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A Generalization of a Theorem of Dye

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The group $\boldsymbol{M}_U$ of unitary operators in a ring $\boldsymbol{M}$ of operators in a Hilbert space is viewed as a uniform space in the uniform structure induced by the weak topology of $\boldsymbol{M}$. Recently it was shown by H. A. Dye ([1], Theorem 2) that if $\boldsymbol{M}$ and $\boldsymbol{N}$ are finite non-atomic rings of operators and ϕ is a group isomorphism of $\boldsymbol{M}_U$ and $\boldsymbol{N}_U$ which is unimorphic in the relative weak uniform structures on $\boldsymbol{M}_U$ and $\boldsymbol{N}_U$, then ϕ has a unique extension to the sum of a linear and a conjugate linear *-isomorphism of $\boldsymbol{M}$ and $\boldsymbol{N}$. We shall show that the statement is also true without the condition "finite". The proof will be carried out, basing on his preliminary results, by two steps; the first for abelian rings and the second for the general case.

1. Let $\boldsymbol{M}$ be a ring of operators on a Hilbert space. $\boldsymbol{M}_P$ stands for the set of projections in $\boldsymbol{M}$, and $\boldsymbol{M}_U$ the set of unitary operators in $\boldsymbol{M}$. $\boldsymbol{M}$ is said to be non-atomic if for every non-zero $P \in \boldsymbol{M}_P$ there exists a non-zero $Q \in \boldsymbol{M}_P$ with $Q < P$. Unless otherwise stated, the rings considered will be non-atomic. For any normal state t of $\boldsymbol{M}$ it will be non-atomic in the sense that $t(P) > 0$ ($P \in \boldsymbol{M}_P$) implies $t(P) > t(Q) > 0$ for some $Q \in \boldsymbol{M}$ with $Q < P$. For if we define t_P by the equation $t_P(A) = t(PAP)$ for every $A \in \boldsymbol{M}$, then t_P is also a normal state with carrier projection $P_0 \leq P$. Since $\boldsymbol{M}$ is non-atomic, we can select a $Q \in \boldsymbol{M}_P$ with $0 < Q < P_0$. Then $t(Q) = t_P(Q)$ and $0 < t(Q) < t(P)$, as desired. The following lemma is a generalization of a theorem of Liapounov [2] and is a direct consequence of a result of Dye ([1], Lemma 2.3).

Lemma 1. *Let $t_1, \cdots, t_n$ be ultraweakly continuous linear forms of a non-atomic ring $\boldsymbol{M}$. Then the span in the unitary space of the n-tuples $(t_1(P), \cdots, t_n(P))$, $P \in \boldsymbol{M}_P$, is a convex closed set and coincides with the span of $(t_1(A), \cdots, t_n(A))$, where A ranges over the set of positive semi-definite operators in the unit sphere $\Sigma_{\boldsymbol{M}}$ of $\boldsymbol{M}$.*

Since, for any self-adjoint A with norm ≤ 1, $\frac{1}{2}(\mathrm{I} - A)$ is positive semi-definite with norm $\leqq 1$, the lemma shows us (Cf. the proof of Lemma 2.4 of [I]) that the span of the n-tuples $(t_1(A), \cdots, t_n(A))$, where A ranges over the set of self-adjoint operators in the unit sphere of $\boldsymbol{M}$, is a convex closed set and coincides with the span of $(t_1(U), \cdots, t_n(U))$, where U

ranges over the set of self-adjoint unitary operators in $\boldsymbol{M}$. For any $A \in \boldsymbol{M}$ with norm ≤ 1, let VH be its polar decomposition, V being partially isometric and H self-adjoint. From the preceding discussion we have $t_i(A)=t_i(VU)$, $i=1,\cdots n$, for some $U \in \boldsymbol{M}_U$. As VU is partially isometric, so the span of $(t_1(A),\cdots, t_n(A))$, $A \in \Sigma_{\boldsymbol{M}}$, coincides with that of $(t_1(V),\cdots, t_n(V))$, where V ranges over the set of partially isometric operators in $\boldsymbol{M}$ (the set of unitary operators if $\boldsymbol{M}$ is finite). Dye ([1], Theorem 1) proved the following theorem:

In any non-atomic ring $\boldsymbol{M}$, the weak closure of $\boldsymbol{M}_U$ is the entire unit sphere $\Sigma_{\boldsymbol{M}}$ of $\boldsymbol{M}$.

