JOURNAL OF SCIENCE OF THE HIROSHIMA UNIVERSITY, SER. A, VOL. 20, NO. 1, MAY, 1956

Reduction of Periodic System to Autonomous One by Means of One-Parameter Group of Transformations

By

Minoru URABE

(Received Jan. 25, 1956)

§ 1. Introduction

Between the periodic system

$$\frac{dx_i}{dt}=X_i(x,t)=\sum_{j=1}^{n}c_{ij}(t)x_j+\sum_{\mathfrak{p}}{}''c_{i\mathfrak{p}}(t)x_1^{p_1}\cdots x_n^{p_n}\quad(i=1,2,\cdots,n)\tag{1.1}$$

and the autonomous system

$$\frac{dx_i}{dt}=\xi_i(x)=\sum_{j=1}^{n}c_{ij}x_j+\sum_{\mathfrak{p}}{}''c_{i\mathfrak{p}}x_1^{p_1}\cdots x_n^{p_n}\quad(i=1,2,\cdots,n)\tag{1.2}$$

where $c_{ij}(t)$, $c_{i\mathfrak{p}}(t)$ are continuous for $-\infty<t<\infty$ and periodic with period $\omega>0$ and $\sum_{\mathfrak{p}}''$ denotes summation over $\mathfrak{p}=(p_1, p_2, \cdots, p_n)$ where $p_1, p_2, \cdots, p_n$ are non-negative integers such that $s(\mathfrak{p})=p_1+p_2+\cdots+p_n\geqq 2$, there is known till now a considerable amount of parallel properties. But, for lack of the principle connecting the two systems, such parallel properties must have been proved till now on each system respectively, even though the proof may be carried on in parallel. In this paper, we would establish such a principle.

For this purpose, we consider to transform the system (1.1) to the system (1.2) by the transformation of the form

$$x_i=F_i(y,t)=\sum_{j=1}^{n}k_{ij}(t)y_j+\sum_{\mathfrak{p}}{}''k_{i\mathfrak{p}}(t)y_1^{p_1}\cdots y_n^{p_n}\quad(i=1,2,\cdots,n),\tag{1.3}$$

where $k_{ij}(t)$, $k_{i\mathfrak{p}}(t)$ are continuous for $-\infty<t<\infty$ and periodic with period ω. If this is possible, then the systems (1.1) and (1.2) correspond to each other by the correspondence (1.3), namely a principle connecting both systems is established. But, as is seen later, transformation like (1.3) is not always possible. In this paper, we show that *transformation like* (1.3) *is possible when and only when the certain finite transformation can be imbedded in a one-paramter group of transformations.*

This result can be applied to various problems. The problem of reducing the periodic system to that of the simplest form is one of them. As an example of application of our result, this problem is attacked from our standpoint. As another important application of our result, the so-called *stroboscopic method* used often in practical application [1, 2, 3][1] is studied and its mathematical legality which does not seem to have been given thoroughly till now is discussed in detail from our standpoint.

§ 2. General theory

1. Necessity

We assume that the periodic system (1.1) can be transformed to the autonomous system (1.2) by the transformation of the form (1.3). Since (1.3) expresses a transformation in the neighborhood of $y_i=0$, it is required that $\det.|k_{ij}(t)|\neq 0$ for $-\infty<t<\infty$. In the sequel, we assume this. Let the inverse of (1.3) be

$$y_i=G_i(x, t) \quad (i=1, 2, \cdots, n) , \tag{2.1}$$

then, evidently, $G_i(x, t)$ are of the forms

$$G_i(x, t)=\sum_{j=1}^{n} l_{ij}(t)x_j+\sum_{\mathfrak{p}}'' l_{i\mathfrak{p}}(t)x_1^{p_1}\cdots x_n^{p_n} \quad (i=1, 2, \cdots, n) , \tag{2.2}$$

where $l_{ij}(t)$, $l_{i\mathfrak{p}}(t)$ are continuous for $-\infty<t<\infty$ and periodic with period ω. From (2.1) follows

$$\frac{dy_i}{dt}=\sum_{j=1}^{n}\frac{\partial G_i(x, t)}{\partial x_j}X_j(x, t)+\frac{\partial G_i(x, t)}{\partial t} .$$

If we write the functions of the right-hand side as $Y_i(x, t)$, then, evidently, $Y_i\{F(y, t), t\}$ are regular with respect to y_j and vanish for $y_j=0$. Consequently, the system (1.1) is certainly transformed to the regular system of the form (1.2) by the transformation (1.3).

By means of the functions $F_i(y)=F_i(y, 0)$ and $G_i(x)=G_i(x, 0)$, we consider the transformation

$$y_i=G_i(z) \text{ or } z_i=F_i(y) \quad (i=1, 2, \cdots, n) . \tag{2.3}$$

By this transformation, the form of the system (1.2) does not alter. Consequently, superposing the transformations (1.3) and (2.3), the initial system (1.1) is transformed to the system of the form (1.2) by the transformation of the form (1.3) satisfying $F_i(y, 0)=y_i$, because

$$F_i\{G(z), 0\}=F_i\{G(z)\}=z_i .$$

Thus, without loss of generality, we may suppose that the transformation (1.3) satisfy $F_i(y, 0)=y_i$. In the sequel, we assume this.

Let the solution of (1.1) taking a set of values x_i for $t=0$ be $\varphi_i(x, t)$ $(i=1, 2, \cdots, n)$. Then, by the assumption, the set of functions

$$\psi_i(x, t)=G_i\{\varphi(x, t), t\} \quad (i=1, 2, \cdots, n) \tag{2.4}$$

becomes a solution of (1.2) and moreover it is valid that

$$\psi_i(x, 0)=G_i\{\varphi(x, 0), 0\}=G_i(x, 0)=x_i .$$

Consequently $\psi_i(x, t)$ becomes a solution of (1.2) such that $\psi_i(x, 0)=x_i$. Then, for $t=\omega$, from periodicity of $F_i(y, t)$ follows

1) The numbers in crotchets denote the references listed at the end of the paper.

$$\begin{aligned}\psi_i(x, \omega) &= G_i\{\varphi(x, \omega), \omega\} \\ &= G_i\{\varphi(x, \omega), 0\} \\ &= \varphi_i(x, \omega) .\end{aligned}$$

This expresses that *the transformation*

(2.5) $$x_i' = \varphi_i(x, \omega) \quad (i=1, 2, \cdots, n)$$

is imbedded in a one-paramter group of transformations $x_i' = \psi_i(x, t)$ *with the operator functions*[1] $\xi_i(x)$.

From (2.4), it is evident that

(2.6) $$\varphi_i(x, t) = F_i\{\psi(x, t), t\} \quad (i=1, 2, \cdots, n) .$$

Now, if we put $y_i = \psi_i(x, t)$, then $x_i = \psi_i(y, -t)$, because $x_i' = \psi_i(x, t)$ is a transformation belonging to a continuous group with canonical parameter t. Then, from (2.6) follows

$$\varphi_i\{\psi(y, -t), t\} = F_i(y, t) .$$

This expresses that *the transformation* (1.3) *by which the system* (1.1) *is transformed to the system of the form* (1.2) *must be*

(2.7) $$x_i = \varphi_i\{\psi(y, -t), t\} \quad (i=1, 2, \cdots, n) ,$$

provided that $F_i(y, 0) = y_i$.

2. Sufficiency

We assume that the transformation (2.5) is imbedded in a certain one-parameter group of transformations with the operator functions of the forms of the right-hand side of (1.2). Let the operator functions of this group be $\xi_i(x)$ $(i=1, 2, \cdots, n)$ and the transformation belonging to this group be

(2.8) $$x_i' = \psi_i(x, t) \quad (i=1, 2, \cdots, n) ,$$

where t is a canonical parameter of the group. According to (2.7), we consider the functions

(2.9) $$F_i(y, t) = \varphi_i\{\psi(y, -t), t\} \quad (i=1, 2, \cdots, n) .$$

Since $\varphi_i(x, t)$ and $\psi_i(x, t)$ are respectively the solutions of the differential equations (1.1) and (1.2) such that $\varphi_i(x, 0) = \psi_i(x, 0) = x_i$, the functions $\varphi_i(x, t)$ and $\psi_i(x, t)$ are expanded with respect to x_j as follows:

$$\begin{cases} \varphi_i(x, t) = \sum_{j=1}^{n} a_{ij}(t) x_j + \sum_{\mathfrak{p}}'' a_{i\mathfrak{p}}(t) x_1^{p_1} \cdots x_n^{p_n} , \\ \psi_i(x, t) = \sum_{j=1}^{n} \tilde{a}_{ij}(t) x_j + \sum_{\mathfrak{p}}'' \tilde{a}_{i\mathfrak{p}}(t) x_1^{p_1} \cdots x_n^{p_n} . \end{cases}$$

Consequently, from (2.9), the functions $F_i(y, t)$ are expanded with respect to y_j like (1.3) and the coefficients $k_{ij}(t)$ of the linear terms of their expansions become $\sum_{k=1}^{n} a_{ik}(t) \tilde{a}_{kj}(-t)$.

