J. Sci. Hiroshima Univ. Ser. A
Vol. 24, No. 3, December, 1960

# *On Skew Products of Groups*

Noboru Nishigôri

(Received September 20, 1960)

## § 1. Introduction

In his paper [4][1], L. Rédei has made a systematic study of the skew products in the theory of groups. There, chiefly he has discussed two types of the skew products which were defined by the following:

$$H \imath \Gamma:\ (a, \alpha)(b, \beta)=(ab,\ a^b\alpha^b\beta),$$

and

$$H 2 \Gamma:\ (a, \alpha)(b, \beta)=(ab^\alpha,\ \alpha^b\beta),$$

respectively.

Furthermore, he has treated the skew product which was defined by the following:

$$H \odot \Gamma:\ (a, \alpha)(b, \beta)=(a^\beta b^\alpha,\ \alpha^b\beta^a),$$

but without sufficient results. After that, L. Rédei and A. Stöhr have studied a special case of this product in [6], also F. Rühs has studied this skew product $H \odot \Gamma$ in [7].

In this paper, we shall generalize this skew product $H \odot \Gamma$ and study the properties of the group which is defined by this skew product. The results we get in this paper are comprehensive of the results obtained by L. Rédei, A. Stöhr and F. Rühs.

Let $H$ and $\Gamma$ be two groups which have the elements $a, b, c, \cdots; \alpha, \beta, \gamma, \cdots$ respectively, and $e$ and $\varepsilon$ be the unit elements of $H$ and $\Gamma$ respectively. The *skew product* of the groups $H$ and $\Gamma$ is the set of all ordered pairs $(a, \alpha)$ with $a \in H$, $\alpha \in \Gamma$, and between those pairs $(a, \alpha)$ the following product is defined:

$$(1.1) \qquad H \divideontimes \Gamma:\ (a, \alpha)(b, \beta)=(aR(\beta) \cdot L(\alpha)b,\ \alpha\mathcal{R}(b) \cdot \mathcal{L}(a)\beta),$$

where $R$ and $L$ are two mappings of $\Gamma$ into a group of the permutations on the elements of $H$, and similarly $\mathcal{R}$ and $\mathcal{L}$ are two mappings of $H$ into a permutation group on the elements of $\Gamma$. And they satisfy the conditions $aR(\varepsilon)=L(\varepsilon)a=a$ and $\alpha\mathcal{R}(e)=\mathcal{L}(e)\alpha=\alpha$.

More generally, in (1.1) we may regard $aR(\beta)$, $L(\alpha)b$; $\alpha\mathcal{R}(b)$, $\mathcal{L}(a)\beta$ as the functions of two variables. So, in this definition, if we add the the conditions

---

(1) Numbers in brackets refer to the references at the end of the paper.

$aR(\alpha)=L(\alpha)a$ and $\alpha\mathcal{R}(a)=\mathcal{L}(a)\alpha$, $a \in H$, $\alpha \in \Gamma$, we may obtain the definition of the skew prouct $H\odot\Gamma$ defined by L. Rédei in [4] (cf. p. 479 Remark 1).

In § 2, we inquire the necessary and sufficient condition in order that the skew product may be a group. In § 3 by obtaining the group $G_1$ which is isomorphic to the group $H\divideontimes\Gamma$, we show that essentially the group $H\divideontimes\Gamma$ is a special case of Rédei's skew product $H{\scriptstyle 2}\Gamma$. In § 4, we first seek for the necessary and sufficient condition for the skew product $H\divideontimes\Gamma$ to be a semi-direct product of the groups $H$ and $\Gamma$, and then for the skew product to be the direct product of $H$ and $\Gamma$. In §§5 and 6, we investigate a special case of the group $H\divideontimes\Gamma$, that is, the case in which all the permutations $R(\beta)$, $L(\alpha)$; $\mathcal{R}(b)$, $\mathcal{L}(a)$ that appear in the definition (1.1) of the skew product $H\divideontimes\Gamma$ are the automorphisms of $H$ and $\Gamma$ respectively. In this case we use the notation $H\otimes\Gamma$ instead of $H\divideontimes\Gamma$. First, we inquire the necessary and sufficient condition for the skew product $H\otimes\Gamma$ to be a group, next we show that the group $H\otimes\Gamma$ is the extension of a fixed subgroup $(H_0, \Gamma_0)$. Furthermore, the product of the group $G_1$ which is defined in § 3 becomes as follows:

$$(a, \alpha)\,(b, \beta)=(a\cdot F(\alpha)b\cdot l(b, \alpha^{-1}, b), \sigma(\alpha, b^{-1}, \alpha)\cdot\alpha\mathcal{F}(b)\cdot\beta), \tag{1.2}$$

where $F(\alpha)$ and $\mathcal{F}(b)$ are the automorphisms of $H$ and $\Gamma$ respectively, and $l(a, \alpha, b)$ and $\sigma(\alpha, a, \beta)$ are fixed functions having their values in $H_0$ and $\Gamma_0$ respectively. In § 7, we study the properties of the two functions $l(a, \alpha, b)$ and $\sigma(\alpha, a, \beta)$. And conversely, by the use of such functions, we define the new skew product $H\divideontimes\Gamma$ of the groups $H$ and $\Gamma$ the product of which is defined by (1.2). Next, we inquire the necessary and sufficient condition for this skew product to be a group, and investigate the structure of this group. And by the examples of the group $H\divideontimes\Gamma$ and by the example of the group which belongs to the type $H{\scriptstyle 2}\Gamma$ but not to the type $H\divideontimes\Gamma$, we show the existence of the group $H\divideontimes\Gamma$ and that the set of the group $H\divideontimes\Gamma$ is a proper subset of the set of the group $H{\scriptstyle 2}\Gamma$. Furthermore in § 8, we get some properties of the group $H\divideontimes\Gamma$. Finally, in § 9, we clarify in the form of final remarks the relation between the results of this paper and those of L. Rédei and A. Stöhr.

## § 2. Necessary and sufficient condition for skew product to be a group.

In this section, we inquire the necessary and sufficient condition in order that the skew product $H\divideontimes\Gamma$ which is defined by (1.1) may be a group. As we mentioned in § 1, the skew product $H\divideontimes\Gamma$ is defined by the following (cf. p. 477):

$$(a, \alpha)(b, \beta)=(aR(\beta)\cdot L(\alpha)b, \alpha\mathcal{R}(b)\cdot\mathcal{L}(a)\beta), \tag{2.1}$$

where the permutations $R(\beta)$, $\mathcal{R}(b)$ are right hand operators, and $L(\alpha)$ and $\mathcal{L}(a)$ are left hand operators. And the "dots" in the notations $aR(\beta)\cdot L(\alpha)b$ and $\alpha\mathcal{R}(b)\cdot\mathcal{L}(a)\beta$ represent the product of the elements $aR(\beta)$ and $L(\alpha)b$ in

$H$, and that of $\alpha\mathcal{R}(b)$ and $\mathcal{L}(a)\beta$ in $\Gamma$ respectively.

REMARK 1. If we use the functions $M(a, \beta)$, $N(\alpha, b)$; $\mathcal{M}(\alpha, b)$, $\mathcal{N}(a, \beta)$, which have two variables and have their values in $H$ and $\Gamma$ respectively, instead of $aR(\beta)$, $L(\alpha)b$; $\alpha\mathcal{R}(b)$, $\mathcal{L}(a)\beta$, we can still make the following arguments.

Now, we investigate the conditions in order that the skew product may be a group with the unit element $(e, \varepsilon)$.

First, from the fact that the element $(e, \varepsilon)$ is the unit element of the group $H \divideontimes \Gamma$, the following conditions must hold,

$$eR(\alpha)=L(\alpha)e=e, \tag{2.2$_1$}$$

$$\varepsilon\mathcal{R}(a)=\mathcal{L}(a)\varepsilon=\varepsilon. \tag{2.2$_2$}$$

Conversely, if the conditions $(2.2)_1$ and $(2.2)_2$ are satisfied, the element $(e, \varepsilon)$ is the unit element of the group $H \divideontimes \Gamma$.

Next, we inquire the conditions in order that the product which is defined by (2.1) may be associative. If the product is associative, we have

$$(aR(\beta)\cdot L(\alpha)b,\ \alpha\mathcal{R}(b)\cdot\mathcal{L}(a)\beta)(c, \gamma)=(a, \alpha)(bR(\gamma)\cdot L(\beta)c,\ \beta\mathcal{R}(c)\cdot\mathcal{L}(b)\gamma),$$

that is,

$$\begin{aligned}&((aR(\beta)\cdot L(\alpha)b)R(\gamma)\cdot L(\alpha\mathcal{R}(b)\cdot\mathcal{L}(a)\beta)c,\ (\alpha\mathcal{R}(b)\cdot\mathcal{L}(a)\beta)\mathcal{R}(c)\cdot\mathcal{L}(aR(\beta)\cdot L(\alpha)b)\gamma)\\ &=(aR(\beta\mathcal{R}(c)\cdot\mathcal{L}(b)\gamma)\cdot L(\alpha)(bR(\gamma)\cdot L(\beta)c),\ \alpha\mathcal{R}(bR(\gamma)\cdot L(\beta)c)\cdot\mathcal{L}(a)(\beta\mathcal{R}(c)\cdot\mathcal{L}(b)\gamma)).\end{aligned}$$

Therefore, it must hold that

$$\begin{aligned}(aR(\beta)\cdot L(\alpha)b)R(\gamma)\cdot L(\alpha R(b)\cdot\mathcal{L}(a)\beta)c\\ =aR(\beta\mathcal{R}(c)\cdot\mathcal{L}(b)\gamma)\cdot L(\alpha)(bR(\gamma)\cdot L(\beta)c),\end{aligned} \tag{2.3$_1$}$$

$$\begin{aligned}(\alpha\mathcal{R}(b)\cdot\mathcal{L}(a)\beta)\mathcal{R}(c)\cdot\mathcal{L}(aR(\beta)\cdot L(\alpha)b)\gamma\\ =\alpha\mathcal{R}(bR(\gamma)\cdot L(\beta)c)\cdot\mathcal{L}(a)(\beta\mathcal{R}(c)\cdot\mathcal{L}(b)\gamma).\end{aligned} \tag{2.3$_2$}$$

In the condition $(2.3)_1$, if we exchange the letters $a$, $b$, $c$ for $\alpha$, $\beta$, $\gamma$ and $R$, $L$ for $\mathcal{R}$, $\mathcal{L}$ we obtain the condition $(2.3)_2$. In like manner, if we can obtain a condition from the other by the exchange $a\leftrightarrow\alpha$, $b\leftrightarrow\beta$, $c\leftrightarrow\gamma$, $R\leftrightarrow\mathcal{R}$, $L\leftrightarrow\mathcal{L}$, we call those two conditions *dual*.

From the formal symmetry of the definition (2.1), if a condition holds in $H$, then its dual condition holds in $\Gamma$ necessarily. So, hereafter, we write only the calculation of the conditions which hold in $H$.

First, if we consider the cases where (i) $a=c=e$, (ii) $a=b=e$, (iii) $b=c=e$, (iv) $a=e$ $\beta=\gamma=\varepsilon$, (v) $c=e$, $\alpha=\beta=\varepsilon$ respectively in the condition $(2.3)_1$, then corresponding to each case, we have the following conditions.

$$(L(\alpha)b)R(\gamma)=L(\alpha)(bR(\gamma)), \tag{2.4$_1$}$$

$$(2.5)_1 \qquad L(\alpha\cdot\beta)c=L(\alpha)(L(\beta)c),$$

$$(2.6)_1 \qquad (aR(\beta))R(\gamma)=aR(\beta\cdot\gamma),$$

$$(2.7)_1 \qquad L(\alpha)b\cdot L(\alpha\mathcal{R}(b))c=L(\alpha)(b\cdot c),$$

$$(2.8)_1 \qquad (a\cdot b)R(\gamma)=aR(\mathcal{L}(b)\gamma)\cdot bR(\gamma).$$

By the duality, we have

$$(2.4)_2 \qquad (\mathcal{L}(a)\beta)\mathcal{R}(c)=\mathcal{L}(a)(\beta\mathcal{R}(c)),$$

$$(2.5)_2 \qquad \mathcal{L}(a\cdot b)\gamma=\mathcal{L}(a)(\mathcal{L}(b)\gamma),$$

$$(2.6)_2 \qquad (\alpha\mathcal{R}(b))\mathcal{R}(c)=\alpha\mathcal{R}(b\cdot c),$$

$$(2.7)_2 \qquad \mathcal{L}(a)\beta\cdot\mathcal{L}(aR(\beta))\gamma=\mathcal{L}(a)(\beta\cdot\gamma),$$

$$(2.8)_2 \qquad (\alpha\cdot\beta)\mathcal{R}(c)=\alpha\mathcal{R}(L(\beta)c)\cdot\beta\mathcal{R}(c).$$

