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On Locally Connected Continua[1] which are not Separated by Any Arc

By

Tadashi TANAKA

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If M is a connected set and L is an arc of M such that the set $M-L$ is not connected, then L will be called a *cut arc* of M. The first section of this paper is concerned with conditions that locally connected continua have no cut arc, and the second section deals with locally connected continua having no cut arc in 2-dimensional Euclidean space. In this paper we shall use the notions and terminologies in G. T. Whyburn's "Analytic Topology" [1].

LEMMA 1.1. *In order that a locally connected continuum M may have no cut arc, the following condition is necessary:*

if p is any point of M and L is any arc in the interior of some arc of M, then there exists a simple closed curve in M containing p and L.

PROOF. Suppose that M has no cut arc. Let $a'b'$ be any arc, ab be any arc in the interior of $a'b'$ and p be any point of M. We may suppose p is not on $a'b'$. Then there exists an arc cd in M such that p is an interior point of cd and $a'b'\cdot cd=0$ (cf. [1], p. 78). Since the set $M-ab'$ is a locally compact, locally connected, connected set by hypothesis, there exists an arc $a'p$ in $M-ab'$ joining a' and p. Then $a'p$ contains an arc $a''c'$ such that $a''c'\cdot(aa'+cd)=a''+c'$, where a'' is on aa' and c' on cd. Similarly we obtain an arc $b'd$ in $M-(ba''+a''c'+c'p)$ joining b' and d and hence an arc $b''d'$ such that $b''d'\cdot(bb'+pd)=b''+d'$ where b'' is on bb'' and d' on pd. Then the set $b''a''+a''c'+c'p+pd'+d'b''$ is a simple closed curve in M containing p and ab. Thus Lemma 1.1 is proved.

LEMMA 1.2. *In order that a locally connected continuum M may have no cut arc, the following condition is sufficient:*

if p is any point of M and L is any arc of M, then there exists a simple closed curve in M containing p and L.

PROOF. Suppose that M satisfies the condition and has a cut arc ab. Now take two points p and q in different components of $M-ab$. Let J be a simple closed curve in M containing p and ab. Then there does not exist a simple closed curve in M containing an arc $ab+bp$ of J and a

1) A continuum means a locally compact, connected and closed set.

point q, contrary to supposition. Thus Lemma 1.2 is proved.

It is easily known by examples, however, that in order that M has no cut arc the condition in Lemma 1.1 is not sufficient and the condition in Lemma 1.2 is not necessary.

As an immediate consequence of Lemmas 1.1 and 1.2, we get

THEOREM 1. *Let M be a locally connected continuum in which every arc may be extended into both directions (cf. [2]). In order that M may have no cut arc, each one of the conditions in Lemmas 1.1 and 1.2 is both necessary and sufficient.*

It is obvious that there does not exist a locally connected continuum having no cut arc in 1-dimensional Euclidean space E^1. But for such locally connected continua in 2-dimensional Euclidean space E^2, we get the following results.

LEMMA 2.1. *Let M be a locally connected continuum having no cut arc. If M has a free arc as a subset, then M is a simple closed curve.*

PROOF. Let ab be a free arc in M. Then the set $M-\widehat{ab}$ is a locally connected continuum. For, since M has no cut arc and ab is a free arc in M, $M-\widehat{ab}$ is a continuum. In order to prove $M-\widehat{ab}$ is locally connected, it suffices to prove that $M-\widehat{ab}$ is locally connected at a and b. Now since M is locally connected, we can take a small enough region R containing a such that $R-a=R\cdot\widehat{ab}+R\cdot(M-ab)$ is a separation. Then the set $R\cdot(M-ab)+a$ is connected and also open in $M-\widehat{ab}$. Therefore $M-\widehat{ab}$ is locally connected at a. Similarly $M-\widehat{ab}$ is also locally connected at b.