Now, if we substitute the expansions of $\varphi_i(x, t)$ and $\psi_i(x, t)$ into (1.1) and (1.2) respectively and compare the coefficients of x_j in both sides, we have

1) For brevity, when the operator of a one-parameter group of transformations is $\sum_{i=1}^{n} \xi_i(x) \frac{\partial}{\partial x_i}$, we call the functions $\xi_i(x)$ the operator functions of this group.

$$\begin{cases} \dfrac{da_{ij}(t)}{dt} = \sum_{k=1}^{n} c_{ik}(t) a_{kj}(t) , \\ \dfrac{d\tilde{a}_{ij}(t)}{dt} = \sum_{k=1}^{n} c_{ik} \tilde{a}_{kj}(t) , \end{cases} \qquad (i, j = 1, 2, \cdots, n) .$$

Moreover, from $\varphi_i(x, 0) = \psi_i(x, 0) = x_i$, $a_{ij}(0) = \tilde{a}_{ij}(0) = \delta_{ij}$[1], consequently $\{a_{ij}(t)\}$ and $\{\tilde{a}_{ij}(t)\}$ $(j = 1, 2, \cdots, n)$ become respectively the fundamental systems of the solutions of the linear differential equations

$$\begin{cases} \dfrac{dx_i}{dt} = \sum_{k=1}^{n} c_{ik}(t) x_k , \\ \dfrac{dx_i}{dt} = \sum_{k=1}^{n} c_{ik} x_k , \end{cases} \qquad (i = 1, 2, \cdots, n) .$$

Then det.$|a_{ij}(t)|$ and det.$|\tilde{a}_{ij}(t)|$ do not become zero for any t, consequently, we see that, for $-\infty < t < \infty$, det.$|k_{ij}(t)| \neq 0$.

Since $\varphi_i(x, t)$ is a solution of the periodic system (1.1), it is valid that

(2.10) $$\varphi_i\{\varphi(x, \omega), t\} = \varphi_i(x, \omega + t) \quad (i = 1, 2, \cdots, n) .$$

Since (2.8) is a transformation belonging to the continuous group of transformations with canonical parameter t, it is valid that

(2.11) $$\psi_i\{\psi(y, t_1), t_2\} = \psi_i(y, t_1 + t_2) \quad (i = 1, 2, \cdots, n) .$$

Then, from (2.9) follows

(2.12) $$\begin{aligned} F_i(y, t+\omega) &= \varphi_i\{\psi(y, -t-\omega), t+\omega\} \\ &= \varphi_i[\varphi\{\psi(y, -t-\omega), \omega\}, t] \qquad \text{(by (2.10))} \\ &= \varphi_i[\varphi\{\psi(\psi(y, -t), -\omega), \omega\}, t] \quad \text{(by (2.11))} . \end{aligned}$$

Now, by the assumption, it holds that

$$\psi_i(x, \omega) = \varphi_i(x, \omega) \quad (i = 1, 2, \cdots, n) .$$

Consequently, since $x_i' = \psi_i(x, -\omega)$ is an inverse of

$$x_i' = \psi_i(x, \omega) = \varphi_i(x, \omega)$$

because of (2.11), it is valid that

$$\varphi_i\{\psi(\psi(y, -t), -\omega), \omega\} = \psi_i(y, -t) .$$

Then, from (2.12) follows

$$F_i(y, t+\omega) = \varphi_i[\psi(y, -t), t] = F_i(y, t) ,$$

namely $F_i(y, t)$ become periodic with respect to t with period ω.

Thus the functions $F_i(y,t)$ become the functions of the same character as those of the right-hand side of (1.3).

If we put $x_i = \psi_i(y, -t)$ in (2.9), then, since $y_i = \psi_i(x, t)$ from (2.11), we have

(2.13) $$\begin{aligned} \varphi_i(x, t) &= F_i(y, t) \\ &= F_i\{\psi(x, t), t\} . \end{aligned}$$

Now, by the assumption, $\psi_i(x, t)$ satisfy the equations

(2.14) $$\frac{dx_i}{dt} = \xi_i(x) \quad (i = 1, 2, \cdots, n)$$

---

1) δ_{ij} is Kronecker's delta.

and $\psi_i(x, 0)=x_i$, in other words, $\psi_i(x, t)$ is a solution of the autonomous system (2.14) such that $\psi_i(x, 0)=x_i$. Then, (2.13) says that the solution $\varphi_i(x, t)$ of (1.1) is transformed to the solution $\psi_i(x, t)$ of (2.14) by the transformation

$$x_i = F_i(y, t) \quad (i=1, 2, \cdots, n)\,, \tag{2.15}$$

in other words, *the periodic system* (1.1) *is transformed to the autonomous system* (2.14) *by the transformation* (2.15), *which is of the form* (1.3) by the preceding results.

Since $\varphi_i(x, 0)=\psi_i(x, 0)=x_i$, from (2.9), it is evident that $F_i(y, 0)=y_i$. Then, by the preceding section, *the transformation* (2.15) *for* $F_i(y, t)$ *defined by* (2.9) *is a unique one satisfying* $F_i(y, 0)=y_i$ *by which* (1.1) *is transformed to* (2.14).

3. Conclusion

The results obtained in the preceding two sections are summarized as follows:

The necessary and sufficient condition that the periodic system (1.1) *can be transformed to the autonomous system of the form* (1.2) *by the transformation of the form* (1.3) *is that the transformation* (2.5) *can be imbedded in a one-parameter group of transformations with the operator functions regular in the origin and vanishing there.*

When the condition is fulfilled, there exists one and only one transformation of the form (1.3) *such that* $F_i(y, 0)=y_i$, *by which the system* (1.1) *is tranformed to an autonomous system of the form* (1.2).

§ 3. Reduction of linear periodic system

Given the linear periodic system

$$\frac{dx_i}{dt}=\sum_{j=1}^{n} c_{ij}(t)x_j \quad (i=1, 2, \cdots, n)\,. \tag{3.1}$$

Let us consider the fundamental system $\{\mathring{a}_{ij}(t)\}$ $(j=1, 2, \cdots, n)$ of the solutions such that $\mathring{a}_{ij}(0)=\delta_{ij}$. Then the solution $\varphi_i(x, t)$ of (3.1) such that $\varphi_i(x, 0)=x_i$ is expressed as follows:

$$\varphi_i(x, t)=\sum_{j=1}^{n} \mathring{a}_{ij}(t)x_j \quad (i=1, 2, \cdots, n)\,. \tag{3.2}$$

Consequently the transformation (2.5) becomes a linear transformation

$$x'_i=\varphi_i(x, \omega)=\sum_{j=1}^{n} \mathring{a}_{ij}(\omega)x_j\,,$$

or, in matrix form,

$$\vec{x}'=\mathring{A}(\omega)\vec{x}\,, \tag{3.3}$$

where $\mathring{A}(\omega)=||\mathring{a}_{ij}(\omega)||$ and $\vec{x}$, $\vec{x}'$ are the column vectors.

Since $\{\mathring{a}_{ij}(t)\}$ $(j=1, 2, \cdots, n)$ is a fundamental system, $\det.|\mathring{A}(\omega)|\neq 0$, consequently there exists a matrix $B=||b_{ij}||$ such that

(3.4) $$e^{\omega B}=\mathring{A}(\omega)\ .$$
This expresses that the transformation (3.3) is imbedded in the one-parameter group $\mathfrak{G}$ of transformations

(3.5) $$\vec{x}'=e^{tB}\vec{x}\ .$$
The operator functions $\xi_i(x)$ of $\mathfrak{G}$ are evidently

(3.6) $$\xi_i(x)=\sum_{j=1}^{n}b_{ij}x_j \quad (i=1, 2, \cdots, n)\ .$$

Thus, by the general theory, we see that *the linear periodic system* (3.1) *is tranformed to the linear autonomous system*

(3.7) $$\frac{dx_i}{dt}=\sum_{j=1}^{n}b_{ij}x_j \quad (i=1, 2, \cdots, n)$$

by the linear periodic transformation

(3.8) $$\vec{x}=\mathring{A}(t)e^{-tB}\vec{y}$$

where $A(t)=||\mathring{a}_{ij}(t)||$. This is a well known result. The characteristic roots of B are called the *characteristic exponents* of the system (3.1).

§ 4. Reduction of nonlinear periodic system

1. Preliminaries

Given the nonlinear periodic system (1.1). Then, from the preceding paragraph, it is readily seen that, by the linear periodic transformation of the form (3.8), the system (1.1) is transformed to the system of the form

(4.1) $$\frac{dx_i}{dt}=X_i(x, t)=\sum_{j=1}^{n}b_{ij}x_j+\sum_{\mathfrak{p}}{}'' b_{i\mathfrak{p}}(t)x_1^{p_1}\cdots x_n^{p_n} \quad (i=1, 2, \cdots, n)\ ,$$

where $b_{i\mathfrak{p}}(t)$ are continuous for $-\infty<t<\infty$ and periodic with period ω.

Owing to (3.4), the characteristic roots μ_i $(i=1, 2, \cdots, n)$ of $B=||b_{ij}||$ are determined from the characteristic roots λ_i $(i=1, 2, \cdots, n)$ of $\mathring{A}(\omega)$ by

(4.2) $$e^{\omega\mu_i}=\lambda_i$$

or

(4.3) $$\mu_i=\frac{1}{\omega}\log\lambda_i =\frac{1}{\omega}\log|\lambda_i|+\frac{\sqrt{-1}}{\omega}\operatorname{Arg}\lambda_i\ .$$

The characteristic roots μ_i of B are called the *characteristic exponents* of the nonlinear system (1.1) as is so for the linear system.

By the suitable linear transformation with constant coefficients, the system (4.1) is transformed to the system of the same form with B of Jordan's canonical form. But, by this linear transformation, the characteristic roots μ_i of B do not alter, consequently, without loss of generality, we may suppose that, in (4.1), B is of Jordan's canonical form and the characteristic roots of B are μ_i $(i=1,2,\cdots,n)$ determined from λ_i by (4.3). In the sequel, we assume this.