Next, for $a=e$, $\beta=\varepsilon$, the condition $(2.3)_1$ is written as follows:

$$(L(\alpha)b)R(\gamma)\cdot L(\alpha\mathcal{R}(b))c=L(\alpha)(bR(\gamma)\cdot c).$$

Applying $(2.7)_1$ to the right side of the above condition, it must hold that

$$(L(\alpha)b)R(\gamma)\cdot L(\alpha\mathcal{R}(b))c=L(\alpha)(bR(\gamma))\cdot L(\alpha\mathcal{R}(bR(\gamma)))c.$$

So, by $(2.4)_1$, we have the condition

$$(2.9)_1 \qquad L(\alpha\mathcal{R}(b))c=L(\alpha\mathcal{R}(bR(\gamma)))c.$$

By the duality, follows

$$(2.9)_2 \qquad \mathcal{L}(aR(\beta))\gamma=\mathcal{L}(aR(\beta\mathcal{R}(c)))\gamma.$$

Similarly, if we consider the case $c=e$, $\beta=\varepsilon$ in $(2.3)_1$ and apply $(2.4)_1$ and $(2.8)_1$, we have the following condition:

$$(2.10)_1 \qquad aR(\mathcal{L}(L(\alpha)b)\gamma)=aR(\mathcal{L}(b)\gamma).$$

By the duality, we obtain

$$(2.10)_2 \qquad \alpha\mathcal{R}(L(\mathcal{L}(a)\beta)c)=\alpha\mathcal{R}(L(\beta)c).$$

For the case $c=b^{-1}$, the condition $(2.7)_1$ becomes

$$(2.11)_1 \qquad (L(\alpha)b)^{-1}=L(\alpha\mathcal{R}(b))b^{-1},$$

and from $(2.7)_2$, we obtain

$$(2.11)_2 \qquad (\mathcal{L}(a)\beta)^{-1}=\mathcal{L}(aR(\beta))\beta^{-1}.$$

In the same way, from $(2.8)_1$ and $(2.8)_2$, follow

$$(2.12)_1 \qquad (bR(\gamma))^{-1} = b^{-1}R(\mathscr{L}(b)\gamma),$$

and

$$(2.12)_2 \qquad (\beta\mathscr{R}(c))^{-1} = \beta^{-1}\mathscr{R}(L(\beta)c).$$

Again, from the condition $(2.3)_1$ we obtain the following condition, for $b=e$ and $\alpha=\gamma=\varepsilon$,

$$aR(\beta)\cdot L(\mathscr{L}(a)\beta)c = aR(\beta\mathscr{R}(c))\cdot L(\beta)c.$$

By $(2.11)_1$ and $(2.12)_1$, we obtain the condition

$$(2.13)_1 \qquad L(\mathscr{L}(a)\beta)c\cdot L(\beta\mathscr{R}(c))c^{-1} = a^{-1}R(\mathscr{L}(a)\beta)\cdot aR(\beta\mathscr{R}(c)).$$

By the duality, we obtain

$$(2.13)_2 \qquad \mathscr{L}(L(\alpha)b)\gamma\cdot\mathscr{L}(bR(\gamma))\gamma^{-1} = \alpha^{-1}\mathscr{R}(L(\alpha)b)\cdot\alpha\mathscr{R}(bR(\gamma)).$$

Further, for the case $\alpha=\gamma=\varepsilon$ in $(2.3)_1$, we obtain

$$aR(\beta)\cdot b\cdot L(\mathscr{L}(a)\beta)c = aR(\beta\mathscr{R}(c))\cdot b\cdot L(\beta)c.$$

So, it follows that

$$(2.14) \qquad b\cdot(L(\mathscr{L}(a)\beta)c\cdot L(\beta\mathscr{R}(c))c^{-1}) = (a^{-1}R(\mathscr{L}(a)\beta)\cdot aR(\beta\mathscr{R}(c)))\cdot b.$$

Therefore, by $(2.13)_1$ and (2.14), we obtain

$$(2.15) \qquad L(\mathscr{L}(a)\beta)c\cdot L(\beta\mathscr{R}(c))c^{-1} \in Z(H),$$

where $Z(H)$ is the center of the group $H$.

Furthermore, if we consider the cases where (i) $b=e$, $\alpha=\varepsilon$ (ii) $b=e$, $\gamma=\varepsilon$, corresponding to each case, the condition $(2.3)_1$ becomes as follows:

$$(2.16) \qquad aR(\beta\cdot\gamma)\cdot L(\mathscr{L}(a)\beta)c = aR(\beta\mathscr{R}(c)\cdot\gamma)\cdot L(\beta)c,$$

$$(2.17) \qquad aR(\beta)\cdot L(\alpha\cdot\mathscr{L}(a)\beta)c = aR(\beta\mathscr{R}(c))\cdot L(\alpha\cdot\beta)c.$$

From (2.16), it holds that

$$L(\mathscr{L}(a)\beta)c\cdot L(\beta\mathscr{R}(c))c^{-1} = a^{-1}R(\mathscr{L}(a)(\beta\cdot\gamma))\cdot aR(\beta\mathscr{R}(c)\cdot\gamma).$$

So, it follows that

$$(2.18) \qquad L(\mathscr{L}(a)\beta)c\cdot L(\beta\mathscr{R}(c))c^{-1} = (a^{-1}R(\mathscr{L}(a)\beta)\cdot aR(\beta\mathscr{R}(c)))R(\gamma).$$

For,

$$\begin{aligned}
&(a^{-1}R(\mathscr{L}(a)\beta)\cdot aR(\beta\mathscr{R}(c)))R(\gamma)\\
&\quad = a^{-1}R(\mathscr{L}(a)\beta)R(\mathscr{L}(aR(\beta\mathscr{R}(c)))\gamma)\cdot aR(\beta\mathscr{R}(c)\cdot\gamma)\\
&\quad = a^{-1}R(\mathscr{L}(a)\beta\cdot\mathscr{L}(aR(\beta))\gamma)\cdot aR(\beta\mathscr{R}(c)\cdot\gamma)
\end{aligned}$$

$$=a^{-1}R(\mathscr{L}(a)(\beta\cdot\gamma))\cdot aR(\beta\mathscr{R}(c)\cdot\gamma).$$

Therefore, from $(2.13)_1$ and (2.18), we have

$$L(\mathscr{L}(a)\beta)c\cdot L(\beta\mathscr{R}(c))c^{-1}\in K_R, \tag{2.19}$$

where $K_R=\{a;\ aR(\alpha)=a,\ \forall\alpha\ \in\Gamma\}$.
Just as we obtain (2.19) from (2.16), we have the following condition from (2.17):

$$L(\mathscr{L}(a)\beta)c\cdot L(\beta\mathscr{R}(c))c^{-1}\in K_L, \tag{2.20}$$

where $K_L=\{a;\ L(\alpha)a\ =a,\ \forall\alpha\in\Gamma\}$.
Thus, from (2.15), (2.19) and (2.20), we have

$$L(\mathscr{L}(a)\beta)c\cdot L(\beta\mathscr{R}(c))c^{-1}\in Z(H)\cap K_R\cap K_L. \tag{2.21$_1$}$$

Further, we prove the following condition:

$$L(\mathscr{L}(a)\beta)c\cdot L(\beta\mathscr{R}(c))c^{-1}\in kern\ \mathscr{R}\cap kern\ \mathscr{L}, \tag{2.22}$$

where $kern\mathscr{R}=\{\alpha;\ \alpha\mathscr{R}(a)=\alpha,\ \forall\alpha\in\Gamma\}$, $kern\mathscr{L}=\{a;\ \mathscr{L}(a)\alpha=\alpha,\ \forall\alpha\in\Gamma\}$.

For,

$$\mathscr{L}(L(\mathscr{L}(a)\beta)c\cdot L(\beta\mathscr{R}(c))c^{-1})\alpha=\mathscr{L}(a^{-1}R(\mathscr{L}(a)\beta)\cdot aR(\beta\mathscr{R}(c)))\alpha$$
$$=\mathscr{L}(a^{-1}R(\mathscr{L}(a)\beta)\cdot aR(\beta))\alpha=\alpha.$$

Similarly, we have

$$\alpha\mathscr{R}(L(\mathscr{L}(a)\beta)c\cdot L(\beta\mathscr{R}(c))c^{-1})=\alpha.$$

Therefore, it holds that

$$L(\mathscr{L}(a)\beta)c\cdot L(\beta\mathscr{R}(c))c^{-1}\in Z(H)\cap K_R\cap K_L\cap kern\mathscr{R}\cap kern\mathscr{L}. \tag{2.23$_1$}$$

By the duality, we have

$$\mathscr{L}(L(\alpha)b)\gamma\cdot\mathscr{L}(bR(\gamma))\gamma^{-1}\in Z(\Gamma)\cap K_{\mathscr{R}}\cap K_{\mathscr{L}}\cap kernR\cap kernL, \tag{2.23$_2$}$$

where $Z(\Gamma)$ is the center of $\Gamma$,

$$K_{\mathscr{R}}=\{\alpha\ ;\ \alpha\mathscr{R}(a)=\alpha,\ \forall a\in H\},\quad K_{\mathscr{L}}=\{\alpha;\ \mathscr{L}(a)\alpha=\alpha,\ \forall a\in H\},$$

$$kernR=\{\alpha;\ aR(\alpha)=a,\ \forall a\in H\},\ kern\,L=\{\alpha;\ L(\alpha)a=a,\ \forall a\in H\}.$$

Thus, we have the system of the necessary conditions, that is, $(2.2)_1$, $(2.4)_1$, $(2.5)_1$, $(2.6)_1$, $(2.7)_1$, $(2.8)_1$, $(2.13)_1$, $(2.23)_1$ and their duals.

Remark 2. The conditions $(2.9)_1$ and $(2.10)_1$ follow from the conditions $(2.5)_1$, $(2.6)_1$ and the duals of $(2.7)_1$, $(2.8)_1$, $(2.13)_1$, and $(2.23)_1$.

For,

$$L(\alpha\mathscr{R}(bR(\gamma)))c=L(\alpha\mathscr{R}(b)\cdot(\alpha\mathscr{R}(b))^{-1}\cdot\alpha\mathscr{R}(bR(\gamma)))c=L(\alpha\mathscr{R}(b)).$$

Similarly, the condition $(2.10)_1$ follows from $(2.6)_1$, $(2.7)_2$ and $(2.23)_2$.

REMARK 3. The following two systems of the conditions are equivalent.

①: $(2.2)_1$, $(2.4)_1$, $(2.5)_1$, $(2.6)_1$, $(2,7)_1$, $(2.8)_1$, $(2.13)_1$, $(2.23)_1$ and their duals.

②: $(2.2)_1$, $(2.4)_1$, $(2.5)_1$, $(2.6)_1$, $(2.7)_1$, $(2.8)_1$, $(2.9)_1$, $(2.10)_1$, $(2.13)_1$, $(2.21)_1$ and their duals.

For, the system ② follows from the system ① according to Remark 2, $(2.23)_1$ and $(2.23)_2$. Conversely,the system ① follows from the system ② according to (2.22).

Conversely, we prove that the system ① of the conditions is sufficient for the product to be associative.

