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Distinguished Elements in a Space of Distributions

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1. Let E be a space of distributions on R^n, i.e. a locally convex space which is contained in $(\mathscr{D}')$ as a linear subspace with a finer topology: that is to say, the injection i: $E\to(\mathscr{D}')$ is continuous.

Definition 1. A distribution $T\in E$ is called *distinguished*, if there exists a sequence of positive numbers $\{\lambda_{\vec{p}}\}$ such that $\{\lambda_{\vec{p}}D^{\vec{p}}T\}$ converges to zero in E as $|\vec{p}|\to\infty$. This is equivalent to saying that there exists a bounded sequence of positive numbers $\{\lambda_{\vec{p}}\}$ such that $\{\lambda_{\vec{p}}D^{\vec{p}}T\}$ is bounded in E. We denote by E^d the set of distinguished elements of E.

With slight modifications of the proof of Theorem 4 of [6] we can show that any distinguished element of E belongs to $(\mathscr{E})$.

Definition 2. Let E, F be two spaces of distributions such that $F\subset E$. If the conditions $E^d=F$ and $F^d=F$ are satisfied, then we say that (E, F) makes a *distinguished pair*.

In this paper we give a fairly general criterion for distinguished elements available for usual spaces of distributions.

Throughout this paper we adopt the notations of L. Schwartz [7] unless otherwise specifically mentioned.

Theorem 1. *Let E, F be two spaces of distributions satisfying the following conditions* (1), (2), (3):

(1) *F is a quasi-complete space contained in $(\mathscr{E})$ with the continuous injection $F\to(\mathscr{E})$.*

(2) *F has a fundamental system of $(\mathscr{E})$-closed absolutely convex neighbourhoods of O in F, that is to say, F has a fundamental system of absolutely convex neighbourhoods of O closed in the topology induced by $(\mathscr{E})$.*

(3) *For any element $S\in E$, $\varphi\in(\mathscr{D})$, we have $S*\varphi\in F$ and the mapping $(S, \varphi)\to S*\varphi$ of $E\times(\mathscr{D})\to F$ is separately continuous.*

Then any distinguished element T of E belongs to F, and the sequence $\{T*\rho_k\}$ converges to T in F, where $\{\rho_k\}$ is any sequence of regularizations ($\rho_k\in(\mathscr{D})$, $\check{\rho}_k=\rho_k$, $\rho_k\geqq O$, $\int\rho_k(x)dx=1$, *and the support of ρ_k tends to O in R^n as $k\to\infty$*).

Proof. $\{\rho_k\}$ is bounded in $(\mathscr{E}')$, so that we may assume that there exists a ball K with center O in R^n which contains the supports of ρ_k, $k=1, 2,\cdots$. Let V be any given $(\mathscr{E})$-closed absolutely convex neighbourhood of O in F, B a bounded set of E which absorbs each derivative $D^{\vec{p}}T$. Such a B exists from the definition of distinguished element T. Then the condition (3) guarantees the existence of a neighbourhood U of zero in $(\mathscr{D}_{2K})$ such that

$$B*U\subset V \tag{4}$$

where U is written in the form $U=U_s\cap(\mathscr{D}_{2K})$ with a neighbourhood U_s of zero in a $(\mathscr{D}^s_{2K})$, s being a positive integer. We note that $\{\rho_k\}$ is bounded in $(\mathscr{D}'^0_K)$.

Choose a parametrix $u\in(\mathscr{D}^s_K)$ of an iterated Laplacian Δ^p for an appropriate positive integer p; $\delta=\Delta^p u+\xi$, $\xi\in(\mathscr{D}_K)$. Since $(\mathscr{D}_K)$ is dense in $(\mathscr{D}^s_K)$, it follows that we can find a sequence of functions $u_i\in(\mathscr{D}^s_K)$, tending to u in $(\mathscr{D}^s_K)$. Hence we have for any k

$$u_i*\rho_k\to u*\rho_k \text{ in } (\mathscr{D}_{2K}) \text{ as } i\to\infty. \tag{5}$$