Case I. The case where the real parts of the characteristic exponents are either all negative or all positive

2. Correspondence between λ_i and μ_i

From (4.2), $|\lambda_i|$'s are either all <1 or all >1. Among $\lambda_1, \cdots, \lambda_n$, there may occur the relation of the form

$$\lambda_i = \lambda_1^{p_1}\lambda_2^{p_2}\cdots\lambda_n^{p_n} \tag{4.4}$$

for a certain $\mathfrak{p}=(p_1, p_2, \cdots, p_n)$ where $p_1, p_2, \cdots, p_n$ are non-negative integers such that $s(\mathfrak{p})=p_1+p_2+\cdots+p_n\geqq 2$. We arrange λ_i's so that, for $i=1, 2, \cdots, s$ $(s\leqq n)$, the relation (4.4) does not occur and, for $i=s+1, \cdots, n$, the relations (4.4) certainly occur and $|\lambda_{s+1}|\geqq|\lambda_{s+2}|\geqq\cdots\geqq|\lambda_n|$ or $|\lambda_{s+1}|\leqq|\lambda_{s+2}|\leqq\cdots\leqq|\lambda_n|$ according as $|\lambda_i|<1$ or >1. Then it is easily seen that the relations (4.4) which really occur are of the form

$$\lambda_i = \lambda_1^{p_1}\lambda_2^{p_2}\cdots\lambda_{i-1}^{p_{i-1}}\,. \tag{4.5}$$

When (4.5) occurs for λ_i, from (4.2), there occurs a relation

$$\mu_i = p_1\mu_1 + p_2\mu_2 + \cdots + p_{i-1}\mu_{i-1} + \frac{2\pi}{\omega} m_i\sqrt{-1} \tag{4.6}$$

for μ_i, where m_i is a integer. Now, as is seen from (4.3), the imaginary part $\Im(\mu_i)$ of μ_i has arbitrarity of integral multiple of $2\pi\sqrt{-1}/\omega$. Then, can we determine each $\Im(\mu_i)$ so that, corresponding to any relation (4.5), the relation (4.6) may always hold for $m_i=0$, namely there may always hold the relation

$$\mu_i = p_1\mu_1 + p_2\mu_2 + \cdots + p_{i-1}\mu_{i-1}\,? \tag{4.7}$$

This is not always possible and there is needed a condition. This condition was sought for previously by the present writer[1]. A necessary and sufficient condition is found in Theorem IV[2] of the paper [4] by the present writer. In the sequel, for brevity, we shall call this condition the condition (μ).

3. Imbedding theorem

Let the solution of (4.1) taking a set of values x_i for $t=0$ be $\varphi_i(x, t)$ $(i=1, 2, \cdots, n)$. Then the functions $\varphi_i(x, t)$ can be expanded with respect to x_j as follows:

$$\varphi_i(x, t) = \sum_{j=1}^{n} a_{ij}(t)x_j + \sum_{\mathfrak{p}}{}'' a_{i\mathfrak{p}}(t)x_1^{p_1}\cdots x_n^{p_n}\,. \tag{4.8}$$

Substitute these into (4.1), then, from comparison of the coefficients of the linear terms in both sides, it follows that

$$\frac{da_{ij}(t)}{dt} = \sum_{k=1}^{n} b_{ik}a_{kj}(t) \quad (i, j=1, 2, \cdots, n)\,,$$

or, in matrix form,

1) [4], Chap. III, pp. 203–210.
2) [4], pp. 209–210.

(4.9) $$\frac{dA(t)}{dt}=BA(t)\,,$$

where $A(t)=||a_{ij}(t)||$. Since $\varphi_i(x, 0)=x_i$, from (4.8), $A(0)$ must be a unit matrix. Then, from (4.9), it must be that

(4.10) $$A(t)=e^{tB}\,.$$

From this, on account of (3.4), follows

$$A(\omega)=e^{\omega B}=\mathring{A}(\omega)\,,$$

consequently the characteristic roots of $A(\omega)$ are λ_i $(i=1, 2, \cdots, n)$ and, by the assumption, $|\lambda_i|$'s are either all <1 or all >1.

By (4.8), in the present case, the transformation (2.5) is of the form

(4.11) $$x_i'=\varphi_i(x; \omega)=\sum_{j=1}^{n} a_{ij}(\omega)x_j+[x]_2 \quad (i=1, 2, \cdots, n)\,,$$

where $[x]_2$ denotes a sum of the terms of the second and higher orders with respect to x_j. Consequently, by the general theory of §2, it becomes a problem to imbed the transformation (4.11) in a one-parameter group of transformations with regular operator functions. This problem was already studied by the present writer. The answer is given in Theorem VIII'[1] of the paper [4] as follows:

When the condition (μ) *is fulfilled and each* $\mathfrak{J}(\mu_i)$ *is chosen so that the relation* (4.7) *may always hold corresponding to any relation* (4.5), *the transformation* (4.11) *can be imbedded in a one-parameter group* $\mathfrak{G}$ *of transformations with regular operator functions* $\xi_i(x)$ *which are determined by*

(4.12) $$\sum_{j=1}^{n} \xi_j(x)\frac{\partial g_i}{\partial x_j}=\mu_i g_i+\delta_i g_{i-1}+\Phi_i(g) \quad (i=1, 2, \cdots, n\,;\ \delta_i=1 \text{ or } 0)\,.$$

Here $\Phi_i(g)$ *are the linear combinations of* $g_1^{p_1}\cdots g_{i-1}^{p_{i-1}}$ *for* $\mathfrak{p}=(p_1, \cdots, p_n)$ *for which* (4.7) *hold, and* $g_i(x)$ *are related to the solutions* $f_i(x)$ *of the equations of Schröder*

(4.13) $$f_i\{\varphi(x, \omega)\}=\lambda_i f_i+\delta_i f_{i-1}+\Psi_i(f) \quad (i=1, 2, \cdots, n)$$

in such a way that

(4.14) $$f_i=\sum_{j=1}^{n} k_{ij}g_j \quad (i=1, 2, \cdots, n)$$

for $||k_{ij}||=K$ *such that* $KA(\omega)K^{-1}$ *is of Jordan's canonical form. In* (4.13), $\Psi_i(f)$ *are the linear combinations of* $f_1^{p_1}\cdots f_{i-1}^{p_{i-1}}$ *for* $\mathfrak{p}=(p_1, \cdots, p_n)$ *for which* (4.5) *hold.*

4. Reduction of the initial system

In this section, we consider the case where the condition (μ) is fulfilled and each $\mathfrak{J}(\mu_i)$ is chosen so that the relation (4.7) may always hold corresponding to any relation (4.5).

Since the absolute values of the characteristic roots λ_i of $A(\omega)$ are either all <1 or all >1, by the previous paper [5] of the present writer,

1) [4], pp. 225–226.

the equations of Schröder (4.13) have the regular solutions of the forms

$$f_i(x)=\sum_{j=1}^{n} k_{ij}x_j+[x]_2 \quad (i=1, 2, \cdots, n)\,.$$

For these solutions, by (4.14), $g_i(x)$ become the functions of the forms

$$g_i(x)=x_i+[x]_2 \quad (i=1, 2, \cdots, n)\,. \tag{4.15}$$

Then, by (4.12), the operator functions $\xi_i(x)$ become

$$\xi_i(x)=\mu_i x_i+\delta_i x_{i-1}+[x]_2 \quad (i=1, 2, \cdots, n)\,. \tag{4.16}$$

Thus, by the general theory of §2, we see that *there exists a transformation of the form* (1.3) *by which the system* (4.1) *is transformed to the autonomous system*

$$\frac{dx_i}{dt}=\xi_i(x) \quad (i=1, 2, \cdots, n) \tag{4.17}$$

for $\xi_i(x)$ *of the forms* (4.16).

Since $\xi_i(x)$ are the functions determined by (4.12) for $g_i(x)$ of the forms (4.15), if we transform the variables x_i to y_i by

$$y_i=g_i(x)\,,$$

the system (4.17) is transformed to the system

$$\frac{dy_i}{dt}=\mu_i y_i+\delta_i y_{i-1}+\Phi_i(y) \quad (i=1, 2, \cdots, n)\,. \tag{4.18}$$

Thus, superposing successively the transformations proposed up to the present in our discussions, we see that *the initial periodic system* (1.1) *is transformed to the autonomous system* (4.18) *by the transformation of the form* (1.3).

5. Remarks

The general case where the condition (μ) is not fulfilled seems to be difficult for treating from the theory of §2. But, if we adopt the method of formal transformation, we can reduce the system (4.1) to the system similar to (4.18) as will be seen later. Reduction itself of (4.1) is not the purpose of this paper, but, taking its significance into account, we shall explain it in the last paragraph as the complement.