From $(2.13)_1$, for $a, c \in H$ and $\beta \in \Gamma$, we have

$$L(\mathscr{L}(a)\beta)c \cdot L(\beta\mathscr{R}(c))c^{-1} = a^{-1}R(\mathscr{L}(a)\beta) \cdot aR(\beta\mathscr{R}(c)).$$

So, from $(2.23)_1$, for $b \in H$ it holds that

$$b \cdot (L(\mathscr{L}(a)\beta)c \cdot L(\beta\mathscr{R}(c))c^{-1}) = (a^{-1}R(\mathscr{L}(a)\beta) \cdot aR(\beta\mathscr{R}(c))) \cdot b.$$

Applying $L(\alpha)$, $\alpha \in \Gamma$ to both sides of the above condition, and by $(2.7)_1$, $(2.13)_1$ and $(2.23)_2$ it follows that

$$L(\alpha)b \cdot (L(\mathscr{L}(a)\beta)c \cdot L(\beta\mathscr{R}(c))c^{-1}) = (a^{-1}R(\mathscr{L}(a)\beta) \cdot aR(\beta\mathscr{R}(c))) \cdot L(\alpha)b.$$

Similarly, applying $R(\gamma)$, $\gamma \in \Gamma$ to both sides of the above condition, and by $(2.8)_1$, $(2.13)_1$ and $(2.23)_1$, we obtain

$$(L(\alpha)b)R(\gamma) \cdot (L(\mathscr{L}(a)\beta)c \cdot L(\beta\mathscr{R}(c))c^{-1}) = (a^{-1}R(\mathscr{L}(a)\beta) \cdot aR(\beta\mathscr{R}(c))) \cdot (L(\alpha)b)R(\gamma).$$

Further, by $(2.13)_1$ and $(2.23)_1$, it holds that

$$\begin{aligned}&(L(\alpha)b)R(\gamma) \cdot L(\alpha\mathscr{R}(b))(L(\mathscr{L}(a)\beta)c \cdot L(\beta\mathscr{R}(c))c^{-1})\\ &= (a^{-1}R(\mathscr{L}(a)\beta) \cdot aR(\beta\mathscr{R}(c)))R(\mathscr{L}(b)\gamma) \cdot (L(\alpha)b)R(\gamma).\end{aligned}$$

Therefore, by $(2.5)_1$, $(2.6)_1$, $(2.7)_1$ and $(2.8)_1$, it follows that

$$\begin{aligned}&(L(\alpha)b)R(\gamma) \cdot L(\alpha\mathscr{R}(b) \cdot \mathscr{L}(a)\beta)c \cdot L(\alpha\mathscr{R}(b)\mathscr{R}(L(\mathscr{L}(a)\beta)c))L(\beta\mathscr{R}(c))c^{-1}\\ &= a^{-1}R(\mathscr{L}(a)\beta)R(\mathscr{L}(aR(\beta\mathscr{R}(c)))\mathscr{L}(b)\gamma) \cdot aR(\beta\mathscr{R}(c) \cdot \mathscr{L}(b)\gamma) \cdot (L(\alpha)b)R(\gamma).\end{aligned}$$

Further, by $(2.4)_1$, $(2.9)_2$, $(2.10)_2$ and $(2.11)_1$, the last of which follows from $(2.7)_1$, and $(2.12)_1$ which follows from $(2.8)_1$, the above condition becomes as follows:

$$\begin{aligned}&(L(\alpha)b)R(\gamma) \cdot L(\alpha\mathscr{R}(b) \cdot \mathscr{L}(a)\beta)c \cdot (L(\alpha\mathscr{R}(b))L(\beta)c))^{-1}\\ &= (aR(\beta)R(\mathscr{L}(b)\gamma))^{-1} \cdot aR(\beta\mathscr{R}(c) \cdot \mathscr{L}(b)\gamma) \cdot L(\alpha)(bR(\gamma)).\end{aligned}$$

So, it holds that

$$\begin{aligned}(2.24)\qquad &aR(\beta)R(\mathscr{L}(b)\gamma) \cdot (L(\alpha)b)R(\gamma) \cdot L(\alpha\mathscr{R}(b) \cdot \mathscr{L}(a)\beta)c\\ &= aR(\beta\mathscr{R}(c) \cdot \mathscr{L}(b)\gamma) \cdot L(\alpha)(bR(\gamma)) \cdot L(\alpha\mathscr{R}(b))L(\beta)c.\end{aligned}$$

By $(2.10)_1$ and $(2.8)_1$, the first two factors of the left side of (2.24) becomes as follows:

$$aR(\beta)R(\mathscr{L}(b)\gamma)\cdot(L(\alpha)b)R(\gamma)=(aR(\beta)\cdot L(\alpha)b)R(\gamma). \tag{2.25}$$

Similarly, the last two factors of the right side of (2.24) becomes as follows:

$$L(\alpha)(bR(\gamma))\cdot L(\alpha\mathscr{R}(b))L(\beta)c=L(\alpha)(bR(\gamma)\cdot L(\beta)c). \tag{2.26}$$

By (2.25) and (2.26), the condition (2.4) becomes as follows:

$$(aR(\beta)\cdot L(\alpha)b)R(\gamma)\cdot L(\alpha\mathscr{R}(b)\cdot\mathscr{L}(a)\beta)c=aR(\beta\mathscr{R}(c)\cdot\mathscr{L}(b)\gamma)\cdot L(\alpha)(bR(\gamma)\cdot L(\beta)c).$$

Consequently, we have proved that the condition $(2.3)_1$ follows from the system ①. In the similar way, we can prove that the condition $(2.3)_2$ follows from the system ①. Thus, we have proved that the product is associative.

Finally, we prove the existence of the inverse element when the system ① is satisfied. For an arbitrary element $(a, \alpha)$ of the skew product $H\divideontimes\Gamma$, we set $(a, \alpha)=(x, \varepsilon)(e, y)$. Then we have $(a, \alpha)^{-1}=(e, y^{-1})(x^{-1}, \varepsilon)$. And from $(a, \alpha)=(xR(y), \mathscr{L}(x)y)$, we have $a=xR(y)$, $\alpha=\mathscr{L}(x)y$. So, by $(2.11)_2$ and $(2.12)_1$, it follows that $a^{-1}=x^{-1}R(\alpha)$, $\alpha^{-1}=\mathscr{L}(a)y^{-1}$ By $(2.6)_1$ and $(2.5)_2$ we have $x^{-1}=a^{-1}R(\alpha^{-1})$, $y^{-1}=\mathscr{L}(a^{-1})\alpha^{-1}$. Therefore, we have the inverse element of $(a, \alpha)$, that is,

$$(a, \alpha)^{-1}=(e, \mathscr{L}(a^{-1})\alpha^{-1})(a^{-1}R(\alpha^{-1}), \varepsilon). \tag{2.27}$$

Thus, we have the following theorem:

THEOREM 1. *The skew product $H\divideontimes\Gamma$ which is defined by the following:*

$$(a, \alpha)(b, \beta)=(aR(\beta)\cdot L(\alpha)b,\ \alpha\mathscr{R}(b)\cdot\mathscr{L}(a)\beta),$$
$$(aR(\varepsilon)=L(\varepsilon)a=a,\ \alpha\mathscr{R}(e)=\mathscr{L}(e)\alpha=\alpha),$$

*is a group with the unit element $(e,\varepsilon)$ if and only if the following system of the conditions is satisfied:*

$$(2.28)\quad\left\{\begin{array}{ll}
①_1\ eR(\alpha)=L(\alpha)e=e, & ①_2\ \varepsilon\mathscr{R}(a)=\mathscr{L}(a)\varepsilon=\varepsilon,\\
②_1\ (L(\alpha)b)R(\beta)=L(\alpha)(bR(\beta)), & ②_2\ (\mathscr{L}(a)\beta)\mathscr{R}(b)=\mathscr{L}(a)(\beta\mathscr{R}(b)),\\
③_1\ bR(\alpha\cdot\beta)=(bR(\alpha))R(\beta), & ③_2\ \beta\mathscr{R}(a\cdot b)=(\beta\mathscr{R}(a))\mathscr{R}(b),\\
④_1\ L(\alpha\cdot\beta)b=L(\alpha)(L(\beta)b), & ④_2\ \mathscr{L}(a\cdot b)\beta=\mathscr{L}(a)(\mathscr{L}(b)\beta),\\
⑤_1\ (b\cdot c)R(\alpha)=bR(\mathscr{L}(c)\alpha)\cdot cR(\alpha), & ⑤_2\ (\beta\cdot\gamma)\mathscr{R}(a)=\beta\mathscr{R}(L(\gamma)a)\cdot\gamma\mathscr{R}(a),\\
⑥_1\ L(\alpha)(b\cdot c)=L(\alpha)b\cdot L(\alpha\mathscr{R}(b))c, & ⑥_2\ \mathscr{L}(a)(\beta\cdot\gamma)=\mathscr{L}(a)\beta\cdot\mathscr{L}(aR(\beta))\gamma,\\
⑦_1\ L(\mathscr{L}(a)\beta)c\cdot L(\beta\mathscr{R}(c))c^{-1} & ⑦_2\ \mathscr{L}(L(\alpha)b)\gamma\cdot\mathscr{L}(bR(\gamma))\gamma^{-1}\\
\quad =a^{-1}R(\mathscr{L}(a)\beta)\cdot aR(\beta\mathscr{R}(c)), & \quad =\alpha^{-1}\mathscr{R}(L(\alpha)b)\cdot\alpha\mathscr{R}(bR(\gamma)),\\
⑧_1\ L(\mathscr{L}(a)\beta)c\cdot L(\beta\mathscr{R}(c))c^{-1}\in Z(H) & ⑧_2\ \mathscr{L}(L(\alpha)b)\gamma\cdot\mathscr{L}(bR(\gamma))\gamma^{-1}\in Z(\Gamma)\\
\quad \cap K_R\cap K_L\cap kern\mathscr{R}\cap kern\mathscr{L}, & \quad \cap K_{\mathscr{R}}\cap K_{\mathscr{R}}\cap kernR\cap kernL.
\end{array}\right.$$

REMARK 4. When the above system of the conditions is satisfied, that is, the skew product $H \divideontimes \Gamma$ is a group, $R$ and $L$ which are two mappings of $\Gamma$ into the permutation group on the elements of $H$ are homomorphisms, because of ③$_1$, ④$_1$ and $aR(\varepsilon)=L(\varepsilon)a=a$. That is, it holds that $\Gamma \sim \{R(\alpha)\}$ and $\Gamma \sim \{L(\alpha)\}$. Similarly, it holds that $H \sim \{\mathcal{R}(a)\}$ and $H \sim \{\mathcal{L}(a)\}$.

## § 3. A group which is isomorphic to group $H \divideontimes \Gamma$.

In this section, we show that the group $H \divideontimes \Gamma$ obtained in § 2 is, in fact, isomorphic to a special case of the group $H \circ \Gamma$ defined by L. Rédei in [4].

The group $H \circ \Gamma$ is the set of all ordered pairs $(a, \alpha)$ with $a \in H$ and $\alpha \in \Gamma$, and its product is defined by the following:

$$(a, \alpha)(b, \beta)=(ab^{\alpha}, \alpha^{b}\beta), \tag{3.1}$$

where $b^{\alpha}$ and $\alpha^{b}$ are fixed functions having their values in $H$ and $\Gamma$ respectively. Then we have the following result:

THEOREM 2. *The group $H \divideontimes \Gamma$ whose product is defined by*

$$(a, \alpha)(b, \beta)=(aR(\beta) \cdot L(\alpha)b,\ \alpha\mathcal{R}(b) \cdot \mathcal{L}(a)\beta), \quad (aR(\varepsilon)=L(\varepsilon)a=a,\ \alpha\mathcal{R}(e)=\mathcal{L}(e)\alpha=\alpha), \tag{3.2}$$

*is isomorphic to a special case of the group $H \circ \Gamma$. And the isomorphism is given by*

$$\Pi:\ (a, \alpha) \rightarrow (aR(\mathcal{L}(a^{-1})\alpha^{-1}),\ \mathcal{L}(a^{-1}R(\alpha^{-1}))\alpha). \tag{3.3}$$

PROOF. We prove the theorem by the method of the transformation (cf. [4]).

The correspondence (3.3) on the group $H \divideontimes \Gamma$ is a permutaion. For, if we define the correspondence $\bar{\Pi}$ by $\bar{\Pi}(a, \alpha)=(aR(\alpha), \mathcal{L}(a)\alpha)$, then $\bar{\Pi} \cdot \Pi(a, \alpha)=(a, \alpha)$, that is, $\bar{\Pi}$ is the inverse of $\Pi$. So, if $(a, \alpha) \neq (b, \beta)$, then it follows $\Pi(a, \alpha) \neq \Pi(b, \beta)$.