But in his demonstration of this theorem some careless arguments are found ([1], p. 59), so that an amendement will be given here. It will suffice to show that any partially isometric operator $V \in \boldsymbol{M}$ lies in the weak closure of $\boldsymbol{M}_U$. Since the carrier projection of any normal state is σ-finite and the theorem is true for any finite ring, it follows from the dimension theory that we may assume that $\boldsymbol{M}$, $E=V^*V$, $F=VV^*$ are σ-finite and properly infinite. In the present case it is sufficient to show that V is a strong limit of a sequence of partially isometric operators $V_n \in \boldsymbol{M}$ such that the restriction of V_n on its initial domain has a unitary extension $U_n \in \boldsymbol{M}$. For V_n is of the form U_nP_n, $P_n \in \boldsymbol{M}_P$ and therefore lies in the weak closure of $\boldsymbol{M}_U$. Now we may write $E=\Sigma_1^\infty E_j$, $F=\Sigma_1^\infty F_j$ where $E_j \sim E$, $F_j \sim F$ and $F_j=VE_jV^*$. Put $E^{(n)}=\Sigma_1^n E_j$ and $F^{(n)}=\Sigma_1^n F_j$. Then $E^{(n)} \sim F^{(n)}$. Clearly $E^{(n)}$, $F^{(n)}$ contains no non-zero central projections in $\boldsymbol{M}$, so that $1-E^{(n)}$, $1-F^{(n)}$ have the same central support in $\boldsymbol{M}$. In any ring of operators two σ-finite properly infinite projections in the ring are equivalent if they have the same central support (Cf. the arguments of the proof of [3], Lemma 5). Therefore $1-E^{(n)} \sim 1-F^{(n)}$, and there exist unitary operators U_n in $\boldsymbol{M}$ such that $VE^{(n)}=U_nE^{(n)}$ for all n. It is easy to see that $V_n=VE^{(n)}$ meet all the requirements just mentioned. The proof is complete.

2. Let $\boldsymbol{M}$ and $\boldsymbol{N}$ be non-atomic rings of operators, and ϕ be a group isomorphism of $\boldsymbol{M}_U$ and $\boldsymbol{N}_U$ which is unimorphic in the relative weak uniform structures on $\boldsymbol{M}_U$ and $\boldsymbol{N}_U$. $\Sigma_{\boldsymbol{M}}$ and $\Sigma_{\boldsymbol{N}}$ are the weak closures of $\boldsymbol{M}_U$ and $\boldsymbol{N}_U$ respectively as indicated in 1, so that ϕ has a unique extension to a weak homeomorphism between $\Sigma_{\boldsymbol{M}}$ and $\Sigma_{\boldsymbol{N}}$ which will also be denoted by the same symbol ϕ. Dye ([1], Lemma 3.3) showed that thus extended ϕ has the following properties:

(i) $\phi(A)^*=\phi(A^*)$ for all $A \in \Sigma_{\boldsymbol{M}}$;

(ii) $\phi(AB)=\phi(A)\phi(B)$ for all $A, B \in \Sigma_{\boldsymbol{M}}$;

(iii) ϕ is a completely additive mapping of $\boldsymbol{M}_P$ on $\boldsymbol{N}_P$;

(iv) $\phi(\alpha 1)=\alpha 1$ for each α, $0 \leq \alpha \leq 1$;

(v) there exists a central projection R in $\boldsymbol{N}$ such that $\phi(\lambda 1)=\lambda R + \bar{\lambda}(1-R)$ for each complex number λ of absolute value ≤ 1.

Now we define a further extension of ϕ on $\boldsymbol{M}$ which is also denoted by the same symbol ϕ. Let A be any operator in M. We choose a positive number α such that $\|\alpha A\| \leq 1$, and we define $\phi(A)=\frac{1}{\alpha}\phi(\alpha A)$ which is determined uniquely independent of the manner of taking α. Clearly ϕ is a one-to-one mapping of $\boldsymbol{M}$ on $\boldsymbol{N}$. It is a simple matter to verify that (i), (ii) hold for all operators in $\boldsymbol{M}$, (iv) for each real α, (v) for each complex number λ, and ϕ is a norm preserving mapping. We shall show that ϕ is the direct sum of a linear and a conjugate linear *-isomorphism of $\boldsymbol{M}$ and $\boldsymbol{N}$. To this end first we show the following

Lemma 2. *If $\boldsymbol{M}$ and $\boldsymbol{N}$ are abelian and non-atomic, then any group isomorphism ϕ mentioned above has a unique extension to the direct sum of a linear and a conjugate linear *-isomorphism of $\boldsymbol{M}$ and $\boldsymbol{N}$.*