Case II. The case where the characteristic exponents are all pure imaginary

6. Imbedding theorem

In the present case, from (4.2) follows

$$|\lambda_i|=1 \quad (i=1, 2, \cdots, n)\,.$$

We consider the case where the solution $\varphi_i^0(x,t)$ of (1.1) such that $\varphi_i^0(x,0)=x_i$ is majorized[1]. Since, by §3, (4.1) is a transform of (1.1) by the transformation of the form (1.3) such that $F_i(y, 0)=y_i$, from the discussions

1) Here we mean that there exists a set of functions $\Phi_i(x)$ regular in $x_j=0$ such that $\varphi_i^0(x,t) \ll \Phi_i(x)$ for $-\infty<t<\infty$. For reference, see [7] and [8].

in 1. of §2, the solution $\varphi_i(x,t)$ of (4.1) such that $\varphi_i(x,0)=x_i$ also becomes a transform of $\varphi_i^0(x,t)$ by the same transformation. Then, from majorizedness of $\varphi_i^0(x,t)$ follows majorizedness of $\varphi_i(x,t)$, because the majorizedness of a set of functions is invariant for the transformation of the form (1.3).

Since the solution $\varphi_i(x,t)$ is expanded as (4.8) and (4.10) is valid, B must be of the diagonal form because of majorizedness of $\varphi_i(x,t)$. Then, from (4.10) and (4.11), in the present case, the transformation (2.5) becomes

(4.19) $$T: x_i'=\varphi_i(x,\omega)=\lambda_i x_i+[x]_2 \quad (i=1,2,\cdots,n).$$

Since $\varphi_i(x,\omega+t)=\varphi_i\{\varphi(x,\omega),t\}$, the transformation T^k is expressed by $x_i'=\varphi_i(x,k\omega)$. Then, from majorizedness of $\varphi_i(x,t)$ follows majorizedness of the group of transformations $\{T^k\}$ $(k=0,\pm1,\pm2,\cdots)$. Then, by Theorem 3[1] of the paper [7], there exists a set of regular functions $f_i(x)$ $(i=1,2,\cdots,n)$ of the form

(4.20) $$f_i(x)=x_i+[x]_2$$

satisfying the equations of Schröder

(4.21) $$f_i\{\varphi(x,\omega)\}=\lambda_i f_i(x) \quad (i=1,2,\cdots,n).$$

This expresses that T becomes the linear transformation

(4.22) $$z_i'=\lambda_i z_i \quad (i=1,2,\cdots,n)$$

by the transformation of the variables

(4.23) $$z_i=f_i(x) \quad (i=1,2,\cdots,n).$$

The transformation (4.22) is evidently imbedded in a one-parameter group $\mathfrak{G}$ of transformations

(4.24) $$z_i'=e^{t\mu_i}z_i \quad (i=1,2,\cdots,n)$$

because of (4.2). From (4.24), it is evident that the operator functions of $\mathfrak{G}$ are

(4.25) $$\xi_i(z)=\mu_i z_i \quad (i=1,2,\cdots,n).$$

7. Reduction of the initial system

From (4.20), it is seen that, by the transformation (4.23), the system (4.1) is transformed to the system of the form

(4.26) $$\frac{dz_i}{dt}=Z_i(z,t)=\mu_i z_i+[z;t]_2 \quad (i=1,2,\cdots,n).$$

Now, by the preceding section, in terms of the variables z_i, the transformation T is expressed as (4.22), which is imbedded in a one-parameter group $\mathfrak{G}$ of transformations with operator functions

$$\xi_i(z)=\mu_i z_i \quad (i=1,2,\cdots,n).$$

Then, since the solution $\psi_i(y,t)$ of

(4.27) $$\frac{dy_i}{dt}=\xi_i(y)=\mu_i y_i \quad (i=1,2,\cdots,n)$$

such that $\psi_i(y,0)=y_i$ is $y_i e^{t\mu_i}$, by the general theory of §2, it is seen that the system (4.26) is transformed to the system (4.27) by the transformation of the form (1.3) as follows:

1) [7], p. 271. Also see Theorem 1 in [8], p. 469.

(4.28) $z_i=F_i(y,t)=\tilde{\varphi}_i\{y_j e^{-t\mu_j},t\}\quad (i,j=1,2,\cdots,n)$,
where $\tilde{\varphi}_i(z,t)$ is the solution of (4.26) such that $\tilde{\varphi}_i(z,0)=z_i$. Since (4.26) is a transform of (4.1) by (4.23), it is evident that $\tilde{\varphi}_i(z,t)=f_i[\varphi\{f^{-1}(z),t\}]$.

Superposing the transformations proposed up to the present, ultimately we see that, *when the solution* $\varphi_i^0(x,t)$ *of the initial periodic system* (1.1) *is majorized, the initial system* (1.1) *can be transformed to the autonomous system of the form* (4.27) *by the transformation of the form* (1.3).

This result is the first half of Theorem 3[1] of the paper [8], which has been proved first by Y. Sibuya [9] and later in a different method by the present writer [8]. Here is presented another new proof, which is far simpler than either of the previous two.

§ 5. Stroboscopic method

1. Preliminaries

Given the real system

$$\frac{dx_i}{dt}=\varepsilon X_i(x,t,\varepsilon)\quad (i=1,2,\cdots,n). \tag{5.1}$$

Here, for simplicity, we assume that $X_i(x,t,\varepsilon)$ *are*

1° *integral with respect to* x_j;

2° *continuous and periodic with period* $\omega>0$ *with respect to* t *for* $-\infty<t<\infty$;

3° *analytic with respect to* ε *in* $\varepsilon=0$.

The so-called *stroboscopic method* [1, 2, 3] advocates that, *for sufficienty small* $|\varepsilon|$,

1° *the system* (5.1) *is approximately equivalent to the system*

$$\frac{dx_i}{dt}=\frac{\varepsilon}{\omega}\int_0^\omega X_i(x,\tau,0)d\tau\quad (i=1,2,\cdots,n) \tag{5.2}$$

called "stroboscopic image" by N. Minorsky [1];

2° *the periodic solution of* (5.1) *with period* ω *corresponds to the critical point of* (5.2);

3° *the stability of the periodic solution of* (5.1) *is same as that of the corresponding critical point of* (5.2).

But, for the present writer, such a proposal does not seem to have been justified thoroughly till now in strict mathematical manner. In this paragraph, making use of the idea of imbedding a finite transformation in a one-parameter group, we would give a mathematical justification for the above proposal.

2. Legality within the first order with respect to ε

Let the solution of (5.1) taking a set of values x_i for $t=0$ be $\varphi_i(x,t,\varepsilon)$ $(i=1,2,\cdots,n)$. Then, by the assumption on $X_i(x,t,\varepsilon)$, $\varphi_i(x,t,\varepsilon)$ can be expanded as follows:

(5.3) $$\varphi_i(x,t,\varepsilon)=\varphi_i^{(0)}(x,t)+\varepsilon\varphi_i^{(1)}(x,t)+\varepsilon^2\varphi_i^{(2)}(x,t)+\cdots .$$

Here, from the initial conditions, it is evident that

(5.4) $$\varphi_i^{(0)}(x,0)=x_i\ ,\qquad \varphi_i^{(1)}(x,0)=\varphi_i^{(2)}(x,0)=\cdots=0\ .$$

Substituting (5.3) into (5.1) and comparing the coefficients of powers of ε, we have:

(5.5) $$\begin{cases} \dfrac{d\varphi_i^{(0)}}{dt}=0\ , \\ \dfrac{d\varphi_i^{(1)}}{dt}=X_i(\varphi^{(0)},t,0)\ , \\ \cdots\cdots . \end{cases}$$

Then, from (5.4), it follows that

(5.6) $$\begin{cases} \varphi_i^{(0)}(x,t)=x_i\ , \\ \varphi_i^{(1)}(x,t)=\displaystyle\int_0^t X_i(x,\tau,0)d\tau\ , \\ \cdots\cdots . \end{cases}$$

Consequently the expansions (5.3) become

(5.7) $$\varphi_i(x,t,\varepsilon)=x_i+\varepsilon\int_0^t X_i(x,\tau,0)d\tau+\cdots \quad (i=1,2,\cdots,n)\ .$$

Then, except for the second and higher orders with respect to ε, the transformation

(5.8) $$x_i'=\varphi_i(x,\omega,\varepsilon) \quad (i=1,2,\cdots,n)$$

is imbedded in a one-parameter group $\mathfrak{G}$ of transformations with the operator functions

(5.9) $$\xi_i(x)=\frac{\varepsilon}{\omega}\int_0^\omega X_i(x,\tau,0)d\tau \quad (i=1,2,\cdots,n)\ .$$

Consequently, except for the second and higher orders with respect to ε, by the general theory of §2, the system (5.1) is transformed to the system

(5.10) $$\frac{dy_i}{dt}=\xi_i(y)=\frac{\varepsilon}{\omega}\int_0^\omega X_i(y,\tau,0)d\tau \quad (i=1,2,\cdots,n)$$

by the transformation

(5.11) $$x_i=F_i(y,t,\varepsilon)=\varphi_i[\psi(y,-t,\varepsilon),t,\varepsilon] \quad (i=1,2,\cdots,n)$$

where $\psi_i(y,t,\varepsilon)$ is a solution of (5.10) such that $\psi_i(y,0,\varepsilon)=y_i$. This result says that *the initial system* (5.1) *is equivalent to* (5.2) *within the first order with respect to* ε.

If the solution $\varphi_i(x,t,\varepsilon)$ is periodic with period ω, then it must be that

(5.12) $$\varphi_i(x,\omega,\varepsilon)=x_i \quad (i=1,2,\cdots,n)\ ,$$

consequently, from (5.7), for the initial values x_i's of a periodic solution of (5.1), $\int_0^\omega X_i(x,\tau,0)d\tau$ must at least be of the first order with respect to ε. Then, within the first order with respect to ε, $y_i=x_i$ where x_i's are the initial values of a periodic solution of (5.1) becomes a solution of (5.10), consequently, from uniqueness of solutions of (5.10), the solution $\psi_i(y,t,\varepsilon)$ corresponding to the periodic solution $\varphi_i(x,t,\varepsilon)$ must be $\psi_i(y,t,\varepsilon)=x_i$ because $\psi_i(y,0,\varepsilon)=\varphi_i(x,0,\varepsilon)=x_i$. This says that, *within the first order with* *respect to* ε, *to the periodic solution of* (5.1) *with period* ω *coresponds a critical point of* (5.2). *The converse is evident from* (5.11).