In order to prove that $H \divideontimes \Gamma \cong H \circ \Gamma$, it is sufficient to show that $(a, \alpha) \circ (b, \beta) = \Pi(\Pi^{-1}(a, \alpha)\Pi^{-1}(b, \beta))$ have the form (3.1). For the arbitrary two elements $(a, \alpha)$ and $(b, \beta)$ of the group, it holds that $\Pi^{-1}(a, \alpha)=(aR(\alpha), \mathcal{L}(a)\alpha)$, $\Pi^{-1}(b, \beta)=(bR(\beta), \mathcal{L}(b)\beta)$. So, we have

$$\begin{aligned}&\Pi^{-1}(a, \alpha)\Pi^{-1}(b, \beta)\\ &\quad=(aR(\alpha)R(\mathcal{L}(b)\beta) \cdot L(\mathcal{L}(a)\alpha)bR(\beta),\ \mathcal{L}(a)\alpha\mathcal{R}(bR(\beta)) \cdot \mathcal{L}(aR(\alpha))\mathcal{L}(b)\beta).\end{aligned}$$

Now, if we set

$$\bar{a}=aR(\alpha)R(\mathcal{L}(b)\beta),\ \bar{b}=L(\mathcal{L}(a)\alpha)bR(\beta),\ \bar{\alpha}=\mathcal{L}(a)\alpha\mathcal{R}(bR(\beta)),\ \bar{\beta}=\mathcal{L}(aR(\alpha))\mathcal{L}(b)\beta,$$

the above product has the form:

$$\Pi^{-1}(a, \alpha)\ \Pi^{-1}(b, \beta)=(\bar{a}\cdot\bar{b}, \bar{\alpha}\cdot\bar{\beta}).$$

Consequently, we have the following new product

$$(3.4)\qquad \begin{aligned}(a, \alpha)\circ(b, \beta)&=\Pi(\bar{a}\cdot\bar{b}, \bar{\alpha}\cdot\bar{\beta})\\ &=(((\bar{b}^{-1}\cdot\bar{a}^{-1})R(\bar{\beta}^{-1}\cdot\bar{\alpha}^{-1}))^{-1}, (\mathscr{L}(\bar{b}^{-1}\cdot\bar{a}^{-1})(\bar{\beta}^{-1}\cdot\bar{\alpha}^{-1}))^{-1}).\end{aligned}$$

We calculate $((\bar{b}^{-1}\cdot\bar{a}^{-1})R(\bar{\beta}^{-1}.\bar{\alpha}^{-1}))^{-1}$. First, we have

$$\begin{aligned}(\bar{b}^{-1}\cdot\bar{a}^{-1})R(\bar{\beta}^{-1}\cdot\bar{\alpha}^{-1})&=\bar{b}^{-1}R(\mathscr{L}(\bar{a}^{-1})(\bar{\beta}^{-1}\cdot\bar{\alpha}^{-1}))\cdot\bar{a}^{-1}R(\bar{\beta}^{-1}\cdot\bar{\alpha}^{-1})\\ &=\bar{b}^{-1}R(\mathscr{L}(\bar{a}^{-1})\bar{\beta}^{-1}\cdot\mathscr{L}(\bar{a}^{-1}R(\bar{\beta}^{-1}))\bar{\alpha}^{-1})\cdot\bar{a}^{-1}R(\bar{\beta}^{-1}\cdot\bar{\alpha}^{-1}).\end{aligned}$$

Next, it holds that

$$\begin{aligned}(\bar{a}^{-1}R(\bar{\beta}^{-1}\cdot\bar{\alpha}^{-1}))^{-1}&=\bar{a}R(\mathscr{L}(\bar{a}^{-1})(\bar{\beta}^{-1}\cdot\bar{\alpha}^{-1}))\\ &=\bar{a}R(\mathscr{L}(\bar{a}^{-1})\bar{\beta}^{-1}\cdot\mathscr{L}(\bar{a}^{-1}R(\bar{\beta}^{-1}))\bar{\alpha}^{-1}).\end{aligned}$$

Consequently, we have

$$(3.5)\qquad \begin{aligned}&((\bar{b}^{-1}\cdot\bar{a}^{-1})R(\bar{\beta}^{-1}\cdot\bar{\alpha}^{-1}))^{-1}\\ &=\bar{a}R(\mathscr{L}(\bar{a}^{-1})\bar{\beta}^{-1}\cdot\mathscr{L}(\bar{a}^{-1}R(\bar{\beta}^{-1}))\bar{\alpha}^{-1})\cdot\\ &\qquad\cdot(\bar{b}^{-1}R(\mathscr{L}(\bar{a}^{-1})\bar{\beta}^{-1}\cdot\mathscr{L}(\bar{a}^{-1}R(\bar{\beta}^{-1}))\bar{\alpha}^{-1}))^{-1}.\end{aligned}$$

In order to calculate this, we begin with the calculation of $\mathscr{L}(\bar{a}^{-1})\bar{\beta}^{-1}$ and $\mathscr{L}(\bar{a}^{-1}R(\bar{\beta}^{-1}))\bar{\alpha}^{-1}$.

$$\begin{aligned}\mathscr{L}(\bar{a}^{-1})\bar{\beta}^{-1}&=\mathscr{L}((aR(\alpha\cdot\mathscr{L}(b)\beta))^{-1})(\mathscr{L}(aR(\alpha))\mathscr{L}(b)\beta)^{-1}\\ &=\mathscr{L}((aR(\alpha\cdot\mathscr{L}(b)\beta))^{-1})\mathscr{L}(aR(\alpha\cdot\mathscr{L}(b)\beta))(\mathscr{L}(b)\beta)^{-1} &&\text{(by }(2.11)_2)\\ &=(\mathscr{L}(b)\beta)^{-1}.\end{aligned}$$

$$\begin{aligned}&\mathscr{L}(\bar{a}^{-1}R(\bar{\beta}^{-1}))\bar{\alpha}^{-1}\\ &=\mathscr{L}((aR(\alpha))^{-1}R(\mathscr{L}(aR(\alpha)\cdot b)\beta)R((\mathscr{L}(aR(\alpha)\cdot b)\beta)^{-1}))(\mathscr{L}(a)\alpha)^{-1}\mathscr{R}(L(\alpha)bR(\beta))\\ &=\mathscr{L}((aR(\alpha))^{-1})\mathscr{L}(aR(\alpha))\alpha^{-1}\mathscr{R}(L(\alpha)bR(\beta)) &&\text{(by }(2.11)_2)\\ &=\alpha^{-1}\mathscr{R}(L(\alpha)bR(\beta))\\ &=(\alpha\mathscr{R}(bR(\beta)))^{-1}.\end{aligned}$$

Now, we calculate the first factor of the right side of (3.5).

$$\begin{aligned}&\bar{a}R(\mathscr{L}(\bar{a}^{-1})\bar{\beta}^{-1}\cdot\mathscr{L}(\bar{a}^{-1}R(\bar{\beta}^{-1}))\bar{\alpha}^{-1})\\ &\quad=aR(\alpha)R(\mathscr{L}(b)\beta)R((\mathscr{L}(b)\beta)^{-1}\cdot(\alpha\mathscr{R}(bR(\beta)))^{-1})\\ &\quad=aR(\alpha\cdot\alpha^{-1}\mathscr{R}(L(\alpha)bR(\beta))) &&\text{(by }(2.12)_2)\\ &\quad=aR(\alpha\cdot(\alpha\mathscr{R}(b))^{-1}).\end{aligned}$$

The last of the above is obtained from the following:

$$(3.6)\qquad cR(\alpha\mathscr{R}(bR(\gamma)))=cR(\alpha\mathscr{R}(b)\cdot(\alpha\mathscr{R}(b))^{-1}\cdot\alpha\mathscr{R}(bR(\gamma)))=cR(\alpha\mathscr{R}(b)).$$

Next, we proceed to calculate the second factor of the right side of (3.5).

$$\bar{b}^{-1}R(\mathscr{L}(\bar{a}^{-1})\bar{\beta}^{-1}\cdot\mathscr{L}(\bar{a}^{-1}R(\bar{\beta}^{-1}))\bar{\alpha}^{-1})$$
$$=L(\mathscr{L}(a)\alpha\mathscr{R}(b))(bR(\beta))^{-1}\,R((\mathscr{L}(b)\beta)^{-1}\cdot(\alpha\mathscr{R}(bR(\beta)))^{-1})$$
$$=L(\mathscr{L}(a)\alpha\mathscr{R}(b))b^{-1}R((\alpha\mathscr{R}(bR(\beta)))^{-1}).$$

Therefore, we have the following inverse element of the above:

$$(\bar{b}^{-1}R(\mathscr{L}(\bar{a}^{-1})\bar{\beta}^{-1}\cdot\mathscr{L}(\bar{a}^{-1}R(\bar{\beta}^{-1}))\bar{\alpha}^{-1}))^{-1}$$
$$=(L(\mathscr{L}(a)\alpha\mathscr{R}(b))b^{-1}R((\alpha\mathscr{R}(bR(\beta)))^{-1}))^{-1}$$
$$=(L(\mathscr{L}(a)\alpha\mathscr{R}(b))b^{-1})^{-1}R(\mathscr{L}(L(\mathscr{L}(a)\alpha\mathscr{R}(b))b^{-1})(\alpha\mathscr{R}(bR(\beta)))^{-1})$$
$$=L(\mathscr{L}(a)\alpha)bR(\mathscr{L}(b^{-1})\alpha^{-1}\mathscr{R}(L(\alpha)bR(\beta)))$$
$$=L(\mathscr{L}(a)\alpha)bR(\mathscr{L}(b^{-1})\alpha^{-1}\mathscr{R}(L(\alpha)b)). \qquad \text{(by (3.6))}$$

Consequently from (3.5), we have

$$((\bar{b}^{-1}\cdot\bar{a}^{-1})R(\bar{\beta}^{-1}\cdot\bar{\alpha}^{-1}))^{-1}$$
$$=aR(\alpha\cdot(\alpha\mathscr{R}(b))^{-1})\cdot L(\mathscr{L}(a)\alpha)bR(\mathscr{L}(b^{-1})\alpha^{-1}\mathscr{R}(L(\alpha)b)). \tag{3.7}$$

Similarly, we have

$$(\mathscr{L}(\bar{b}^{-1}\cdot\bar{a}^{-1})(\bar{\beta}^{-1}\cdot\bar{\alpha}^{-1}))^{-1}$$
$$=\mathscr{L}(L(\alpha\mathscr{R}(b))b^{-1}R(\alpha^{-1}))\alpha\mathscr{R}(bR(\beta))\cdot\mathscr{L}((L(\alpha)b)^{-1}\cdot b)\beta. \tag{3.8}$$

Therefore, by (3.7) and (3.8), we obtain the form of the product (3.4), that is,

$$(a,\alpha)\circ(b,\beta)$$
$$=(aR(\alpha\cdot(\alpha\mathscr{R}(b))^{-1})\cdot L(\mathscr{L}(a)\alpha)bR(\mathscr{L}(b^{-1})(\alpha\mathscr{R}(b))^{-1}),$$
$$\mathscr{L}((L(\alpha)b)^{-1}R(\alpha^{-1}))\alpha\mathscr{R}(bR(\beta))\cdot\mathscr{L}((L(\alpha)b)^{-1}\cdot b)\beta). \tag{3.9}$$

Thus, the group $H\times\Gamma$ is isomorphic to the group $G_1$ the product of which is defined by (3.9). This group $G_1$ has the unit element $(e, \varepsilon)$. Further, in the group $G_1$, the associativity of the product holds. So, for the three elements $(a, \varepsilon)$, $(b, \beta)$, and $(c, \gamma)$ of $G_1$, it holds that

$$((a,\varepsilon)(b,\beta))\,(c,\gamma)=(a,\varepsilon)\,((b,\beta)(c,\gamma)).$$

From this it follows that

$$(a\cdot b)R(\beta\cdot(\beta\mathscr{R}(c))^{-1})\cdot L(\mathscr{L}(a\cdot b)\beta)cR(\mathscr{L}(c^{-1})(\beta\mathscr{R}(c))^{-1})$$
$$=a\cdot bR(\beta\cdot(\beta\mathscr{R}(c))^{-1})\cdot L(\mathscr{L}(b)\beta)cR(\mathscr{L}(c^{-1})(\beta\mathscr{R}(c))^{-1}).$$

Here, $a$ and $b$ are the two arbitrary elements of $H$, so for $a=b^{-1}$ we have:

$$bR(\beta\cdot(\beta\mathscr{R}(c))^{-1})\cdot L(\mathscr{L}(b)\beta)cR(\mathscr{L}(c^{-1})(\beta\mathscr{R}(c))^{-1})$$
$$=b\cdot L(\beta)cR(\mathscr{L}(c^{-1})(\beta\mathscr{R}(c))^{-1}).$$

In the group $\Gamma$, in the similar way, we have:

$$\mathscr{L}((L(\alpha)b)^{-1}R(\alpha^{-1}))\alpha\mathcal{R}(bR(\beta))\cdot\mathscr{L}((L(\alpha)b)^{-1}\cdot b)\beta$$
$$=\mathscr{L}((L(\alpha)b)^{-1}R(\alpha^{-1}))\alpha\mathcal{R}(b)\cdot\beta.$$

Therefore, the product (3.9) has the following form:

$$(3.10)\qquad (a,\alpha)\circ(b,\beta)$$
$$=(a\cdot L(\alpha)bR(\mathscr{L}(b^{-1})\alpha^{-1}\mathcal{R}(L(\alpha)b)),\ \mathscr{L}(L(\alpha\mathcal{R}(b))b^{-1}R(\alpha^{-1}))\alpha\mathcal{R}(b)\cdot\beta).$$

Thus, the product of the group $G_1$ is defined by (3.10). In this definition, if we set

$$F(\alpha, b)=L(\alpha)bR(\mathscr{L}(b^{-1})\alpha^{-1}\mathcal{R}(L(\alpha)b)),$$
$$\mathscr{F}(\alpha, b)=\mathscr{L}(L(\alpha\mathcal{R}(b))b^{-1}R(\alpha^{-1}))\alpha\mathcal{R}(b),$$

$F(\alpha, b)$ and $\mathscr{F}(\alpha, b)$ are the functions of two variables having their values in $H$ and $\Gamma$ respectively, and the definition (3.10) has the following form:

$$(a,\alpha)\circ(b,\beta)=(a\cdot F(\alpha, b),\ \mathscr{F}(\alpha, b)\cdot\beta).$$

Therefore, the group $G_1$ is a special case of the group $H^2\Gamma$ which is defined by L. Rédei, and is isomorphic to the group $H\divideontimes\Gamma$. Thus, we have proved Theorem 2.