Accordingly there exists an arc L in $M-\widehat{ab}$ joining a and b. Then $(M-\widehat{ab})-L=0$, for otherwise L separates $(M-\widehat{ab})-L$ and $\widehat{ab}$ in M, contrary to supposition. Thus $M=ab+L$ is a simple closed curve.

THEOREM 2. *Let M be a locally connected continuum in E^2 having no cut arc. Then the following results hold:*

(i) *if M is compact, then M is a simple closed curve;*

(ii) *if M is non-compact, then $M=E^2$.*

PROOF. Proof of (i). Let D be a complementary domain of M. Then the boundary $F(D)$ of D is a compact, locally connected continuum of which every true cyclic element is a simple closed curve (cf. [1], pp. 106–107). Moreover, $F(D)$ consists of just one true cyclic element. For, if $F(D)$ contains no true cyclic element, then $F(D)$ is a dendrite and $F(D)=M$. Therefore M has a cut arc, contrary to supposition. Hence $F(D)$ contains at least one true cyclic element E. If $F(D)$ contains a point p of M not belonging to E, then there exists a cut point q of $F(D)$ which separates p and a point of E in $F(D)$. The point q is also a cut point of M, contrary to supposition.

Now let us suppose, on the contrary, that M is not a simple closed curve. Let D be the unbounded complementary domain of M. Then since $F(D)$ is a simple closed curve and M is not a simple closed one, there exists a point a of M which does not belong to $F(D)$. Let ab be an arc of M such that $ab\cdot(a+F(D))=a+b$, where b is on $F(D)$.

Case I. The case where there exists an arc ac in M joining a and a point c of $F(D)$ and not containing b. In this case it is easily shown that the set $ab+ac$ contains a cut arc of M, contrary to supposition.

Case II. The case where every arc in M joining a and $F(D)$ contains b. In the last case it is easily shown that b is a cut point of M, contrary to supposition. Thus (i) is proved.

Proof of (ii). Let us suppose, on the contrary, that $M\neq E^2$. Then there exist a point p of E^2-M and a boundary point q of the complementary domain of M containing p. Now let J be the circle with center q and radius the distance between p and q, and then let p_1pp_2 be the arc of J such that $p_1pp_2\cdot M=p_1+p_2$.

Our proof is now divided into the following two cases.

Case I. The case where $p_1=p_2$. In this case $p_1(=p_2)$ is a cut point of M. For the interior of J contains q, the exterior of J contains a point of M as M is non-compact, and p_1 is only one point of M on J. Therefore p_1 is a cut point of M, contrary to supposition.

Case II. The case where $p_1\neq p_2$. Let p_1p_2 be an arc in M joining p_1 and p_2, and then let us make a simple closed curve $p_1p_2+p_1pp_2$. Then the interior of $p_1p_2+p_1pp_2$ does not contain any point of M, for otherwise the arc p_1p_2 would be a cut arc of M. Let R_1 and R_2 be small enough regions in M containing p_1 and p_2, respectively. Take two point p_1' and p_2' which do not belong to p_1p_2 in R_1 and R_2, respectively. Since M is non-compact, by Lemma 2.1 we can take such points as p_1' and p_2'. Let $p_1'p_2'$ be an arc in $M-p_1p_2$ joining p_1' and p_2', p_1p_1' an arc in R_1 joining p_1 and p_1', and p_2p_2' an arc in R_2 joining p_2 and p_2'. Then the set $p_1p_1'+p_1'p_2'+p_2'p_2$ contains an arc L joining p_1 and p_2 and is different from p_1p_2. Then the interior of the simple closed curve $L+p_1pp_2$ contains a point of M. Therefore L is a cut arc of M, contrary to supposition. Thus (ii) is proved.

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Mathematical Institute, Hiroshima University

References

[1] G. T. Whyburn, Analytic Topology, Amer. Math. Soc. Colloquium publications, vol. **28**, New York, 1942.

[2] W. L. Ayres, Continuous curve in which every arc may be extended, Bull. Amer. Math. Soc., vol. **53** (1929).