Since the system (5.2) is a transform of (5.1) within the first order with respect to ε, *the stability of the periodic solution of* (5.1) *is evidently same as that of the corresponding critical point of* (5.2) *within the first order with respect to* ε.

Thus, ultimately we see that *the stroboscopic method is mathematically correct so long as it is used within the first order with respect to* ε.

Since the solution $\psi_i(y, t, \varepsilon)$ of (5.10) is expanded with respect to ε as follows:

(5.13) $$\psi_i(y, t, \varepsilon) = y_i + \xi_i(y)t + \cdots ,$$

from (5.7), within the first order with respect to ε, the transformation (5.11) becomes

(5.14) $$\begin{aligned} x_i = F_i(y, t) &= y_i - \xi_i(y)t + \varepsilon\int_0^t X_i(y, \tau, 0)d\tau \\ &= y_i + \varepsilon\left[\int_0^t X_i(y, \tau, 0)d\tau - \frac{t}{\omega}\int_0^\omega X_i(y, \tau, 0)d\tau\right] . \end{aligned}$$

This is an explicit form of the transformation under which the system (5.1) is equivalent to the autonomous system (5.2) within the first order with respect to ε.

3. Strict legality of periodic solution and its stability

The periodic solution with period ω of (5.1) is a solution such that its initial values x_i satisfy (5.12). By (5.3), the equations (5.12) are written as

(5.15) $$\varphi_i^{(1)}(x, \omega) + \varepsilon\varphi_i^{(2)}(x, \omega) + \cdots = 0 \quad (i = 1, 2, \cdots, n) .$$

Consequently, for sufficiently small $|\varepsilon|$, the real solution $x_i = \alpha_i$ of (5.12) is found in the neighborhood of the real solution $x_i = \alpha_i^0$ of the equations

(5.16) $$\varphi_i^{(1)}(x, \omega) = 0 \quad (i = 1, 2, \cdots, n) ,$$

which, from (5.6) and (5.9), are equivalent to

(5.17) $$\xi_i(x) = 0 \quad (i = 1, 2, \cdots, n) .$$

Put

(5.18) $$x_i = \alpha_i^0 + c_i ,$$

then, from analyticity of $\varphi_i^{(1)}(x, \omega)$, $\varphi_i^{(2)}(x, \omega)$, $\cdots\cdots$ due to the assumptions on $X_i(x, t, \varepsilon)$, for sufficiently small $|c_i|$, (5.15) are written as follows:

(5.19) $$\sum_{j=1}^{n} \frac{\partial\varphi_i^{(1)}(\alpha^0, \omega)}{\partial\alpha_j^0} c_j + \varepsilon\varphi_i^{(2)}(\alpha^0, \omega) + \cdots = 0 \quad (i = 1, 2, \cdots, n)$$

on account of $\varphi_i^{(1)}(\alpha^0, \omega) = 0$. Then, when

(5.20) $$J = \det.\left|\frac{\partial\varphi_i^{(1)}(\alpha^0, \omega)}{\partial\alpha_j^0}\right| \neq 0 ,$$

the real quantities c_i are determined uniquely so that (5.19) may hold, in other words, when (5.20) holds, the real solution $x_i = \alpha_i$ of (5.12) is determined uniquely in the neighborhood of the real solution $x_i = \alpha_i^0$ of (5.16), namely, the periodic solution of (5.1) is determined uniquely corresponding to the real solution $x_i=\alpha_i^0$ of (5.16) which is a critical point of the stroboscopic image (5.2).

The stability of such a periodic solution of (5.1) is determined by convergence of iteration of the transformation

$$r_i'=\varphi_i(\alpha+r,\omega,\varepsilon)-\alpha_i \quad (i=1,2,\cdots,n) \tag{5.21}$$

for sufficiently small $|r_i|$. Since α_i is a solution of (5.12), the above transformation is of the form

$$r_i'=\sum_{j=1}^{n}\frac{\partial\varphi_i(\alpha,\omega,\varepsilon)}{\partial\alpha_j}r_j+[r]_2 \quad (i=1,2,\cdots,n)\,. \tag{5.22}$$

Now, by (5.3), it is valid that

$$\frac{\partial\varphi_i(\alpha,\omega,\varepsilon)}{\partial\alpha_j}=\delta_{ij}+\varepsilon\frac{\partial\varphi_i^{(1)}(\alpha,\omega)}{\partial\alpha_j}+[\varepsilon]_2 \quad (i,j=1,2,\cdots,n)\,. \tag{5.23}$$

Since c_i satisfying (5.19) are of the order of ε because of (5.20), substituting $\alpha_i=\alpha_i^0+c_i$ into (5.23), we have

$$\frac{\partial\varphi_i(\alpha,\omega,\varepsilon)}{\partial\alpha_j}=\delta_{ij}+\varepsilon\frac{\partial\varphi_i^{(1)}(\alpha^0,\omega)}{\partial\alpha_j^0}+[\varepsilon]_2\,,$$

from which the matrix $\left\|\frac{\partial\varphi_i(\alpha,\omega,\varepsilon)}{\partial\alpha_j}\right\|$ can be expressed as

$$\left\|\frac{\partial\varphi_i(\alpha,\omega,\varepsilon)}{\partial\alpha_j}\right\|=\exp.\left\|\varepsilon\frac{\partial\varphi_i^{(1)}(\alpha^0,\omega)}{\partial\alpha_j^0}+[\varepsilon]_2\right\|.$$

Then, if we write the characteristic roots of $\left\|\frac{\partial\varphi_i(\alpha,\omega,\varepsilon)}{\partial\alpha_j}\right\|$ and $\left\|\frac{\partial\varphi_i^{(1)}(\alpha^0,\omega)}{\partial\alpha_j^0}\right\|$ as λ_i and μ_i $(i=1,2,\cdots,n)$ respectively, it is valid that

$$\lambda_i=e^{\varepsilon\mu_i}\cdot e^{[\varepsilon]_2} \quad (i=1,2,\cdots,n)\,. \tag{5.24}$$

From this follows that, when $|\varepsilon|$ is sufficiently small, $|\lambda_i|<1$ or >1 according as $\varepsilon\Re(\mu_i)<0$ or >0.

Now, from (5.6) and (5.10), the critical point $x_i=\alpha_i^0$ of (5.2) is stable or unstable according as $\varepsilon\Re(\mu_i)<0$ for $i=1,2,\cdots,n$ or $\varepsilon\Re(\mu_i)>0$ for at least one i of $1,2,\cdots,n$. Consequently, so long as the stabity is decided according to the signs of $\varepsilon\Re(\mu_i)$, the stability of periodic solution of (5.1) is same as that of the corresponding critical point of (5.2).

Now, the condition (5.20) is equivalent to

$$\det.\left|\frac{\partial}{\partial\alpha_j^0}\frac{\varepsilon}{\omega}\int_0^\omega X_i(\alpha^0,\tau,0)d\tau\right|\neq 0\,,$$

consequently, when the condition (5.20) is valid, the critical point $x_i=\alpha_i^0$ of (5.2) becomes an elementary critical point. Thus, summarizing the above results, we have the conclusion:

To each elementary critical point of the stroboscopic image (5.2), *there corresponds one and only one periodic solution with period* ω *of the initial system* (5.1) *and, so long as the stability is decided according to the signs of characteristic roots of the matrix composed of coefficients of linear parts in expansions of the right-hand sides of* (5.2) *in the critical point, the stability of the periodic solution of* (5.1) *is same as that of the corresponding critical point of* (5.2).

This conclusion says that the stroboscopic method certainly stands to reason for determination of the periodic solution (in mathematical proper sense) and for decision of the stability of that solution, when the stroboscopic image of the initial system has only elementary critical points and moreover, in each critical point, the real parts of the characteristic roots of the matrix composed of coefficients of linear parts in its expansions are all negative or at least one of them is positive.

4. Remarks on approximation

When the condition (5.20) does not hold, the real quantities c_i satisfying (5.19) do not always exist. Consequently, when (5.20) does not hold, corresponding to the critical point of the stroboscopic image, there does not always exist a periodic solution of the initial system.

But, when $|\varepsilon|$ is sufficiently small, the solution corresponding to $x_i = \alpha_i^0$ is periodic approximately—within the first order with respect to ε, since $\varphi_i(\alpha_0, \omega, \varepsilon) - \alpha_i^0$ are of the second and higher orders with respect to ε because of (5.3). Consequently, corresponding to any critical point of the stroboscopic image, even if there may not exist a mathematically—strictly—periodic solution of the initial system, there always exists an approximately—within the first order with respect to ε—periodic solution. This is the result obtained in the approximate process—in the reasonings within the first order with respect to ε.