Remark 5. In the above proof, the conditions $(2.10)_1$, (2.11) and (2.12) follow from the system of the conditions (2.28) (cf. (2.11), (2.12) and Remark 2).

## § 4. Relation of group $H\divideontimes\Gamma$ to group $H\times\Gamma$ and $H$Ⓢ$\Gamma$.

In this section, we inquire the necessary and sufficient condition for the group $H\divideontimes\Gamma$ to be a semi-direct product, and then for the group $H\divideontimes\Gamma$ to be the direct product of the groups $H$ and $\Gamma$.

The semi-direct product $H$Ⓢ$\Gamma$ of the groups $H$ and $\Gamma$ is constituted of all ordered pairs $(a, \alpha)$ with $a\in H$ and $\alpha\in\Gamma$, the product of which is defined by $(a,\alpha)(b,\beta)=(a\cdot F(\alpha)b,\ \alpha\cdot\beta)$, where $F$ is a homomorphism of $\Gamma$ into the subgroup of the group of all automorphisms of $H$. This product is restated as follows: the semi-direct product $H$Ⓢ$\Gamma$ is $H\Gamma$ where $H$ is a normal subgroup of $H$Ⓢ$\Gamma$ and $\Gamma$ is a subgroup of $H$Ⓢ$\Gamma$ with trivial intersection with $H$. (We may take $\Gamma$ as a normal subgroup and $H$ as a subgroup, and in this case, the definition of the product is $(a,\alpha)(b,\beta)=(a\cdot b,\ \alpha\mathscr{F}(b)\cdot\beta)$).

Now, if we express the set of all the elements $(a, \varepsilon)$ by $(H, \varepsilon)$, $(H, \varepsilon)$ is a subgroup of the group $H\divideontimes\Gamma$. Similarly, $(e, \Gamma)$ is a subgroup of $H\divideontimes\Gamma$. And it holds that $H\cong(H,\varepsilon)$, $\Gamma\cong(e,\Gamma)$ and $(H,\varepsilon)\cap(e,\Gamma)=\{(e,\varepsilon)\}$.

We proceed to inquire the condition in order that the group $H\divideontimes\Gamma$ may be isomorphic to the semi-direct product of the groups $H$ and $\Gamma$. For that purpose it is sufficient to get the condition for $H \divideontimes \Gamma$ to be $(H, \varepsilon)$Ⓢ$(e, \Gamma)$. As we mentioned above, the group $H \divideontimes \Gamma$ is $(H, \varepsilon)$Ⓢ$(e, \Gamma)$ if and only if $(H, \varepsilon)$ or $(e, \Gamma)$ is a normal subgroup of $H \divideontimes \Gamma$, so we inquire the condition that the subgroup $(H, \varepsilon)$ is normal in $H \divideontimes \Gamma$.

For the arbitrary elements $(a, \alpha) \in H \divideontimes \Gamma$ and $(b, \varepsilon) \in (H, \varepsilon)$, we have

$$\begin{aligned}&(a, \alpha)(b, \varepsilon)(a, \alpha)^{-1}\\&=(aR(\mathscr{L}(b \cdot a^{-1})\alpha^{-1}) \cdot L(\alpha)bR(\mathscr{L}(a^{-1})\alpha^{-1}) \cdot L(\alpha\mathscr{R}(b) \cdot \mathscr{L}(a \cdot L(\alpha)b \cdot a^{-1})\alpha^{-1})a^{-1}R(\alpha^{-1}),\\&\quad(\alpha\mathscr{R}(b) \cdot \mathscr{L}(a \cdot L(\alpha)b \cdot a^{-1})\alpha^{-1})\mathscr{R}(a^{-1}R(\alpha^{-1}))).\end{aligned}$$

Therefore, in order that $(H, \varepsilon)$ may be a normal subgroup of $H \divideontimes \Gamma$, the following condition must hold;

$$(\alpha\mathscr{R}(b) \cdot \mathscr{L}(a \cdot L(\alpha)b \cdot a^{-1})\alpha^{-1})\mathscr{R}(a^{-1}R(\alpha^{-1}))=\varepsilon. \tag{4.1}$$

So, it must hold that

$$\mathscr{L}(L(\alpha)b)\alpha^{-1}=\alpha^{-1}\mathscr{R}(L(\alpha)b).$$

As the elements $b \in H$ and $\alpha \in \Gamma$ are arbitrary, so the above condition may be written as follows:

$$\mathscr{L}(c)\beta=\beta\mathscr{R}(c), \qquad \forall c \in H,\ \forall \beta \in \Gamma. \tag{4.2}$$

Conversely, we assume that $\mathscr{L}(c)\beta=\beta\mathscr{R}(c)$, $c \in H$, $\beta \in \Gamma$. By this assumption, from the condition $②_2$ of (2.28), we have the condition

$$\mathscr{L}(a \cdot b)\beta=\mathscr{L}(b \cdot a)\beta. \tag{4.3}$$

Now, by the assumption, it holds that

$$\mathscr{L}(L(\alpha)b)\alpha^{-1}=\alpha^{-1}\mathscr{R}(L(\alpha)b),$$

for $b \in H$ and $\alpha \in \Gamma$. So, by (4.3), it holds that

$$\mathscr{L}(a \cdot L(\alpha)b \cdot a^{-1})\alpha^{-1}=\alpha^{-1}\mathscr{R}(L(\alpha)b),$$

that is,

$$\alpha\mathscr{R}(b) \cdot \mathscr{L}(a \cdot L(\alpha)b \cdot a^{-1})\alpha^{-1}=\varepsilon.$$

So, we have

$$(\alpha\mathscr{R}(b) \cdot \mathscr{L}(a \cdot L(\alpha)b \cdot a^{-1})\alpha^{-1})\mathscr{R}(a^{-1}R(\alpha^{-1}))=\varepsilon.$$

Therefore, the subgroup $(H, \varepsilon)$ is a normal subgroup of $H \divideontimes \Gamma$.

That is, the necessary and sufficient condition in order that $(H, \varepsilon)$ may be a normal subgroup of $H \divideontimes \Gamma$ is $\mathscr{L}(c)\beta=\beta\mathscr{R}(c)$ for $c \in H$, $\beta \in \Gamma$.

In the similar way, we can show that the condition for $(e, \Gamma)$ to be a normal subgroup of $H \divideontimes \Gamma$ is

$$L(\beta)c=cR(\beta), \qquad \forall c \in H, \forall \beta \in \Gamma. \tag{4.4}$$

Next, we inquire the condition that the group may be isomorphic to the direct product of the groups $H$ and $\Gamma$. The group is isomorphic to the direct product of $H$ and $\Gamma$ if and only if $(H, \varepsilon)$ and $(e, \Gamma)$ are normal subgroups of $H \divideontimes \Gamma$. So, by the above results, the necessary and sufficient condition in order that $H \divideontimes \Gamma$ is isomorphic to $H \times \Gamma$ (direct product) is that the conditions (4.2) and (4.4) hold. Thus, we have the proposition:

PROPOSITION 1. *If and only if the condition* $L(\alpha)a=aR(\alpha)$ *or* $\mathscr{L}(a)\alpha=\alpha\mathscr{R}(a)$, $a \in H$, $\alpha \in \Gamma$ *holds, the group* $H \divideontimes \Gamma$ *is isomorphic to the semi-direct product of the groups* $H$ *and* $\Gamma$.

*Further, if and only if the conditions* $L(\alpha)a=aR(\alpha)$ *and* $\mathscr{L}(a)\alpha=\alpha\mathscr{R}(a)$, $a \in H$, $\alpha \in \Gamma$ *hold simultaneously, the group* $H \divideontimes \Gamma$ *is isomorphic to the direct product of the groups* $H$ *and* $\Gamma$.

REMARK 6. As we mentioned in § 1, the definition of the group $H \odot \Gamma$ follows from the definition of the group $H \divideontimes \Gamma$ under the conditions $L(\alpha)a=aR(\alpha)$ and $\mathscr{L}(a)\alpha=\alpha\mathscr{R}(a)$. Therefore, by the proposition 1, the group $H \odot \Gamma$ is isomorphic to the direct product $H \times \Gamma$. This is the results obtained by F. Rühs in his paper [7].

REMARK 7. When the condition $\mathscr{L}(c)\gamma=\gamma\mathscr{R}(c)$ holds, the definition (3.10) of the product of the group $G_1$ is reduced to the form:

$$(a, \alpha)(b, \beta)=(a \cdot L(\alpha)bR(\alpha^{-1}), \alpha \cdot \beta)^{(2)}. \tag{4.5}$$

For, by the condition $\mathscr{L}(c)\gamma=\gamma\mathscr{R}(c)$ the definition (3.10), viz.,

$$\begin{aligned}&(a, \alpha)(b, \beta)\\ &\quad=(a \cdot L(\alpha)bR(\mathscr{L}(b^{-1})\alpha^{-1}\mathscr{R}(L(\alpha)b)), \mathscr{L}(L(\alpha\mathscr{R}(b))b^{-1}R(\alpha^{-1}))\alpha\mathscr{R}(b) \cdot \beta)\end{aligned}$$

is reduced to the form:

$$(a, \alpha)(b, \beta)=(a \cdot L(\alpha)R(\alpha^{-1}), \mathscr{L}(L(\alpha)b^{-1}R(\alpha^{-1}) \cdot b)\alpha \cdot \beta).$$

Further, it holds that $\mathscr{L}(L(\alpha)b^{-1}R(\alpha^{-1}) \cdot b)\alpha=\alpha$.

For,

$$\begin{aligned}&\mathscr{L}(L(\alpha)b^{-1}R(\alpha^{-1}) \cdot b)\alpha\\ &=\mathscr{L}(b^{-1}R(\alpha^{-1}))\mathscr{L}(b)\alpha \cdot (\mathscr{L}(b^{-1}R(\alpha^{-1}))\mathscr{L}(b)\alpha)^{-1} \cdot \mathscr{L}(L(\alpha)b^{-1}R(\alpha^{-1}))\mathscr{L}(b)\alpha\\ &=\mathscr{L}(b^{-1}R(\alpha^{-1}))\mathscr{L}(b)\alpha \cdot (\alpha\mathscr{R}(b^{-1}R(\alpha^{-1})))^{-1} \cdot \alpha\mathscr{R}(b^{-1}R(\alpha^{-1})R(\mathscr{L}(b)\alpha))\\ &=\mathscr{L}(b^{-1}R(\alpha^{-1}))\mathscr{L}(b)\alpha \cdot (\mathscr{L}(b^{-1}R(\alpha^{-1}))\alpha)^{-1} \cdot \mathscr{L}(b^{-1})\alpha\end{aligned}$$

(2) Hereafter, in the definition of the product of the group $G_1$, we use the notation $(a, \alpha)(b, \beta)$ instead of $(a, \alpha)\circ(b, \beta)$.

$=\mathscr{L}(b^{-1}R(\alpha^{-1}))\mathscr{L}(b)\alpha\cdot\mathscr{L}(b^{-1})\alpha^{-1}\cdot\mathscr{L}(b^{-1})\alpha$

$=\mathscr{L}(b)\alpha\mathscr{R}(b^{-1})\cdot(\mathscr{L}(b)\alpha\mathscr{R}(b^{-1}))^{-1}\cdot\mathscr{L}(b)\alpha\mathscr{R}(b^{-1}R(\alpha^{-1}))\cdot\mathscr{L}(b^{-1})\alpha^{-1}\cdot\mathscr{L}(b^{-1})\alpha$

$=\alpha\cdot\mathscr{L}(L(\mathscr{L}(b)\alpha)b^{-1})\alpha^{-1}\cdot(\mathscr{L}(b^{-1})\alpha^{-1})^{-1}\cdot\mathscr{L}(b^{-1})\alpha^{-1}\cdot\mathscr{L}(b^{-1})\alpha$

$=\alpha\cdot\mathscr{L}(L(\alpha)b^{-1})\alpha^{-1}\cdot\mathscr{L}(b^{-1})\alpha$

$=\alpha.$

Therefore, we have the product (4.5)

$$(a, \alpha)(b, \beta)=(a\cdot L(\alpha)bR(\alpha^{-1}), \alpha\cdot\beta)=(a\cdot F(\alpha)b, \alpha\cdot\beta).$$

Further, if the group is defined by the above product, we can show that the permutation $F(\alpha)$ is an automorphism of $H$ (cf. §9).

Moreover, $\mathscr{L}(c)\gamma=\gamma\mathscr{R}(c)$ and $L(\gamma)c=cR(\gamma)$ hold simultaneously, the definition of the product (4.5) is reduced to the form:

$$(a, \alpha)(b, \beta)=(a\cdot b, \alpha\cdot\beta).$$

## § 5. Skew product $H\divideontimes\Gamma$ defined by automorphisms.

In this section, we consider the special case of the skew product, that is, the case where in the definition of the product of $H\divideontimes\Gamma$:

$$(a, \alpha)(b, \beta)=(aR(\beta)\cdot L(\alpha)b, \alpha\mathscr{R}(b)\cdot\mathscr{L}(a)\beta),$$

$R(\beta)$ and $L(\alpha)$ are the automorphisms of $H$, and $\mathscr{R}(b)$ and $\mathscr{L}(a)$ are the automorphisms of $\Gamma$.