From (4) it follows that $\{\Delta^p T*u_i\}$ is a Cauchy sequence in F_V, hence there exists a positive number N such that for every positive numbers $i, j>N$ we have

$$\Delta^p T*u_i-\Delta^p T*u_j\in V. \tag{6}$$

Since $\Delta^p T$ belongs to $(\mathscr{E})$ (as remarked after Def. 1) and the application $(f, S)\to f*S$ of $(\mathscr{E})\times(\mathscr{E}')\to(\mathscr{E})$ is hypocontinuous, $\Delta^p T*u_i$ tends to $\Delta^p T*u$ in $(\mathscr{E})$ as $i\to\infty$. Then we have for a sufficiently large i

$$\Delta^p T*u_i-\Delta^p T*u\in\bar{V}, \text{ the closure of } V \text{ in } (\mathscr{E}). \tag{7}$$

From (4) and (5) we have also for sufficiently large i depending on k

$$\Delta^p T*(u_i*\rho_k)-\Delta^p T*(u*\rho_k)\in V. \tag{8}$$

On the other hand, since $u_i*\rho_k\to u_i$ uniformly in $(\mathscr{D}^s_{2K})$ as $k\to\infty$, we have

$$\Delta^p T*u_i-\Delta^p T*u_i*\rho_k\in V \text{ for a sufficiently large } k, \text{ and} \tag{9}$$

$$T*\xi-T*\xi*\rho_k\in V \text{ for a sufficiently large } k. \tag{10}$$

Estimating $T-T*\rho_k$ by means of (7), (8), (9) and (10) we have for a sufficiently large k

$$\begin{aligned}
&T-T*\rho_k\\
&\quad=T*(\Delta^p u+\xi)-T*(\Delta^p u+\xi)*\rho_k\\
&\quad=(\Delta^p T*u-\Delta^p T*u*\rho_k)+(T*\xi-T*\xi*\rho_k)\\
&\quad=(\Delta^p T*u-\Delta^p T*u_i)+(\Delta^p T*u_i-\Delta^p T*u_i*\rho_k)\\
&\quad\quad+(\Delta^p T*u_i*\rho_k-\Delta^p T*u*\rho_k)+(T*\xi-T*\xi*\rho_k)\\
&\quad\in\bar{V}+V+V+V\subset 4\bar{V}.
\end{aligned}$$

Then we have for sufficiently large k, k'

$$T*\rho_k-T*\rho_{k'}\in 8\bar{V}\cap F=8V$$

and consequently $\{T*\rho_k\}$ is a Cauchy sequence in F, hence by the quasi-completeness of F, there exists a distribution $S \in F$ such that $T*\rho_k \to S$ in F.

On the other hand, since $T \in (\mathscr{E})$, we have

$$T*\rho_k \to T \text{ in } (\mathscr{E}).$$

Therefore $S=T$. This completes the proof.

Next, before giving several examples of distinguished pairs, we shall give some remarks on the conditions of the theorem.

1° Any space of type $(\boldsymbol{F})$ has the first countability property of Mackey, i. e. for any given sequence of bounded subsets A_k there exists a sequence of positive numbers λ_k such that the union of $\lambda_k A_k$ is bounded [3; p. 69]. Hence if E is a space of type $(\boldsymbol{F})$ and $D^{\vec{p}}T \in E$ for any $\vec{p}$, then T is distinguished, hence moreover if E is stable under differentiations every distribution of E is distinguished.

2° If F has a fundamental system of $(\mathscr{D}')$-closed absolutely convex neighbourhoods $\{U\}$ of 0, then it is clear that U is $(\mathscr{E})$-closed.

3° Suppose $E*(\mathscr{D}) \subset F$. If E is barrelled and F is a permitted space or its dual, then the bilinear mapping $(T, \varphi) \to T*\varphi$ of $E \times (\mathscr{D}) \to F$ becomes hypocontinuous [10; p. 21].

4° Let $\mathfrak{A}$ be any bounded subset of E. If there exists a sequence of positive numbers $\lambda_{\vec{p}}$ such that the union of $\lambda_{\vec{p}} D^{\vec{p}}\mathfrak{A}$ is bounded, then under the conditions (1), (2) and (3) $T*\rho_k \to T$ uniformly as $k \to \infty$ when T runs through $\mathfrak{A}$. In fact, our proof of the theorem is applied to this case with necessary modifications.