When (5.20) does not hold, even if there may exist a periodic solution of (5.1), it is not always unique. Either (5.20) holds or not, suppose that, corresponding to the critical point $x_i = \alpha_i^0$ of the stroboscopic image, there exists a unique periodic solution of the initial system. Then, as is stated before, the stability of such a periodic solution is determined by convergence of iteration of the transformation of the form (5.22) and, when $\varepsilon\Re(\mu_i) < 0$ for $i = 1, 2, \cdots, n$ or $\varepsilon\Re(\mu_i) > 0$ for at least one i of $1, 2, \cdots, n$, the stability of the periodic solution is same as that of the corresponding critical point. But, when $\varepsilon\Re(\mu_i) \leqq 0$ for $i = 1, 2, \cdots, n$ and $\varepsilon\Re(\mu_i) = 0$ certainly occurs for at least one i, from (5.24), for i such that $\varepsilon\Re(\mu_i) = 0$, it may be that $|\lambda_i| < 1$, $= 1$ or > 1 owing to $e^{[\varepsilon]^2}$, though $|\lambda_i| < 1$ for i such that $\varepsilon\Re(\mu_i) < 0$. Therefore, in such a case, the stability of the periodic solution does not necessarily coincide with that of the corresponding critical point. Consequently such a case is not one where the former approximate process — reasonings within the first order with respect to ε — is effective.

The case where $\varepsilon\Re(\mu_i) \leqq 0$ and $\varepsilon\Re(\mu_i) = 0$ certainly occurs, is a critical case for the stroboscopic image. In such a case, as is well known, the stability of a critical point can not be decided only by the feature of linear parts of the expansions of the system in the critical point. Now, the solution $\psi_i(x, t, \varepsilon)$ $(i = 1, 2, \cdots, n)$ of the stroboscopic image (5.2) such that $\psi_i(x, 0, \varepsilon) = x_i$ can be sought for readily from the theory of continuous group of transformations as follows:

$$\psi_i(x, t, \varepsilon)=x_i+t\cdot\frac{\varepsilon}{\omega}\varphi_i^{(1)}(x, \omega)+\frac{t^2}{2!}\cdot\frac{\varepsilon^2}{\omega^2}\sum_{j=1}^{n}\varphi_j^{(1)}(x, \omega)\frac{\partial\varphi_i^{(1)}(x,\omega)}{\partial x_j}+\cdots$$
$$(i=1, 2, \cdots, n),$$

because $\int_0^\omega X_i(x, \tau, 0)d\tau=\varphi_i^{(1)}(x, \omega)$ from (5.6). Then it follows that

$$\psi_i(x, \omega, \varepsilon)=x_i+\varepsilon\varphi_i^{(1)}(x, \omega)+\frac{\varepsilon^2}{2}\sum_{j=1}^{n}\varphi_j^{(1)}(x, \omega)\frac{\partial\varphi_i^{(1)}(x, \omega)}{\partial x_j}+\cdots.$$

If we put $x_i=\alpha_i^0+r_i$, the above formula can be written as follows:

$$\psi_i(\alpha^0+r, \omega, \varepsilon)-\alpha_i^0=r_i+\varepsilon\left\{\sum_{j=1}^{n}\frac{\partial\varphi_i^{(1)}(\alpha^0, \omega)}{\partial\alpha_j^0}r_j+[r]_2\right\}+\frac{\varepsilon^2}{2}[r]_1. \tag{5.25}$$

Then convergence of iteration of the tranformation

$$r_i'=\psi_i(\alpha^0+r, \omega, \varepsilon)-\alpha_i^0,$$

namely the stabitity of the critical point $x_i=\alpha_i^0$ depends on the terms $\frac{\varepsilon^2}{2}[r]_1$ in the right-hand side of (5.25) when the stability depends on the terms $[r]_2$ in the braces, because $|r_i|$ can be smaller than $|\varepsilon|$. This expresses that, when $\varepsilon\Re(\mu_i)\leqq 0$ and $\varepsilon\Re(\mu_i)=0$ certainly occurs, the stability of a critical point of the stroboscopic image depends on the terms of the second and higher orders with respect to ε, in other words, the stability of the critical point is not determined within the first order with respect to ε. Consequently, such a case is certainly not contained in the former approximate discussions.

The case where $\varepsilon\Re(\mu_i)=0$ certainly occurs is used to be excluded in practical application because of the structural stability[1]. Besides, in practical application, any system is supposed to have only elementary critical point on analogous account of structural stability, consequently, by 3. of this paragraph, corresponding to any critical point of the stroboscopic image, there exists a unique periodic solution of the initial system. Thus we see that, *in practical application, for determination of the periodic solution and for decision of its stability, the stroboscopic method stands to reason also mathematically.*

§ 6. Complement. Reduction of nonlinear periodic system

1. Preliminaries

As is remarked in 5. of §4, in this paragraph, we consider the system (4.1), of which the real parts of the characteristic exponents μ_i are either all negative or all positive. When the real parts of the characteristic exponents are all positive, if we put $t=-\tau$, the system (4.1) is reduced to the system

$$\frac{dx_i}{d\tau}=-X_i(x, -\tau) \quad (i=1, 2, \cdots, n),$$

1) [10], p. 183 and pp. 337-340.

of which the real parts of the characteristic exponents are all negative. Consequently, in the sequel, we consider only the system (4.1), of which the real parts of the characteristic exponents are all negative. Then, from (4.2) follows

$$|\lambda_i|<1 \quad (i=1, 2, \cdots, n)\,.$$

Our method is similar to that used by the present writer [6] for reduction of the nonlinear autonomous system. First, for arbitrary function $\theta(x, t)$ expansible as the sum of the terms of the second and higher orders with respect to x_j with continuous coefficients periodic with period ω, we consider the equation

$$\sum_{j=1}^{n} X_j(x, t)\frac{\partial f}{\partial x_j}+\frac{\partial f}{\partial t}=\mu' f+\theta(x, t) \tag{6.1}$$

and seek for the solution $f(x, t)$ regular in the neighborhood of $x_j=0$, vanishing with x_j and periodic with respect to t with period ω. Here μ' is any one of μ_i's.

2. Case where the relation (4.4) does not occur

In this case, since the condition (μ) is fulfilled automatically, reduction of the system (4.1) is already solved in §4. But, here, in preparation for treating the general case, we study again this case but in the other way —namely making use of formal transformation. We write the system (6.1) as follows:

$$\begin{aligned}\sum\left\{\mu x_1\frac{\partial f}{\partial x_1}+\mu x_2\frac{\partial f}{\partial x_2}+\cdots+\mu x_m\frac{\partial f}{\partial x_m}\right\}+\frac{\partial f}{\partial t}-\mu' f \\ =-\sum\left\{x_1\frac{\partial f}{\partial x_2}+\cdots+x_{m-1}\frac{\partial f}{\partial x_m}\right\} \\ -\sum\left\{v_1\frac{\partial f}{\partial x_1}+\cdots+v_m\frac{\partial f}{\partial x_m}\right\}+\theta\,,\end{aligned} \tag{6.2}$$

where

$\sum$ denote summation over blocks of the canonical form of B;

the indices are those numbered in any representative block;

$v_1, \cdots, v_m$ are the sums of the terms of the second and higher orders with respect to x_j in $X_1, \cdots, X_m$.

First, for (6.2), we seek for the formal solution with respect to x_j with periodic coefficients.

Differentiating both sides of (6.2) with respect to x_i and putting all x_j's zero thereafter, we have:

$$\begin{cases}(\mu-\mu')\dfrac{\partial f}{\partial x_1}+\dfrac{\partial}{\partial t}\dfrac{\partial f}{\partial x_1}=-\dfrac{\partial f}{\partial x_2}\,, \\ (\mu-\mu')\dfrac{\partial f}{\partial x_2}+\dfrac{\partial}{\partial t}\dfrac{\partial f}{\partial x_2}=-\dfrac{\partial f}{\partial x_3}\,, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\end{cases}$$

$$\left\{\begin{aligned}&(\mu-\mu')\frac{\partial f}{\partial x_{m-1}}+\frac{\partial}{\partial t}\frac{\partial f}{\partial x_{m-1}}=-\frac{\partial f}{\partial x_m}\,,\\&(\mu-\mu')\frac{\partial f}{\partial x_m}+\frac{\partial}{\partial t}\frac{\partial f}{\partial x_m}=0\,.\end{aligned}\right.$$

Evidently these equations are all satisfied if we choose $\frac{\partial f}{\partial x_i}$ so that $\frac{\partial f}{\partial x_1}$ corresponding to the block containing μ' may be an in determinate constant and other $\frac{\partial f}{\partial x_i}$'s may be all zero. From this, in the sequel, we suppose that $\frac{\partial f}{\partial x_i}$ are chosen so.