In this case, if the skew product $H\divideontimes\Gamma$ is a group, by Remark 4 $R(\Gamma)$ and $L(\Gamma)$ are the subgroups of the group $\mathfrak{A}(H)$ of all automorphisms of $H$, and it holds that $\Gamma\sim R(\Gamma)$, $\Gamma\sim L(\Gamma)$. Similarly, $\mathscr{R}(H)$ and $\mathscr{L}(H)$ are the subgroups of $\mathfrak{A}(\Gamma)$, and it holds that $H\sim\mathscr{R}(H)$ and $H\sim\mathscr{L}(H)$.

Here, we redefine the skew product in this case. Let $R$ and $L$ be two homomorphisms of $\Gamma$ into the subgroups of $\mathfrak{A}(H)$ and $\mathscr{R}$ and $\mathscr{L}$ be two homomorphisms of $H$ into the subgroups of $\mathfrak{A}(\Gamma)$. We define the skew product $H\otimes\Gamma$ by the following:

(5.1) $$(a, \alpha)(b, \beta)=(aR(\beta)\cdot L(\alpha)b, \alpha\mathscr{R}(b)\cdot\mathscr{L}(a)\beta).$$

Then, by Theorem 1 and Remark 3, we have the proposition:

PROPOSITION 2. *The skew product $H\otimes\Gamma$ the product of which is defined by*

$$(a, \alpha)(b, \beta)=(aR(\beta)\cdot L(\alpha)b, \alpha\mathscr{R}(b)\cdot\mathscr{L}(\alpha)\beta),$$

*where $R(\beta)$, $L(\alpha)$; $\mathscr{R}(b)$, $\mathscr{L}(a)$ are the automorphisms of $H$ and $\Gamma$ respectively, is a group if and only if the following system of the conditions is satisfied:*

$$(5.2)\quad \begin{cases} ①_1\ (L(\alpha)b)R(\beta)=L(\alpha)(bR(\beta)), & ①_2\ (\mathscr{L}(a)\beta)\mathcal{R}(b)=\mathscr{L}(a)(\beta\mathcal{R}(b)), \\ ②_1\ bR(\mathscr{L}(a)\alpha)=bR(\alpha), & ②_2\ \beta\mathcal{R}(L(\alpha)a)=\beta\mathcal{R}(a), \\ ③_1\ L(\alpha\mathcal{R}(a))b=L(\alpha)b, & ③_2\ \mathscr{L}(aR(\alpha))\beta=\mathscr{L}(a)\beta, \\ ④_1\ L(\mathscr{L}(b)\alpha)a\cdot L(\alpha)a^{-1} & ④_2\ \mathscr{L}(L(\beta)a)\alpha\cdot\mathscr{L}(a)\alpha^{-1} \\ \qquad =b^{-1}R(\alpha)\cdot bR(\alpha\mathcal{R}(a)), & \qquad =\beta^{-1}\mathcal{R}(a)\cdot\beta\mathcal{R}(aR(\alpha)), \\ ⑤_1\ L(\mathscr{L}(b)\alpha)a\cdot L(\alpha)a^{-1} & ⑤_2\ \mathscr{L}(L(\beta)a)\alpha\cdot\mathscr{L}(a)\alpha^{-1} \\ \qquad \in Z(H)\cap K_R\cap K_L, & \qquad \in Z(\Gamma)\cap K_{\mathcal{R}}\cap K_{\mathscr{L}}. \end{cases}$$

*where* $Z(H)=$ *center of* $H$, $\qquad Z(\Gamma)=$ *center of* $\Gamma$,

$$K_R=\{a;\ aR(\alpha)=a,\ \forall\alpha\in\Gamma\},\qquad K_{\mathcal{R}}=\{\alpha;\ \alpha\mathcal{R}(a)=\alpha,\ \forall a\in H\},$$
$$K_L=\{a;\ L(\alpha)a=a,\ \forall\alpha\in\Gamma\},\qquad K_{\mathscr{L}}=\{\alpha;\ \mathscr{L}(a)\alpha=\alpha,\ \forall a\in H\}.$$

Further, in this case we have the following condition by $②_2$, $③_2$ and $④_1$ of (5.2),

$$(5.3)\qquad L(\mathscr{L}(b)\alpha)a\cdot L(\alpha)a^{-1}\in \mathit{kern}\ \mathcal{R}\cap\ \mathit{kern}\ \mathscr{L},$$

where *kern* $\mathcal{R}$ and *kern* $\mathscr{L}$ are the kernels of the homomorphisms $\mathcal{R}$ and $\mathscr{L}$ respectively.

By the duality, we have

$$(5.4)\qquad \mathscr{L}(L(\beta)a)\alpha\cdot\mathscr{L}(a)\alpha^{-1}\in \mathit{kern}\ R\cap \mathit{kern}\ L.$$

In this case, the product (3.10) of the group $G_1$ which is isomorphic to the group $H\otimes\Gamma$ is reduced to the form:

$$(5.5)\qquad (a,\alpha)(b,\beta)=(a\cdot L(\alpha)bR(\alpha^{-1}\mathcal{R}(b)),\ \mathscr{L}(L(\alpha)b^{-1})\alpha\mathcal{R}(b)\cdot\beta).$$

For, using the conditions $②_1$, $②_2$, $③_1$, $③_2$, from the definition (3.10), we obtain easily (5.5).

Hereafter, we use the notation $G_{①}$ which represents the group $G_1$ that is isomorphic to the group $H\otimes\Gamma$ and belongs to the type $H^2\Gamma$. So, the product of the group $G_{①}$ is defined by (5.5).

Finally, we show the example of the group $H\otimes\Gamma$.

EXAMPLE 1. Let $H$ and $\Gamma$ be two cyclic groups of the order 8, and $a$ and $\alpha$ be the generators of $H$ and $\Gamma$ respectively. The automorphism of $H$ (and $\Gamma$) is uniquely determined by determing the corresponding element to its generator. So, we define two automorphisms $\varphi_1$ and $\varphi_2$ of $H$, and two automorphisms $\psi_1$ and $\psi_2$ of $\Gamma$ as follows:

$$\varphi_1:\ a\to a^3,\ \varphi_2:\ a\to a^5;\ \psi_1:\ \alpha\to\alpha^3,\ \psi_2:\ \alpha\to\alpha^5.$$

Now, we define two homomorphisms $L$ and $R$ of $\Gamma$ into $\mathfrak{A}(H)$, and two homomorphisms $\mathscr{L}$ and $\mathcal{R}$ of $H$ into $\mathfrak{A}(\Gamma)$ as follows:

$$L:\ \Gamma\to\{I,\ \varphi_1\},\ R:\ \Gamma\to\{I,\ \varphi_2\};\ \mathscr{L}:\ H\to\{I,\ \psi_2\},\ \mathcal{R}:\ H\to\{I,\ \psi_1\},$$

where $I$ is the identity automorphism.

Then we have $kern\ L = kern\ R = \{\alpha^2\}$ and $kern\ \mathscr{L} = kern\ \mathscr{R} = \{a^2\}$. And, we can easily see that the automorphisms $L(\alpha^s)$, $R(\alpha^t)$; $\mathscr{L}(a^m)$, $\mathscr{R}(a^n)$, $(s, t, m, n = 0, 1, 2, 3, 4, 5, 6, 7)$, which are determined by those homomorphisms, satisfy the system of the conditions (5.2). Therefore, the skew product defined by

$$(a^m, \alpha^s)(a^n, \alpha^t) = (a^m R(\alpha^t) \cdot L(\alpha^s)a^n, \alpha^s \mathscr{R}(a^n) \cdot \mathscr{L}(a^m)\alpha^t)$$

is a group of the type $H \otimes \Gamma$.

## § 6. Structure of group $H \otimes \Gamma$.

In this section, we investigate the structure of the group $H \otimes \Gamma$ defined by the automorphisms.

Now, let $H_0$ be a subgroup of $H$ which is generated by all the elements $aR(\alpha\mathscr{R}(b)) \cdot a^{-1}R(\alpha)$, $a, b \in H$, $\alpha \in \Gamma$. $H_0$ is contained in the center of $H$, so it is a normal subgroup of $H$. Similarly, the subgroup $\Gamma_0$ generated by all the elements $\mathscr{L}(L(\alpha)a)\beta \cdot \mathscr{L}(a)\beta^{-1}$, $a \in H$, $\alpha, \beta \in \Gamma$ is a normal subgroup of $\Gamma$. Dividing into two cases, that is, (i) $H_0 = \{e\}$ and $\Gamma_0 = \{\varepsilon\}$ and (ii) the case other than (i), we investigate the structure of the group $H \otimes \Gamma$.

Case 1. $H_0 = \{e\}$ and $\Gamma_0 = \{\varepsilon\}$.

In this case, the system of the conditions (5.2) is reduced to the following form:

$$(6.1)\quad \begin{cases} ①_1\ (L(\alpha)b)R(\beta) = L(\alpha)(bR(\beta)), & ①_2\ (\mathscr{L}(a)\beta)\mathscr{R}(b) = \mathscr{L}(a)(\beta\mathscr{R}(b)), \\ ②_1\ bR(\mathscr{L}(a)\alpha) = bR(\alpha), & ②_2\ \beta\mathscr{R}(L(\alpha)a) = \beta\mathscr{R}(a), \\ ③_1\ L(\alpha\mathscr{R}(a))b = L(\alpha)b, & ③_2\ \mathscr{L}(aR(a))\beta = \mathscr{L}(a)\beta, \\ ④_1\ bR(\alpha\mathscr{R}(a)) = bR(\alpha), & ④_2\ \beta\mathscr{R}(aR(\alpha)) = \beta\mathscr{R}(a), \\ ⑤_1\ L(\mathscr{L}(a)\alpha)b = L(\alpha)b, & ⑤_2\ \mathscr{L}(L(\alpha)a)\beta = \mathscr{L}(a)\beta. \end{cases}$$

REMARK 8. From the system of the conditions (6.1), by adding the conditions $L(\alpha)a = aR(\alpha)$, $\mathscr{L}(a)\alpha = \alpha\mathscr{R}(a)$, we can obtain the sufficient conditions in order that the skew product $H \odot \Gamma$ may be a group, which is obtained by L. Rédei in [4].

In this case 1, the definition of the product of the group $G_{①}$ defined in § 5 is given by the following:

$$(6{\cdot}2)\quad (a, \alpha)(b, \beta) = (a \cdot L(\alpha)bR(\alpha^{-1}), \mathscr{L}(b^{-1})\alpha\mathscr{R}(b) \cdot \beta)$$

In this definition, if we set $F(\alpha)b = L(\alpha)bR(\alpha^{-1})$ and $\alpha\mathscr{F}(b) = \mathscr{L}(b^{-1})\alpha\mathscr{R}(b)$, from (6.2) we have

$$(6.3)\quad (a, \alpha)(b, \beta) = (a \cdot F(\alpha)b, \alpha\mathscr{F}(b) \cdot \beta),$$

where $F(\alpha)$ and $\mathscr{F}(b)$ are the automorphisms of $H$ and $\Gamma$ respectively.

By the notation $H ② \Gamma$, we represent the group the product of which is defined by (6.3). It is clear that the group $H \circledtwo \Gamma$ is a special case of the group $H 2 \Gamma$.

Then, in the case 1, the group $H \otimes \Gamma$ is isomorphic to the group $H \circledtwo \Gamma$. So, by Szép's Theorem (cf. [10]), we have the theorem:

THEOREM 3. *If* $\{aR(\alpha\mathcal{R}(b))\cdot a^{-1}R(\alpha)\}=\{e\}$ *and* $\{\mathcal{L}(L(\alpha)a)\beta\cdot\mathcal{L}(a)\beta^{-1}\}\doteq\{\varepsilon\}$, *it holds that*

$$H\otimes\Gamma/(H_1,\ \Gamma_1)\cong H/H_1\times\Gamma/\Gamma_1 \text{ (direct product)}$$
$$(H_1,\ \Gamma_1)\cong H_1\times\Gamma_1 \text{ (direct product)}$$

*where* $H_1=\{a;\ \mathcal{L}(a^{-1})\alpha\mathcal{R}(a)=\alpha,\ \forall\alpha\in\Gamma\}$ *and* $\Gamma_1=\{\alpha;\ L(\alpha)aR(\alpha^{-1})=a,\ \forall a\in H\}$ *and* $(H_1,\ \Gamma_1)$ *is a subgroup of* $H\otimes\Gamma$ *which is generated by all the elements* $(a,\ \alpha)$ *with* $a\in H_1,\ \alpha\in\Gamma_1$.

Case 2. The case where at least one of $H_0$ and $\Gamma_0$ is not the group constituted of only one element.