2. Examples. We shall show that each of the following pairs forms a distinguished pair: $((\mathscr{D}'), (\mathscr{E}))$, $((\mathscr{O}_M), (\mathscr{O}_c))$, $((\mathscr{S}'), (\mathscr{O}_c))$, $((\mathscr{B}'), (\mathscr{B}))$, $((\mathscr{D}'_{L^p}), (\mathscr{D}_{L^p}))$(for $1 \leq p < \infty$), $((\mathscr{O}'_{c'}), (\mathscr{S}))$, $((\mathscr{E}'), (\mathscr{D}))$.

The spaces $(\mathscr{E})$, $(\mathscr{B})$, $(\mathscr{D}_{L^p})$ (for $1 \leq p < \infty$), $(\mathscr{S})$ are spaces of type $(\boldsymbol{F})$ stable under differentiations. Therefore by the remark 1°, we have $(\mathscr{E})^d = (\mathscr{E})$, $(\mathscr{B})^d = (\mathscr{B})$, $(\mathscr{D}_{L^p})^d = (\mathscr{D}_{L^p})$ (for $1 \leq p < \infty$), $(\mathscr{S})^d = (\mathscr{S})$. Any element of $(\mathscr{D})$ is an element of some $(\mathscr{D}_K)$, K being a compact subset of R^n. $(\mathscr{D}_K)$ is a space of type $(\boldsymbol{F})$, so that $(\mathscr{D})^d = (\mathscr{D})$. Next we shall show that $(\mathscr{O}_c)^d = (\mathscr{O}_c)$. Before this we note that the space $(\mathscr{O}_c)$, the dual of $(\mathscr{O}'_c)$, consists of the elements written in the form $S*\varphi$, $S \in (\mathscr{S}')$, $\varphi \in (\mathscr{S})$. $(\mathscr{O}'_c)$ is the space of convolution operators: $(\mathscr{S}) \to (\mathscr{S})$, with the topology induced by $\mathscr{L}_b((\mathscr{S}), (\mathscr{S}))$. This induced topology coincides with the topology of simple convergence, that is, the topology of $\mathscr{L}_s((\mathscr{S}), (\mathscr{S}))$. In fact, any neighbourhood of zero in $(\mathscr{O}'_c)$ contains a neighbourhood of the form $\{T: |\langle T*A, A'\rangle| \leq 1\}$ with a bounded set A of $(\mathscr{S})$ and an equicontinuous set A' of $(\mathscr{S}')$. If we can show that A may be written in the from $A = B*g$ with a bounded set B in $(\mathscr{S})$ and a function g of $(\mathscr{S})$, our assertion is through. This follows from the following

Lemma 1. *Any bounded set A of $(\mathscr{S})$ can be written in the form $A = B*g$* or

$A=Bg$, where B is a bounded set of $(\mathscr{S})$ and g is an element of $(\mathscr{S})$.

PROOF. Fourier transformation shows us that it suffices to show that A may be written in the form $A=Bg$ with the described properties. $sup_{\varphi \in A} |D^{\vec{p}}\varphi|^{1/k}$ is a rapidly decreasing function for any $\vec{p}$ and k. Since Mackey's first countability axiom holds in any space of type $(\boldsymbol{F})$, we can choose a sequence of positive numbers $\lambda_{\vec{p},k}$ such that $\lambda_{\vec{p},k}\ sup_{\varphi \in A} |D^{\vec{p}}\varphi|^{1/k}$ is uniformly rapidly decreasing. Then it follows from a lemma of Chevalley [2; p. 127] that there exists a strict positive function $g \in (\mathscr{S})$ such that $\lambda_{\vec{p},k}\ sup_{\varphi \in A} |D^{\vec{p}}\varphi|^{1/k} \leqq g$. Hence

$$|D^{\vec{p}}\varphi/g^{k-1}| \leqq g/\lambda^{k}_{\vec{p},k}$$

On the other hand $D^{\vec{p}}(\varphi/g)$ is written in the form

$$P(\varphi, D^1\varphi, \cdots, D^{\vec{p}}\varphi, g, D^1 g, \cdots, D^{\vec{p}}g)/g^{|\vec{p}|+1}$$

where $P(\cdots)$ is a polynomial linear with respect to $\varphi, D^1\varphi, \cdots, D^{\vec{p}}\varphi$. Therefore $|D^{\vec{p}}(\varphi/g)| \leqq M_{\vec{p}}g$ with constants $M_{\vec{p}}$ independent of elements of A, hence we can write A in the form $A=Bg$. This completes the proof.