In order to determine the derivatives of higher orders of f, we differentiate both sides of (6.2) $p_1, p_2, \cdots, p_m$ times ($p=\sum(p_1+\cdots p_m)\geqq 2$) with respect to $x_1, x_2, \cdots, x_m$ respectively and thereafter put all x_j's zero. Then we have:

$$\text{(6.3)}\quad \Big(\sum_i p_i\mu_i-\mu'\Big)\frac{\partial^p f}{\Pi\partial x_1^{p_1}\cdots\partial x_m^{p_m}}+\frac{\partial}{\partial t}\frac{\partial^p f}{\Pi\partial x_1^{p_1}\cdots\partial x_m^{p_m}}$$
$$=-\sum\Big\{p_1\frac{\partial^p f}{\cdots\partial x_1^{p_1-1}\partial x_2^{p_2+1}\partial x_3^{p_3}\cdots}+\cdots+p_{m-1}\frac{\partial^p f}{\cdots\partial x_{m-1}^{p_{m-1}-1}\partial x_m^{p_m+1}\cdots}\Big\}$$
$$+L(t)\,,$$

where $L(t)$ is a linear expression of the derivatives of at most $(p-1)$-th order with periodic coefficients. Putting

$$\text{(6.4)}\qquad \kappa=\sum_i p_i\mu_i-\mu'$$

and writing $\frac{\partial^p f}{\Pi\partial x_1^{p_1}\cdots\partial x_m^{p_m}}$ briefly as $f_{\mathfrak{p}}$, we express the equation (6.3) briefly as follows:

$$\text{(6.5)}\qquad \frac{df_{\mathfrak{p}}}{dt}+\kappa f_{\mathfrak{p}}=L(f_{\mathfrak{q}},t)\,,$$

where $f_{\mathfrak{q}}$ are the derivatives of f appearing in the right-hand side of (6.3) and $L(f_{\mathfrak{q}},t)$ is a linear expression of $f_{\mathfrak{q}}$'s with periodic coefficients. The equation (6.5) is readily integrated as follows:

$$\text{(6.6)}\qquad f_{\mathfrak{p}}=e^{-\kappa t}\Big[\int_0^t e^{\kappa\tau}L(f_{\mathfrak{q}},\tau)d\tau+c_{\mathfrak{p}}\Big]\,,$$

where $c_{\mathfrak{p}}$ is an integration constant. Then, if $f_{\mathfrak{q}}$'s are all periodic with period ω, it follows that

$$f_{\mathfrak{p}}(t+\omega)=e^{-\kappa\omega}e^{-\kappa t}\Big[\int_0^\omega e^{\kappa\tau}L(f_{\mathfrak{q}},\tau)d\tau+\int_\omega^{\omega+t}e^{\kappa\tau}L(f_{\mathfrak{q}},\tau)d\tau+c_{\mathfrak{p}}\Big]$$
$$=e^{-\kappa t}\Big[e^{-\kappa\omega}\int_0^\omega e^{\kappa\tau}Ld\tau+\int_0^t e^{\kappa\tau}Ld\tau+c_{\mathfrak{p}}e^{-\kappa\omega}\Big]\,.$$

Consequently, in order that $f_{\mathfrak{p}}$ may be periodic with period ω, it is necessary and sufficient that

$$\text{(6.7)}\qquad c_{\mathfrak{p}}e^{-\kappa\omega}+e^{-\kappa\omega}\int_0^\omega e^{\kappa\tau}Ld\tau=c_{\mathfrak{p}}\,.$$

Now, if $e^{-\kappa\omega}=1$, from (6.4), the relation of the form (4.4) is valid. This is contrary to the assumption. Therefore $e^{-\kappa\omega}\neq 1$, consequently $c_{\mathfrak{p}}$ satisfying (6.7) is uniquely determined, in other words, the periodic solution $f_{\mathfrak{p}}$ of (6.5) is uniquely determined if $f_{\mathfrak{q}}$'s are given periodically. Then, from the form of (6.3), it is readily seen that, if the constant value of $\frac{\partial f}{\partial x_1}$ corresponding to the block containing μ' is arbitrarily given, the derivatives of the second and higher orders of f are successively uniquely determined so that they may be periodic with period ω. Thus we see that the formal periodic solution $f(x, t)$ of (6.2) of the form

$$f(x, t)=ax_1+[x; t]_2 \tag{6.8}$$

can be uniquely sought for, when the value of a is given arbitrarily.

In the following, by means of the method of majorant, we shall prove convergence of the formal solution $f(x, t)$ obtained now. Since $v_1, \cdots, v_m$ and θ are bounded in the sufficiently small neighborhood of $x_j=0$ for $-\infty<t<\infty$, there exist the regular functions $V(x)$ and $\Theta(x)$ such that $v_i \ll V(x)$, $\theta \ll \Theta(x)$ and $V(0)=V'(0)=0$, $\Theta(0)=\Theta'(0)=0$. Taking sufficiently small positive number ε, we consider the equation

$$\varepsilon\sum\left\{x_1\frac{\partial F}{\partial x_1}+\cdots+x_m\frac{\partial F}{\partial x_m}\right\}-\varepsilon F=\sum\left\{x_1\frac{\partial F}{\partial x_2}+\cdots+x_{m-1}\frac{\partial F}{\partial x_m}\right\} \tag{6.9}$$
$$+V\sum\left\{\frac{\partial F}{\partial x_1}+\cdots+\frac{\partial F}{\partial x_m}\right\}+\Theta .$$

As is proved by the present writer [6], the above equation has a regular solution of the form

$$F=Ax_1+[x]_2 , \tag{6.10}$$

where A is an arbitrary constant. We take A so that $|a|\leqq A$.

Now the values of the derivatives of F for $x_j=0$ are obtained analogously as those of f differentiating (6.9) with respect to x_j and putting $x_j=0$ thereafter. The formulas for determining these derivatives are of the forms

$$\kappa_0 F_{\mathfrak{p}}=L(F_{\mathfrak{q}}) \tag{6.11}$$

where $\kappa_0=\varepsilon(p-1)$. Since $v_i \ll V$ and $\theta \ll \Theta$, if $|f_{\mathfrak{q}}|\leqq F_{\mathfrak{q}}$, then it is valid that

$$|L(f_{\mathfrak{q}}, t)|\leqq L(F_{\mathfrak{q}}) . \tag{6.12}$$

Now, since $\Re(\mu_i)<0$ $(i=1, 2, \cdots, n)$, there exists a positive integer M such that, for any set $\mathfrak{p}$ of p_i's such that $p=s(\mathfrak{p})=\sum p_i\geqq M$, $\Re(\kappa)<0$. For a set $\mathfrak{p}$ such that $s(\mathfrak{p})<M$, from (6.11) and (6.12), we can take ε so small that $|f_{\mathfrak{p}}|<F_{\mathfrak{p}}$, because $0<\kappa_0<\varepsilon(M-1)$. For a set $\mathfrak{p}$ such that $s(\mathfrak{p})\geqq M$, if $|f_{\mathfrak{q}}|\leqq F_{\mathfrak{q}}$, then, from (6.6) follows that, for $t\leqq 0$,

$$\begin{aligned}|f_{\mathfrak{p}}|&\leqq e^{-\Re(\kappa)t}\left[\int_t^0 e^{\Re(\kappa)\tau}|L(f_{\mathfrak{q}}, \tau)|d\tau+|c_{\mathfrak{p}}|\right]\\ &\leqq e^{-\Re(\kappa)t}\left[\frac{L(F_{\mathfrak{q}})}{\Re(\kappa)}\{1-e^{\Re(\kappa)t}\}+|c_{\mathfrak{p}}|\right]\\ &=-\frac{L(F_{\mathfrak{q}})}{\Re(\kappa)}+e^{-\Re(\kappa)t}\left[\frac{L(F_{\mathfrak{q}})}{\Re(\kappa)}+|c_{\mathfrak{p}}|\right] .\end{aligned}$$

Since $\Re(\kappa)<0$, as $t\to-\infty$, $e^{-\Re(\kappa)t}\to 0$, from which follows

$$|f_{\mathfrak{p}}|\leqq -\frac{L(F_{\mathfrak{q}})}{\Re(\kappa)}, \tag{6.13}$$

because $f_{\mathfrak{p}}$ is periodic with respect to t. Now, since $\Re(\kappa)=\sum_i p_i\Re(\mu_i)-\Re(\mu')$ <0 and $\Re(\mu_i)$, $\Re(\mu')<0$, we can take ε so small that, for any set $\mathfrak{p}$ such that $s(\mathfrak{p})\geqq M$,

$$\left|\frac{\Re(\kappa)}{p-1}\right|>\varepsilon\ ^{1)}.$$

Then, from (6.11) and (6.13) follows

$$|f_{\mathfrak{p}}|\leqq -\frac{L(F_{\mathfrak{q}})}{\Re(\kappa)}<\frac{L(F_{\mathfrak{q}})}{\kappa_0}=F_{\mathfrak{p}}.$$

Thus, by induction, we see that $|f_{\mathfrak{p}}|<F_{\mathfrak{p}}$ for any set $\mathfrak{p}$. Then, from regularity of F follows convergence of the formal solution $f(x, t)$.

3. General case

Let us consider the general case where the relation (4.4) may occur. We arrange λ_i as in §4 and consider the equations

$$\left\{\begin{aligned} &\sum_{j=1}^{n} X_j(x,t)\frac{\partial f_\alpha}{\partial x_j}+\frac{\partial f_\alpha}{\partial t}=\mu_\alpha f_\alpha+\theta_\alpha(x,t) && (\alpha=1,2,\cdots,s),\\ &\sum_{j=1}^{n} X_j(x,t)\frac{\partial f_\nu}{\partial x_j}+\frac{\partial f_\nu}{\partial t}=\mu_\nu f_\nu+\theta_\nu(x,t)+\Phi_\nu(x,t) && (\nu=s+1,\cdots,n),\end{aligned}\right. \tag{6.14}$$

where $\theta_i(x,t)$ $(i=1,2,\cdots,n)$ are of the same forms as $\theta(x,t)$ of the previous section and $\Phi_\nu(x,t)$ are the linear combinations of $x_1^{p_1}\cdots x_{\nu-1}^{p_{\nu-1}}$ for sets $\mathfrak{p}$ of non-negative integers $p_1,\cdots,p_{\nu-1}$ $(s(\mathfrak{p})\geqq 2)$ satisfying (4.5) with periodic coefficients.