Now, let be

$$\text{(6.4)}\qquad \begin{cases} l(a,\ \alpha,\ b)=aR(\alpha\mathcal{R}(b))\cdot a^{-1}R(\alpha), \\ \sigma(\alpha,\ a,\ \beta)=\mathcal{L}(L(\alpha)a)\beta\cdot\mathcal{L}(a)\beta^{-1}. \end{cases}$$

Then, the definition of the product of the group $G_{①}$ which is defined in § 5 becomes as follows:

$$\text{(6.5)}\qquad (a,\ \alpha)(b,\ \beta)=(a\cdot F(\alpha)b\cdot l(b,\ \alpha^{-1},\ b),\ \sigma(\alpha,\ b^{-1},\ \alpha)\cdot\alpha\mathcal{F}(b)\cdot\beta),$$

where $F(\alpha)b=L(\alpha)bR(\alpha^{-1})$ and $\alpha\mathcal{F}(b)=\mathcal{L}(b^{-1})\alpha\mathcal{R}(b)$ are the automorphisms of the group $H$ and $\Gamma$ respectively. And the group $H\otimes\Gamma$ is isomorphic to the group $G_{①}$ whose product is defined by (6.5).

Now, in the case 2, we investigate the structure of the group $H\otimes\Gamma$. First, we show that $(H_0,\ \Gamma_0)$ which is constituted of all the pairs $(a,\ \alpha)$ with $a\in H_0,\ \alpha\in\Gamma_0$ is a normal subgroup of the group $H\otimes\Gamma$.

For, it is clear that $(H_0,\ \Gamma_0)$ is a subgroup of $H\otimes\Gamma$. And for $(a,\ \alpha)\in H\otimes\Gamma,\ (b_0,\ \beta_0)\in(H_0,\ \Gamma_0)$, it holds that

$$\text{(6.6)}\qquad (a,\ \alpha)(b_0,\ \beta_0)=(a\cdot b_0,\ \alpha\cdot\beta_0)=(b_0\cdot a,\ \beta_0\cdot\alpha)=(b_0,\ \beta_0)(a,\ \alpha).$$

So, $(b_0,\ \beta_0)$ belongs to the center of $H\otimes\Gamma$, that is, $(H_0,\ \Gamma_0)$ is contained in the center of $H\otimes\Gamma$. Therefore, the subgroup $(H_0,\ \Gamma_0)$ is normal.

Further, from (6.6) we can see that

$$(H_0,\ \Gamma_0)\cong H_0\times\Gamma_0 \text{ (direct product)}$$

Moreover, it holds that

$$\text{(6.7)}\qquad H\otimes\Gamma/(H_0,\ \Gamma_0)\cong H/H_0 \circledtwo \Gamma/\Gamma_0.$$

For, as the group $H \otimes \Gamma$ is isomorphic to $G_{①}$ and the subgroups $H_0$ and $\Gamma_0$ are invariant under the automorphisms $R(\alpha)$ and $\mathscr{L}(a)$ respectively, we can regard $(H_0, \Gamma_0)$ as a subgroup of $G_{①}$. So, in order to prove (6.7) it is sufficient to prove that

$$G_{①}/(H_0, \Gamma_0) \cong H/H_0 ② \Gamma/\Gamma_0.$$

If we define the mapping $\psi$ of $G_{①}$ into $H/H_0 ② \Gamma/\Gamma_0$ by $\psi(a, \alpha) = (aH_0, \alpha\Gamma_0)$, then $\psi$ is a homomorphism of $G_{①}$ onto $H/H_0 ② \Gamma/\Gamma_0$.

For, it is clear that $\psi$ is onto. And $F(\alpha\Gamma_0)$ and $\mathscr{G}(bH_0)$ are considered as the automorphisms of $H/H_0$ and $\Gamma/\Gamma_0$ respectively, so it holds

$$\begin{aligned}
&\psi(a, \alpha)(b, \beta) = \psi(a \cdot F(\alpha)b \cdot l(b, \alpha^{-1}, b), \sigma(\alpha, b^{-1}, \alpha) \cdot \alpha\mathscr{G}(b) \cdot \beta) \\
&= (a \cdot F(\alpha)bH_0, \alpha\mathscr{G}(b) \cdot \beta\Gamma_0) \\
&= (aH_0, \alpha\Gamma_0)(bH_0, \beta\Gamma_0)
\end{aligned}$$

Further, $(a, \alpha) \in \mathit{kern}\, \psi$ if and only if $a \in H_0$ and $\alpha \in \Gamma_0$, that is, $(a, \alpha) \in (H_0, \Gamma_0)$. Thus, we have the theorem:

THEOREM 4. *If, in the groups $H$ and $\Gamma$, at least one of the subgroups $H_0$ and $\Gamma_0$ is not constituted of only one element, then it holds that*

$$H \otimes \Gamma/(H_0, \Gamma_0) \cong H/H_0 ② \Gamma/\Gamma_0,$$
$$(H_0, \Gamma_0) \cong H_0 \times \Gamma_0 \ (\textit{direct product})$$

*where* $H_0 = \{aR(\alpha\mathscr{R}(b)) \cdot a^{-1}R(\alpha)\}$ *and* $\Gamma_0 = \{\mathscr{L}(L(\alpha)a)\beta \cdot \mathscr{L}(a)\beta^{-1}\}$.

## § 7. A special type of group $H ② \Gamma$.

In § 6, we have shown that the group $H \otimes \Gamma$ is isomorphic to a special type of the group $H ② \Gamma$, that is, the group $G_{①}$ which is defined by

$$(7.1) \qquad (a, \alpha)(b, \beta) = (a \cdot F(\alpha)b \cdot l(b, \alpha^{-1}, b), \sigma(\alpha, b^{-1}, \alpha) \cdot \alpha\mathscr{G}(b) \cdot \beta),$$

where $F(\alpha)b = L(\alpha)bR(\alpha^{-1})$ and $\alpha\mathscr{G}(b) = \mathscr{L}(b^{-1})\alpha\mathscr{R}(b)$ are the automorphisms of $H$ and $\Gamma$ respectively. And we have defined

$$(7.2) \qquad l(a, \alpha, b) = aR(\alpha\mathscr{R}(b)) \cdot a^{-1}R(\alpha),$$

$$(7.3) \qquad \sigma(\alpha, a, \beta) = \mathscr{L}(L(\alpha)a)\beta \cdot \mathscr{L}(a)\beta^{-1}.$$

In this section, we investigate the properties of the functions $l(a, \alpha, b)$ and $\sigma(\alpha, a, \beta)$. Next, using those functions, conversely we define a new skew product $H ※ \Gamma$ of the groups $H$ and $\Gamma$ which is the same type as $G_{①}$. And we inquire the necessary and sufficient condition in order that the skew product $H ※ \Gamma$ may be a group, and investigate the structure of this group. Finally we show the example of the group which belongs to the type $H ② \Gamma$ but not to the type $H ※ \Gamma$, and the examples of the type $H ※ \Gamma$.

First, we investigate the properties of the function $l(a, \alpha, b)$ and $\sigma(\alpha, a, \beta)$. For the automorphisms $R(\alpha)$ and $\mathscr{L}(a)$ which define the functions $l(a, \alpha, b)$ and $\sigma(\alpha, a, \beta)$, the system of the conditions (5.2) holds. So, from ④$_1$ and ④$_2$ of (5.2) it follows that

$$l(a, \alpha, b)=L(\mathscr{L}(a)\alpha)b \cdot L(\alpha)b^{-1}, \tag{7.4}$$

$$\sigma(\alpha, a, \beta)=\alpha\mathcal{R}(aR(\beta)) \cdot \alpha^{-1}\mathcal{R}(a). \tag{7.5}$$

Further the function $l(a, \alpha, b)$ has the following properties:

$$l(a \cdot c, \alpha, b)=l(a, \alpha, b) \cdot l(c, \alpha, b), \tag{7.6}$$

$$l(a, \alpha \cdot \beta, b)=l(a, \alpha, b) \cdot l(a, \beta, b), \tag{7.7}$$

$$l(a, \alpha, b \cdot c)=l(a, \alpha, b) \cdot l(a, \alpha, c), \tag{7.8}$$

$$l(l(a, \beta, c), \alpha, b)=l(a, \sigma(\alpha, c, \beta), b)=l(a, \alpha, l(c, \beta, b))=e, \tag{7.9}$$

$$l(F(\beta)a, \alpha, b)=l(a, \alpha\mathscr{F}(c), b)=l(a, \alpha, F(\beta)b)= l(a, \alpha, b). \tag{7.10}$$

PROOF of (7.6)

$$\begin{aligned} l(a\cdot c, \alpha, b)&=(a \cdot c)R(\alpha\mathcal{R}(b)) \cdot (a \cdot c)^{-1}R(\alpha) \\ &=aR(\alpha\mathcal{R}(b)) \cdot cR(\alpha\mathcal{R}(b)) \cdot c^{-1}R(\alpha) \cdot a^{-1}R(\alpha) \\ &=l(a, \alpha, b)\cdot l(c, \alpha, b). \end{aligned}$$

PROOF of (7.7)

$$\begin{aligned} l(a, \alpha \cdot \beta, b)&=aR((\alpha \cdot \beta)\mathcal{R}(b)) \cdot a^{-1}R(\alpha \cdot \beta) \\ &=aR(\alpha\mathcal{R}(b) \cdot \beta\mathcal{R}(b)) \cdot a^{-1}R(\alpha \cdot \beta) \\ &=(aR(\alpha\mathcal{R}(b)))R(\beta\mathcal{R}(b)) \cdot a^{-1}R(\alpha \cdot \beta) \\ &=(aR(\alpha) \cdot l(a, \alpha, b))R(\beta\mathcal{R}(b) \cdot a^{-1}R(\alpha \cdot \beta) \\ &=(aR(\alpha))R(\beta) \cdot l(aR(\alpha), \beta, b) \cdot a^{-1}R(\alpha \cdot \beta) \cdot l(a, \alpha, b) \\ &=l(aR(\alpha), \beta, b) \cdot l(a, \alpha, b) \\ &=L(\mathscr{L}(aR(\alpha))\beta)b \cdot L(\beta)b^{-1} \cdot l(a, \alpha, b) \\ &=L(\mathscr{L}(a)\beta)b \cdot L(\beta)b^{-1} \cdot l(a, \alpha, b) \\ &=l(a, \alpha, b) \cdot l(a \beta, b) \end{aligned}$$

The property (7.8) can be proved in the same way as (7.6).

PROOF of (7.9)

$$l(l(a, \beta, c), \alpha,b) =l(a, \beta, c)R(\alpha\mathcal{R}(b)) \cdot (l(a, \beta, c))^{-1}R(\alpha)=e.$$

Similarly, we have $l(a, \sigma(\alpha, c, \beta), b)=l(a, \alpha, l(c, \beta, b))=e$.

PROOF of (7.10)

$$\begin{aligned} l(F(\beta)a, \alpha, b) &= L(\mathscr{L}(F(\beta)a)\alpha)b \cdot L(\alpha)b^{-1} \\ &= L(\mathscr{L}(L(\beta)a)\alpha)b \cdot L(\alpha)b^{-1} \\ &= L(\mathscr{L}(a)\alpha \cdot \sigma(\beta, a, \alpha))b \cdot L(\alpha)b^{-1} \\ &= L(\mathscr{L}(a)\alpha)b \cdot L(\alpha)b^{-1} \\ &= l(a, \alpha, b). \end{aligned}$$

In the similar way, we have $l(a, \alpha, F(\beta)b) = l(a, \alpha, b)$. Moreover,

$$\begin{aligned} l(a, \alpha\mathscr{F}(c), b) &= aR((\alpha\mathscr{F}(c))\mathscr{R}(b)) \cdot a^{-1}R(\alpha\mathscr{F}(c)) \\ &= aR(\alpha\mathscr{R}(c \cdot b)) \cdot a^{-1}R(\alpha\mathscr{R}(c)) \\ &= aR(\alpha) \cdot l(a, \alpha, c \cdot b) \cdot a^{-1}R(\alpha) \cdot l(a^{-1}, \alpha, c) \\ &= l(a, \alpha, b) \cdot l(e, \alpha, c) \\ &= l(a, \alpha, b). \end{aligned}$$

Further the function has the following property:

(7.11) $$l(a^m, \alpha, b) = l(a, \alpha^m, b) = l(a, \alpha, b^m) = (l(a, \alpha, b))^m,$$
(for an integer $m$).