Proof. Let $\Omega_{\boldsymbol{M}}$ and $\Omega_{\boldsymbol{N}}$ be the spectres of $\boldsymbol{M}$ and $\boldsymbol{N}$ respectively. $\boldsymbol{M}$ and $\boldsymbol{N}$ are viewed as the sets of complex-valued continuous functions on $\Omega_{\boldsymbol{M}}$ and $\Omega_{\boldsymbol{N}}$ respectively. It follows from the property (iii) that $\Omega_{\boldsymbol{M}}$ and $\Omega_{\boldsymbol{N}}$ are homeomorphic in such a way that the representing continuous functions of $\boldsymbol{M}_P$ and $\boldsymbol{N}_P$ correspond to each other, since two stonean spaces are homeomorphic if the lattices of their open closed sets are isomorphic. Then the property (v) implies that there exists a unique mapping ψ of $\boldsymbol{M}$ on $\boldsymbol{N}$ which is the direct sum of a linear and a conjugate linear *-isomorphism of $\boldsymbol{M}$ and $\boldsymbol{N}$ and $\psi(\lambda P)=\phi(\lambda P)$ for each $P \in \boldsymbol{M}_P$. Let us define $\phi_1=\psi^{-1}\phi$. Then ϕ_1 is a group automorphism of $\boldsymbol{M}_U$ which is unimorphic in the relative weak uniform structure of $\boldsymbol{M}_U$. If we can show that $\phi_1(U)=U$ for each $U \in \boldsymbol{M}_U$, then ϕ_1 will be identical since $\boldsymbol{M}_U$ is weakly dense in the unit sphere $\Sigma_{\boldsymbol{M}}$ of $\boldsymbol{M}$ and ϕ_1 is weakly continuous in $\Sigma_{\boldsymbol{M}}$. From the fact that the set of products of a finite number of unitary operators of the form $\lambda P+(1-P)$, $P \in \boldsymbol{M}_P$, is weakly dense in $\boldsymbol{M}_U$, it is sufficient to show that $\phi_1(\lambda P+(1-P))=\lambda P+(1-P)$. Now $\phi_1(\lambda P+(1-P))$ $=\phi_1(\lambda P+(1-P))\phi_1(P)+\phi_1(\lambda P+(1-P))\phi_1(1-P)=\phi_1(\lambda P)+\phi_1(1-P)=\lambda P+1-P$. The proof is complete.

We are now in the position to show the following

Theorem. *Let $\boldsymbol{M}$ and $\boldsymbol{N}$ be non-atomic rings of operators. If ϕ is a group isomorphism of $\boldsymbol{M}_U$ on $\boldsymbol{N}_U$ which is unimorphic in the relative weak uniform structures on $\boldsymbol{M}_U$ and $\boldsymbol{N}_U$, then ϕ has a unique extension to the direct sum of a linear and a conjugate linear *-isomorphism of $\boldsymbol{M}$ and $\boldsymbol{N}$.*

Proof. From the properties (ii), (v) the uniqueness is evident if the extension in the theorem exists. As to the extension we may assume by the properties just mentioned that $R=1$ or $R=0$.

Case $R=1$. Let C be any given self-adjoint operator in $\boldsymbol{M}$. Let $\boldsymbol{A}$ be a maximal abelian *-subalgebra of $\boldsymbol{M}$ which contains C. $\boldsymbol{A}$ is clearly a ring of operators and so is also $\phi(\boldsymbol{A})$ by the property (ii). Then by the

preceding lemma we obtain that $\phi(e^{tC})=e^{t\phi(C)}$ and $\phi\left(\frac{1}{t}(e^{tC}-1)\right)=\frac{1}{t}(e^{t\phi(C)}-1)$ $\rightarrow\phi(C)$ uniformly as $t\rightarrow 0$. Now let A, B be any self-adjoint operators in $\boldsymbol{M}$.
Then

$$\phi\left(\frac{1}{t}(e^{\frac{1}{2}tA}e^{tB}e^{\frac{1}{2}tA}-1)\right)=\frac{1}{t}(e^{\frac{1}{2}t\phi(A)}e^{t\phi(B)}e^{\frac{1}{2}t\phi(A)}-1). \tag{1}$$

Now passing to the limit as $t\rightarrow 0$, we obtain from the equation (1) that $\phi(A+B)=\phi(A)+\phi(B)$. We put $\Phi(A+iB)=\phi(A)+i\phi(B)$. It is easy to verify that Φ is a linear one-to-one mapping such that $\Phi(U)=\phi(U)$ on $\boldsymbol{M}_U$. Then from the equations

$$\Phi(e^{itA}e^{itB})=\phi(e^{itA}e^{itB})=\phi(e^{itA})\phi(^{itB})=\Phi(e^{itA})\Phi(e^{itA})$$

we have

$$0=\frac{1}{t^2}\{\Phi(e^{itA}e^{itB})-\Phi(e^{itA})\Phi(e^{itB})\}=\Phi(A)\Phi(B)-\Phi(AB)+t\{\cdots\}, \tag{2}$$

Then passing to the limit as $t\rightarrow 0$, we obtain from the equation (2) that $\Phi(AB)=\Phi(A)\Phi(B)$ for self-adjoint A, B and in turn for all A, B in $\boldsymbol{M}$. Thus Φ is a linear *-isomorphism and therefore weakly continuous in Σ_M. Since $\Phi(U)=\phi(U)$ on $\boldsymbol{M}_U$, it follows that $\Phi(A)=\phi(A)$ for each A in $\boldsymbol{M}$.

Case $R=0$. We can show that ϕ is a conjugate linear *-isomorphism of $\boldsymbol{M}$ on $\boldsymbol{N}$. The proof is carried out by the same way as above with obvious modifications. The details are omitted.

References

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[3] T. Ogasawara, *Topologies on rings of operators*, This journal 19 (1955), 255–272.