By the discussions of the previous section, the formers of (6.14) have the regular periodic solutions of the forms

$$f_\alpha=a_{\alpha_1}x_{\alpha_1}+[x;t]_2.$$

For the latters, the derivatives of f_ν of the first order are determined in the same manner as the formers and the derivatives $f_{\mathfrak{p}}$ of f_ν of the second and higher orders are determined by the formula

$$\frac{df_{\mathfrak{p}}}{dt}+\kappa f_{\mathfrak{p}}=L(f_{\mathfrak{q}},t)+\Phi_{\mathfrak{p}} \tag{6.15}$$

instead of (6.5). The solution $f_{\mathfrak{p}}$ of (6.15) is readily sought for as follows:

$$f_{\mathfrak{p}}=e^{-\kappa t}\left[\int_0^t e^{\kappa\tau}\{L(f_{\mathfrak{q}},\tau)+\Phi_{\mathfrak{p}}\}d\tau+c_{\mathfrak{p}}\right].$$

For a set $\mathfrak{p}$ such that (4.5) does not hold, $e^{-\kappa\omega}\neq 1$, consequently the arbitrary constant $c_{\mathfrak{p}}$ is uniquely determined so that $f_{\mathfrak{p}}$ may be periodic with period ω. For a set $\mathfrak{p}$ such that (4.5) holds, as is seen from (6.7), in order that $f_{\mathfrak{p}}$ may be periodic with period ω, it is necessary and sufficient that

1) This inequality is already used in [6]. For proof, for instance, see [11] p. 6.

(6.16) $$\int_0^\omega e^{\kappa\tau}\{L(f_{\mathfrak{q}}, \tau)+\Phi_{\mathfrak{p}}\}d\tau=0 .$$

Now $\Phi_{\mathfrak{p}}$ is of the form

$$\Phi_{\mathfrak{p}}=(\Pi p_1!\, p_2! \cdots p_m!)c' ,$$

where c' is a coefficient of $\Pi x_1^{p_1} \cdots x_m^{p_m}$ in Φ_ν. Consequently, if we choose c' so that $c'=ce^{-\kappa t}$ where c is a constant, the equation (6.16) becomes

$$\int_0^\omega e^{\kappa\tau}L(f_{\mathfrak{q}}, \tau)d\tau+c\omega(\Pi p_1!\, p_2! \cdots p_m!)=0 ,$$

consequently c is uniquely determined, namely the coefficient c' of Φ_ν is uniquely determined. In this case, however, the integration constant $c_{\mathfrak{p}}$ is undetermined, so we may put $c_{\mathfrak{p}}=0$ for simplicity. In the sequel, we choose $c_{\mathfrak{p}}$ so. Since $e^{-\kappa\omega}=\lambda_\nu/\lambda_1^{p_1}\cdots\lambda_n^{p_n}=1$ owing to the assumption, $e^{-\kappa t}$ becomes periodic, consequently the coefficient c' of Φ_ν becomes periodic. Thus, we see that, if the derivatives $f_{\mathfrak{q}}$'s are known, the coefficient $c'=ce^{-\kappa t}$ of $x_1^p \cdots x_{\nu-1}^{p_{\nu-1}}$ in Φ_ν and consequently the derivatives $f_{\mathfrak{p}}$ of f_ν are uniquely determined. Then, repeating the above process, it is seen that the periodic formal solutions are obtained also for the latters of (6.14). Convergence of these formal solutions is proved quite analogously as that of f_α. Thus we see that *the equations* (6.14) *have the regular periodic solutions of the forms*

(6.17) $$f_i=a_{i_1}x_{i_1}+[x;t]_2 \quad (i=1, 2, \cdots, n) .$$

4. Reduction of the initial system

Making use of the results of the preceding section, it is proved analogously as in [6] that the equations of the forms

(6.18) $$\begin{cases} Xf_{i_1}=\mu_i f_{i_1}+\Phi_{i_1}(f) , \\ Xf_{i_2}=\mu_i f_{i_2}+f_{i_1}+\Phi_{i_2}(f) , \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ Xf_{i_m}=\mu_i f_{i_m}+f_{i_{m-1}}+\Phi_{i_m}(f) \end{cases}$$

where $X\equiv\sum_{j=1}^n X_j(x, t)\frac{\partial}{\partial x_j}+\frac{\partial}{\partial t}$ and $\Phi_{i_l}(f)$ are the sums of terms of the forms $ce^{-\kappa t}f_1^{p_1}\cdots f_{i-1}^{p_{i-1}}$ for sets $\mathfrak{p}$ of non-negative integers $p_1, \cdots, p_{i-1}$ ($s(\mathfrak{p})\geqq 2$) satisfying (4.5), have the regular periodic solutions $f_i(x, t)$ of the forms

(6.19) $$f_i=x_i+[x;t]_2 \quad (i=1, 2, \cdots, n) .$$

Making use of these solutions $f_i(x, t)$, we transform the variables x_j to y_j by

(6.20) $$y_i=f_i(x, t) \quad (i=1, 2, \cdots, n) .$$

Then, from (6.18) follows

(6.21) $$\frac{dy_i}{dt}=\mu_i y_i+\delta_i y_{i-1}+\Phi_i(y) \quad (i=1, 2, \cdots, n) .$$

Thus we have the conclusion:

When the real parts of the characteristic exponents are either all negative or all positive, the system (4.1) *is reduced to the system of the form* (6.21) *by the transformation* (6.20) *defined for* $f_i(x, t)$ *of the forms* (6.19)[1].

5. Remarks

When the condition (μ) is fulfilled, if $\Im(\mu_i)$ are chosen so that $\kappa = \sum_j p_j\mu_j - \mu_i = 0$ for any set $\mathfrak{p}$ ($s(\mathfrak{p}) \geqq 2$) of non-negative integers p_j's for which (4.5) holds, then, for such μ_i, the coefficients $c' = ce^{-\kappa t}$ of $\Phi_i(y)$ become constants, consequently the system (6.21) becomes the autonomous system (4.18). This expresses the validity of the conclusion of §4.

The solution of (6.21) is easily derived as follows:

$$(6.22)\qquad \begin{cases} y_{i_1} = e^{t\mu_i}\{c_{i_1} + P_{i_1}(t)\}\ , \\ y_{i_2} = e^{t\mu_i}\{c_{i_1}t + c_{i_2} + P_{i_2}(t)\}\ , \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ y_{i_m} = e^{t\mu_i}\left\{c_{i_1}\dfrac{t^{m-1}}{(m-1)!} + c_{i_2}\dfrac{t^{m-2}}{(m-2)!} + \cdots + c_{i_{m-1}}t + c_{i_m} + P_{i_m}(t)\right\}, \end{cases}$$

where $c_{i_1}, c_{i_2}, \cdots, c_{i_m}$ are the arbitrary constants and $P_{i_1}(t), \cdots, P_{i_m}(t)$ are the definite polynomials of t. Then, by (6.20), the solution of the initial system becomes a set of functions of the forms

$$(6.23)\qquad x_i = g_i(y_1, \cdots, y_n, t)\quad (i = 1, 2, \cdots, n)\ ,$$

where y_j are the functions given by (6.22) and g_i are the inverse functions of f_i with respect to x_j. Thus we see that *the solution of the initial system is a set of regular functions of the functions* (6.22) *with the coefficients periodic with respect to* t. This is a well known result[2].

References

[1] N. Minorsky, *Sur l'oscillateur non linéaire de Mathieu*. C.R. Acad. Sci. Paris, **232**, 2179–2180 (1951).

[2] N. Minorsky, *La méthode stroboscopique et ses applications*. Bull. Soc. France Mec., **4**, 15–26 (1954).

[3] J. Haag, *Sur certains systèmes d'équations différentielles à solutions périodiques*. Bull. Sci. Math., **70**, 155–172 (1946).

[4] M. Urabe, *Equations of Schröder* (*Continued*). J. Sci. Hiroshima Univ., Ser. A., **15**, 203–233 (1952).

[5] M. Urabe, *Equations of Schröder*. J. Sci. Hiroshima Univ., Ser. A, **15**, 113–131 (1951).

[6] M. Urabe, *On solutions of the linear homogeneous partial differential equations in the vicinity of the singularity*. I, III. J. Sci. Hiroshima Univ., Ser. A, **14**, 1–12 (1950); **15**, 25–37 (1951).

[7] M. Urabe, *Application of majorized group of transformations to functional equations*. J. Sci. Hiroshima Univ., Ser. A, **16**, 267–283 (1952).

1) When the real parts of the characteristic exponents are all positive, we have the reduced form (6.21) with δ_i substituted by $-\delta_i$. But, as is readily seen, this reduced form is transformed to the form of (6.21) by the linear transformation with constant coefficients.

2) For instance, see [12] p. 429.

[8] M. Urabe, *Application of majorized group of transformations to ordinary differential equations with periodic coefficients*. J. Sci. Hiroshima Univ., Ser. A, **19**, 469–478 (1956).

[9] Y. Sibuya, *Sur un système des équations différentielles ordinaires non linéaires à coefficients constants ou périodiques*. J. Fac. Sci. Univ. Tokyo, Sec. I, **7**, 19–32 (1954).

[10] A. A. Andronow and C. E. Chaikin, *Theory of oscillations*. Princeton, 1949.

[11] E. Picard, *Traité d'analyse*, t. III. Paris, 1908.

[12] M. A. Liapounoff, *Problème général de la stabilité du mouvement*. Princeton, 1949.

Department of Mathematics,
Faculty of Science,
Hiroshima University.