Suppose E and F be locally convex spaces. There is in an obvious way an isomorphism between the dual of $\mathscr{L}_s(E, F)$ and the set $E \otimes F'$ [1; *p*. 77]. Applying this fact to $\mathscr{L}_s((\mathscr{S}), (\mathscr{S}))$, any element of $(\mathscr{O}_c)$ is written in the form $\sum_{i=1}^{n} S_i * \varphi_i$ where $S_i \in (\mathscr{S}')$, $\varphi_i \in (\mathscr{S})$. Owing to Lemma 1, we can find a $\varphi \in (\mathscr{S})$ such that $\varphi_i = \psi_i * \varphi$, ψ_i being in $(\mathscr{S})$. Therefore any element of $(\mathscr{O}_c)$ is of the form $S*\varphi$, $S \in (\mathscr{S}')$, $\varphi \in (\mathscr{S})$. The converse is evident.

Grothendieck proved [4; p. 131] that $(\mathscr{O}'_c)$ and its dual $(\mathscr{O}_c)$ are ultrabornological, nuclear and complete. It is easy to see that they are permitted [8; p. 10]. The bilinear mapping $(S, \varphi) \rightarrow S*\varphi$ of $(\mathscr{S}') \times (\mathscr{S}) \rightarrow (\mathscr{O}_c)$ is hypocontinuous [10; p. 21]. For any element $T=S*\varphi \in (\mathscr{O}_c)$, $S \in (\mathscr{S}')$, $\varphi \in (\mathscr{S})$, each derivative $D^{\vec{p}}T$ is written as $S*D^{\vec{p}}\varphi$ and any $\varphi \in (\mathscr{S})$ is distinguished in $(\mathscr{S})$, so that T is distinguished in $(\mathscr{O}_c)$. This implies $(\mathscr{O}_c)^d = (\mathscr{O}_c)$.

For our later use we show

LEMMA 2. *Any bounded set A of $(\mathscr{O}_c)$ is written in the form $A=B*\varphi$, where B is a bounded set of $(\mathscr{S}')$* and φ is *a function of* $(\mathscr{S})$.

PROOF. Any bounded set of $(\mathscr{O}_c)$ is equicontinuous when $(\mathscr{O}_c)$ is considered as the dual of $(\mathscr{O}'_c)$. Any neighbourhood of zero in $(\mathscr{O}'_c)$ contains a neighbourhood of zero given by $U=\{T: |<\varphi*T, B>| \leqq 1\}$ where φ is a function of $(\mathscr{S})$ and B is a compact disc of $(\mathscr{S}')$. It suffices to show that $U^{\circ}=B*\varphi$. Since the mapping $(\varphi, S) \rightarrow S*\varphi$ of $(\mathscr{S}) \times (\mathscr{S}')$ into $(\mathscr{O}'_c)$ is hypocontinuous, it follows that $B*\varphi$ is a compact disc. Therefore $U^{\circ}=B*\varphi$, completing the proof.

Since the strong dual of a semi-reflexive space is barrelled [1; Chap. IV, §3, Prop. 4, p. 88], the spaces $(\mathscr{D}')$, $(\mathscr{S}')$, $(\mathscr{D}'_{L^p})$ (for $1<p<\infty$), $(\mathscr{E}')$ are barrelled. The spaces $(\mathcal{O}_M)$, $(\mathcal{O}'_c)$ are barrelled as shown in Grothendieck [4; p. 131], $(\mathcal{D}'_{L^1})$ is also barrelled as shown in Schwartz [8; p. 126]. To see that $(\mathcal{B}')$ is barrelled, it suffices to show that $(\mathcal{D}_{L^1})$ is quasi-normable. This is because the dual of any quasi-normable space of type (***F***) is barrelled [5; p. 108].

