\}=\{\sigma\}$.

Let us number the elements of $\mathcal{S}$ in such a way as

$$\kappa_0^s=\sigma_{s_1},\ \kappa_0^t=\sigma_{t_1},\ s<t\longrightarrow s_1<t_1,$$
$$\kappa^s_{i_1j_1k_1,\cdots,i_nj_nk_n}=\sigma_{s_1},\ \kappa^t_{i_1j_1k_1,\cdots,i_nj_nk_n}=\sigma_{t_1},\ s<t\longrightarrow s_1<t_1,$$
$$\kappa_0^s=\sigma_{s_1},\ \kappa^t_{i_1j_1k_1,\cdots,i_nj_nk_n}=\sigma_{t_1},\ \sigma_{s_1}\supset\sigma_{t_1}\longrightarrow s_1<t_1,$$
$$\kappa^s_{i_1j_1k_1,\cdots,i_mj_mk_m}=\sigma_{s_1},\ \kappa^t_{i_1j_1k_1,\cdots,i_nj_nk_n}=\sigma_{t_1},\ \sigma_{s_1}\supset\sigma_{t_1}\longrightarrow s_1<t_1,$$
$$\kappa_0^s\ (\text{or } \kappa^s_{i_1j_1k_1,\cdots,i_mj_mk_m})=\sigma_{s_1},\ \mu^t_{i_1'j_1'k_1',\cdots,i_n'}=\sigma_{t_1}\longrightarrow s_1<t_1.$$

For $\sigma_i\in\mathcal{S}$, we can define an automorphism, h_i, of π as in case (A) or (B) or (C), and have a sequence of automorphisms, $h_1, h_2, \cdots$, such that for $j>i$, h_j has no effect on the graph actually operated by h_i, i.e. the set of points, x, such that $h_i(x)\neq x$.

The fact $h_{(\cdots i\cdots 1)}(M)\cong M$ shows that M is a polyhedral inner approximation. Hence we have

THEOREM 3. *Each plane peano continuum, M, is a polyhedral inner approximation, and is homeomorphic with the set constructed by the following process:*

(i) *constructing a segmentwise dendrite, T,*

(ii) *adding 2-simplices, Δ_i's, which construct a null-sequence, to T in such a way as each pair of 2-simplices of the system has at most one point in common and Δ_i's intersect acyclicly with T, that is, $\sum\Delta_i+T=M_1$ contains no simple closed curve whose inner domain is not contained in M_1 and M_1 is connected,*

(iii) *from the interiors of Δ_i's, subtracting (polyhedronwise[8]) Jordan*

(8) A Jordan region is called to be polyhedronwise if its frontier is a polygon.

regions, D_j's, which construct a null-sequence and are mutually exclusive,

(iv) *adding segmentwise dendrites, T_k's, constructing a null-sequence, to the set, S, resulted by* (i), (ii) *and* (iii), *so that, for each k, $T_k \cdot S$ consists of only one point, p_k, and so that $T_k - p_k$ is contained in some D_j and $T_k \cdot T_l = \phi$ for $k \neq l$,*

(v) *applying* (ii), (iii) *and* (iv) *to T_k's,*

(iv) *repeating and continuing* (ii)~(v).

REMARK. In order that a plane peano continuum M is homeomorphic with an Euclidean polyheder, it is necessary and sufficient that $\mathcal{C}_n$, $\mathcal{G}_n$, $\mathcal{T}_n$ and $\mathcal{Q}_n$ are all finite sets, the branch points in each element of $\mathcal{T}_n$ are finite in number and M is m-connected.

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Mathematical Institute,
Hiroshima University