LEMMA 3. *The space $(\mathcal{D}_{L^1})$ is quasi-normable.*

PROOF. From a lemma of Grothendieck [5; lemma 6, p. 107], it is sufficient to show that for any given neighbourhood U of zero in $(\mathcal{D}_{L^1})$ there exists a neighbourhood V of zero in $(\mathcal{D}_{L^1})$ such that for any positive number α we can find a bounded set M: $V\subset\alpha U+M$. Without loss of generality we can take U as a neighbourhood given by $\{\varphi\in(\mathcal{D}_{L^1})\colon \int|D^{\vec p}\varphi|\,dx\leqq 1 \text{ for } |\vec p|\leqq m\}$. We show that we can take V a neighbourhood given by $\{\varphi\in(\mathcal{D}_{L^1})\colon \int|D^{\vec p}\varphi|\,dx\leqq 1$ for $|\vec p|\leqq m+1\}$. For any given positive number α we choose $\xi\in(\mathcal{D})$ with support in $\left\{|x|<\frac{\alpha}{n}\right\}$ such that $\xi(x)\geqq 0$, $\int\xi(x)dx=1$. $V*\xi$ is a bounded set of $(\mathcal{D}_{L^1})$ because

$$\int|(D^{\vec q}\varphi*\xi)(x)|\,dx\leqq\int|\varphi(x)|\,dx\int|D^{\vec q}\xi(x)|\,dx\leqq\int|D^{\vec q}\,\xi(x)|\,dx$$

for any $\varphi\in V$ and $\vec q$.

Next we prove that $\psi=\varphi-\varphi*\xi\in\alpha U$ for any $\varphi\in V$. Let p_U denote the semi-norm corresponding to U, then we have

$$\begin{aligned} p_U(\psi)=p_U(\varphi-\varphi*\xi)&\leqq\int\xi(a)p_U(\varphi-\varphi_a)da\\ &\leqq\int\xi(a)\Big(\sum_{i=1}^{n}\Big|\int_0^a p_U\Big(\Big(\frac{\partial\varphi}{\partial x_i}\Big)_t\Big)dt_i\Big|\Big)\,da\\ &=\int\xi(a)\,|a|\,da<\alpha \end{aligned}$$

Hence $V-V*\xi\subset\alpha U$, as desired. If we put $M=V*\xi$, we see that $V\subset\alpha U+M$, completing the proof.

It is well known that the function spaces considered above are complete and are permitted except $(\mathcal{B})$. However, $(\mathcal{B})$ is the dual of the permitted space $(\mathcal{D}'_{L^1})$. Therefore the remark 3° is applied to the pairs of spaces considered above. Hence the condition (3) is valid for these pairs.

We shall show that the spaces $(\mathcal{E})$, $(\mathcal{O}_c)$, $(\mathcal{B})$, $(\mathcal{D}_{L^p})$ (for $1\leqq p<\infty$), $(\mathcal{S})$ and $(\mathcal{D})$ have a fundamental system of $(\mathcal{E})$-closed absolutely convex neighbourhoods. From the definitions of neighbourhoods of zero, it is evident that $(\mathcal{E})$, $(\mathcal{B})$, $(\mathcal{D}_{L^p})$ (for $1\leqq p<\infty$), $(\mathcal{S})$, $(\mathcal{D})$ have a fundamental system of $(\mathcal{E})$-closed absolutely convex neighbourhoods. As regards $(\mathcal{O}_c)$, the polars of bounded sets of $(\mathcal{O}'_c)$ form a fundamental system of neighbourhoods of zero in $(\mathcal{O}_c)$. On the other hand, any bounded set B of $(\mathcal{O}'_c)$ is contained in a bounded set consisting of elements of $(\mathcal{D})$, because $B\subset\bigcup_k(\alpha_k B)*\rho_k$ where $\{\alpha_k\}$ is a sequence of multiplicators and $\{\rho_k\}$ is a sequence of regularizations, and because $\bigcup_k(\alpha_k B)*\rho_k$ is a bounded set of $(\mathcal{O}'_c)$ consisting of elements of $(\mathcal{D})$.

From these considerations we see that the pairs form distinguished pairs, as desired.

We shall extend the notion of 'distinguished element' to a space of vector valued distributions. Let E, F and G be three locally convex quasi-complete Hausdorff topological vector spaces. The ε-product $E(G)$ is, by definition, $\mathscr{L}_\varepsilon(G'_c, E)$ the space of linear continuous mappings of G'_c into E with the topology of uniform convergence on equicontinuous sets of G'. $E(G)$ is quasi-complete as shown by L. Schwartz [8; p. 29]. In the following theorem we shall assume that E and F are spaces of distributions on R^n. An element $\vec{T} \in E(G)$ is called *distinguished* if for any equicontinuous set $A \subset G'$ the image $<\vec{T}, A>$ satisfies the condition of the remark 4°. A pair $(E(G), F(G))$ is called *distinguished* if $E(G)^d = F(G)$ and $F(G)^d = F(G)$.

THEOREM 2. *Let E and F be quasi-complete spaces of distributions on R^n satisfying the conditions* (1), (2), (3) *of the preceding theorem. Let G be any locally convex quasi-complete space. Then any distinguished element $\vec{T}$ of $E(G)$ lies in $F(G)$ and $\vec{T}*\rho_k \to \vec{T}$ in $F(G)$ as $k\to\infty$, where $\{\rho_k\}$ is any sequence of regularizations.*

PROOF. $<\vec{T}, g'>$ is distinguished in E, so that $<\vec{T}, g'> \in F$ by Theorem 1. If g' runs through any equicontinuous set A of G', $<\vec{T}, g'>*\rho_k \to <\vec{T}, g'>$ uniformly in F as noted in the remark 4°. This means that $\vec{T}*\rho_k \to \vec{T}$ in $F(G)$ since $F(G)$ is quasi-complete as remarked above. This completes the proof.

Examples. The pairs $(\mathscr{D}'(G), \mathscr{E}(G))$, $(\mathscr{O}_M(G), \mathscr{O}_c(G))$, $(\mathscr{S}'(G), \mathscr{O}_c(G))$, $(\mathscr{B}'(G), \mathscr{B}(G))$, $(\mathscr{D}'_{L^p}(G), \mathscr{D}_{L^p}(G))$ (for $1 \leq p < \infty$), $(\mathscr{O}'_c(G), \mathscr{S}(G))$, $(\mathscr{E}'(G), \mathscr{D}(G))$ are all distinguished. Owing to Theorem 2 and the preceding examples it is clear that any distinguished element of $\mathscr{D}'(G)$ (resp. $\mathscr{O}_M(G)$, $\mathscr{S}'(G)$, $\mathscr{B}'(G)$, $\mathscr{D}'_{L^p}(G)$ (for $1 \leq p < \infty$), $\mathscr{S}(G)$, $\mathscr{D}(G)$) lies in $\mathscr{E}(G)$ (resp. $\mathscr{O}_c(G)$, $\mathscr{O}_c(G)$, $\mathscr{B}(G)$, $\mathscr{D}_{L^p}(G)$ (for $1 \leq p < \infty$), $\mathscr{S}(G)$, $\mathscr{D}(G)$), therefore it suffices to show that any element of $\mathscr{E}(G)$ (resp. $\mathscr{O}_c(G)$, $\mathscr{O}_c(G)$, $\mathscr{B}(G)$, $\mathscr{D}_{L^p}(G)$ (for $1 \leq p < \infty$), $\mathscr{S}(G)$, $\mathscr{D}(G)$) is distinguished. If F is a space of type $(\boldsymbol{F})$ stable under differentiations, any element of $F(G)$ is distinguished, since Mackey's first countability property holds for F. Therefore for the end of the proof it is sufficient to show that any element of $\mathscr{O}_c(G)$ is distinguished. By Lemma 2, any bounded set A of $(\mathscr{O}_c)$ is written in the form $B*\varphi$ where B is a bounded set of $(\mathscr{S}')$ and φ is an element of $(\mathscr{S})$. It is easy to see that A satisfies the condition of the remark 4°. This implies that any element of $\mathscr{O}_c(G)$ is distinguished.

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