PROOF. Case 1. $m=0$:
By the definition $l(a, \alpha, b)$ we have

(7.12) $$l(e, \alpha, \mathrm{b}) = l(a, \varepsilon, b) = l(a, \alpha, e) = e.$$

Case 2. $m=-1$:
By (7.6), (7.7), (7.8) and (7.12), we have

(7.13) $$l(a^{-1}, \alpha, b) = l(a, \alpha^{-1}, b) = l(a, \alpha, b^{-1}) = (l(a, \alpha, b))^{-1}.$$

Case 3. $m \neq 0, -1$:
By (7.6), (7.7), (7.8) and (7.13), we have

$$l(a^m, \alpha, b) = l(a, \alpha^m, b) = l(a, \alpha, b^m) = (l(a, \alpha, b))^m.$$

Similarly, we can prove the following properties of the function $\sigma(\alpha, a, \beta)$.

(7.14) $$\sigma(\alpha \cdot \gamma, a, \beta) = \sigma(\alpha, a, \beta) \cdot \sigma(\gamma, a, \beta),$$

(7.15) $$\sigma(\alpha, a \cdot b, \beta) = \sigma(\alpha, a, \beta) \cdot \sigma(\alpha, b, \beta),$$

(7.16) $$\sigma(\alpha, a, \beta \cdot \gamma) = \sigma(\alpha, a, \beta) \cdot \sigma(\alpha, a, \gamma),$$

(7.17) $$\sigma(\sigma(\alpha, b, \gamma), a, \beta) = \sigma(\alpha, l(a, \gamma, b), \beta) = \sigma(\alpha, a, \sigma(\gamma, b, \beta)) = \varepsilon,$$

(7.18) $$\sigma(\alpha\mathscr{F}(b), a, \beta) = \sigma(\alpha, F(\gamma)a, \beta) = \sigma(\alpha, a, \beta\mathscr{F}(b)) = \sigma(\alpha, a, \beta),$$

(7.19) $$\sigma(\alpha^m, a, \beta) = \sigma(\alpha, a^m, \beta) = \sigma(\alpha, a, \beta^m) = (\sigma(\alpha, a, \beta))^m.$$

Now, we proceed to define the new skew product by using the functions which have the above properties.

Let $F$ be a homomorphism of $\Gamma$ into the group $\mathfrak{A}(H)$ of all automorphisms of $H$, and $\mathcal{G}$ be a homomorphism of $H$ into $\mathfrak{A}(\Gamma)$. Then we define:

$$Z(H)=\text{center of } H, \qquad Z(\Gamma)=\text{center of } \Gamma,$$
$$K_F=\{a;\ F(\alpha)a=a,\ \forall\alpha\in\Gamma\}, \qquad K_{\mathcal{G}}=\{\alpha;\ \alpha\mathcal{G}(a)=\alpha,\ \forall a\in H\},$$
$$kern\ \mathcal{G}=\{a;\ \alpha\mathcal{G}(a)=\alpha,\ \forall\alpha\in\Gamma\}, \qquad kern F=\{\alpha;\ F(\alpha)a=a,\ \forall a\in H\},$$
$$\bar{H}_0=Z(H)\cap K_F\cap kern\ \mathcal{G}, \qquad \bar{\Gamma}_0=Z(\Gamma)\cap K_{\mathcal{G}}\cap kern\ F.$$

Further, let $l(a, \alpha, b)$ be a function which has the properties (7.6), (7.7) (7.8), (7.9) and (7.10), having its value in $\bar{H}_0$. Similarly, let $\sigma(\alpha, a, \beta)$ be a function which has the properties (7.14), (7.15), (7.16), (7.17) and (7.18), having its value in $\bar{\Gamma}_0$.

Then, we define the new skew product $H\divideontimes\Gamma$ by the following:

$$(a, \alpha)(b, \beta)=(a\cdot F(\alpha)b\cdot l(b, \alpha^{-1}, b),\ \sigma(\alpha, b^{-1}, \alpha)\cdot\alpha\mathcal{G}(b)\cdot\beta). \tag{7.20}$$

We inquire the conditions in order that the skew product $H\divideontimes\Gamma$ may be a group. From the associativity of the product, for three elements $(a, \alpha)$, $(b,\beta)$ and $(c, \gamma)$ of $H\divideontimes\Gamma$, we have

$$(a\cdot F(\alpha)b\cdot l(b, \alpha^{-1}, b)\cdot\sigma(\alpha, b^{-1}, \alpha)\cdot\alpha\mathcal{G}(b)\cdot\beta)(c, \gamma)$$
$$=(a, \alpha)(b\cdot F(\beta)c\cdot l(c, \beta^{-1}, c),\ \sigma(\beta, c^{-1}, \beta)\cdot\beta\mathcal{G}(c)\cdot\gamma).$$

Consequently, it holds that

$$\begin{aligned} a\cdot F(\alpha)b\cdot l(b, \alpha^{-1}, b)\cdot F(\sigma(\alpha, b^{-1}, \alpha)\cdot\alpha\mathcal{G}(b)\cdot\beta)c\ \cdot \\ \cdot\, l(c, \beta^{-1}\cdot\alpha^{-1}\mathcal{G}(b)\cdot\sigma(\alpha^{-1}, b^{-1}, \alpha), c) \\ =a\cdot F(\alpha)(b\cdot F(\beta)c\cdot l(c, \beta^{-1}, c)) \\ \cdot\, l(b\cdot F(\beta)c\cdot l(c, \beta^{-1}, c), \alpha^{-1}, b\cdot F(\beta)c\cdot l(c, \beta^{-1}, c)). \end{aligned} \tag{7.21}$$

By (7.6), (7.7), (7.8), (7.9) and (7.10), we have

$$F(\alpha\mathcal{G}(b)\cdot\beta)c=F(\alpha\cdot\beta)c\cdot l(b, \alpha^{-1}, c).\cdot l(c, \alpha^{-1}, b).$$

For $\beta=\varepsilon$, it holds that

$$F(\alpha\mathcal{G}(b))c=F(\alpha)c\cdot l(b, \alpha^{-1}, c)\cdot l(c, \alpha^{-1}, b). \tag{7.22}$$

Conversely, suppose that (7.22) holds. Then we can easily prove that (7.21) holds.

Similarly, in the group $\Gamma$ it holds that

$$\alpha\mathcal{G}(F(\beta)a)=\alpha\mathcal{G}(a)\cdot\sigma(\alpha, a^{-1}, \beta)\cdot\sigma(\beta, a^{-1}, \alpha). \tag{7.23}$$

And $(e, \varepsilon)$ is the unit element of $H \divideontimes \Gamma$, and the inverse element of $(a, \alpha)$ is given by

$$(F(\alpha^{-1})a^{-1} \cdot l(a^{-1}, \alpha, a^{-1}), \sigma(\alpha^{-1}, a, \alpha^{-1}) \cdot \alpha^{-1}\mathfrak{F}(a^{-1})).$$

Thus, we have the theorem:

THEOREM 5. *Let $H \divideontimes \Gamma$ be a skew product defined by*

$$(a, \alpha)(b, \beta) = (a \cdot F(\alpha)b \cdot l(b, \alpha^{-1}, b), \sigma(\alpha, b^{-1}, \alpha) \cdot \alpha\mathfrak{F}(b) \cdot \beta),$$

*where $F(\alpha)$ and $\mathfrak{F}(b)$ are the automorphisms of $H$ and $\Gamma$ respectively, and $l(a,\alpha,b)$ and $\sigma(\alpha, a, \beta)$ are the functions having their values in $\bar{H}_0 = Z(H) \cap K_F \cap kern\mathfrak{F}$ and $\bar{\Gamma}_0 = Z(\Gamma) \cap K_{\mathfrak{F}} \cap kern F$ respectively, and further $l(a, \alpha, b)$ and $\sigma(\alpha, a, \beta)$ have the following properties:*

$$l(a \cdot c, \alpha, b) = l(a, \alpha, b) \cdot l(c, \alpha, b), \qquad \sigma(\alpha \cdot \gamma, a, \beta) = \sigma(\alpha, a, \beta) \cdot \sigma(\gamma, a, \beta),$$

$$l(a, \alpha \cdot \beta, b) = l(a, \alpha, b) \cdot l(a, \beta, b), \qquad \sigma(\alpha, a \cdot b, \beta) = \sigma(\alpha, a, \beta) \cdot \sigma(\alpha, b, \beta),$$

$$l(a, \alpha, b \cdot c) = l(a, \alpha, b) \cdot l(a, \alpha, c), \qquad \sigma(\alpha, a, \beta \cdot \gamma) = \sigma(\alpha, a, \beta) \cdot \sigma(\alpha, a, \gamma),$$

$$l(l(a, \beta, c), \alpha, b) = l(a, \sigma(\alpha, c, \beta), b) = l(a, \alpha, l(c, \beta, b)) = e, \qquad \sigma(\sigma(\alpha, b, \gamma), a, \beta) = \sigma(\alpha, l(a, \gamma, b), \beta) = \sigma(\alpha, a, \sigma(\gamma, b, \beta)) = \varepsilon,$$

$$l(F(\beta)a, \alpha, b) = l(a, \alpha\mathfrak{F}(c), b) = l(a, \alpha, F(\beta)b) = l(a, \alpha, b), \qquad \sigma(\alpha\mathfrak{F}(b), a, \beta) = \sigma(\alpha, F(\gamma)a, \beta) = \sigma(\alpha, a, \beta\mathfrak{F}(b)) = \sigma(\alpha, a, \beta).$$

*Then the skew product $H \divideontimes \Gamma$ is a group if and only if the following conditions are satisfied:*

$$F(\alpha\mathfrak{F}(b))a = F(\alpha)a \cdot l(a, \alpha^{-1}, b) \cdot l(b, \alpha^{-1}, a),$$

$$\alpha\mathfrak{F}(F(\beta)a) = \alpha\mathfrak{F}(a) \cdot \sigma(\alpha, a^{-1}, \beta) \cdot \sigma(\beta, a^{-1}, \alpha).$$

In the same way as § 6, we can see that group $H \divideontimes \Gamma$ has the following structure:

If $H_0$ is the subgroup of $H$ generated by all $l(a, \alpha, b)$ and $\Gamma_0$ is the subgroup of $\Gamma$ generated by all $\sigma(\alpha, a, \beta)$, then $(H_0, \Gamma_0)$ is a normal subgroup of $H \divideontimes \Gamma$. And it holds that

$$(H_0, \Gamma_0) \cong H_0 \times \Gamma_0 \qquad \text{(direct product)}$$

$$H \divideontimes \Gamma/(H_0, \Gamma_0) \cong H/H_0 ⓩ \Gamma/\Gamma_0$$

where $H/H_0 ⓩ \Gamma/\Gamma_0$ is a group the product of which is defined by

$$(a_1, \alpha_1)(b_1, \beta_1) = (a_1 \cdot F(\alpha_1)b_1, \alpha_1\mathfrak{F}(b_1) \cdot \beta_1), \; a_1, b_1 \in H/H_0, \; \alpha_1, \beta_1 \in \Gamma/\Gamma_0,$$

and $F(\alpha_1)$ and $\mathfrak{F}(b_1)$ are the automorphisms of $H/H_0$ and $\Gamma/\Gamma_0$ respectively.

Finally we show the example of the group which belongs to the type $H^2\Gamma$, but not to the type $H \divideontimes \Gamma$, and the examples of the group $H \divideontimes \Gamma$. By those examples we can show that the existence of the sequence of the product of the groups:

$$H\times\Gamma\subsetneqq H\circledS\Gamma\subsetneqq H\divideontimes\Gamma\subsetneqq H_2\Gamma.$$

Example 2. Let $G$ be a non-abelian group of order 24 (cf. [1] p. 160), that is

$$G=\{a, b, c;\ a^4=e,\ b^2=e,\ bab=a^{-1},\ c^3=e,\ c^{-1}a^2c=b,\ c^{-1}bc=a^2b,\ a^{-1}ca=c^2a^2b\}.$$

If we set $H=\{a, b\}$ and $\Gamma=\{c\}$, then we have $G=H_2\Gamma$. But this group $G$ does not belong to the type $H\divideontimes\Gamma$.

For, $H$ is a non-abelian group of order 8 and $\Gamma$ is a cyclic group of order 3. So the group $\mathfrak{A}(\Gamma)$ of the automorphisms of $\Gamma$ has order 2, that is, $\mathfrak{A}(\Gamma)=\{I,\varphi\}$, where $I$ is the identity automorphism and $\varphi$ transforms $e$, $c$, $c^2$ to $e, c^2$, $c$ respectively. Therefore, there are only two homomorphisms $\mathscr{F}$ of $H$ into $\mathfrak{A}(\Gamma)$, that is $\mathscr{F}_1$: $H\rightarrow\{I\}$, $\mathscr{F}_2$: $H\rightarrow\{I, \varphi\}$.

Case 1. $\mathscr{F}=\mathscr{F}_1$ and $kern\ F=\{e\}$:

In this case, we have $\bar{\Gamma}_0=\{e\}$. If $G$ belongs to the type $H\divideontimes\Gamma$, then it must hold that $(e, c)\ (a, e)=(F(c)a\cdot l(a, c^{-1}, a), c)$. But in the group $G$, it holds that $ca=a^3c^2$. It is a contradiction.

Case 2. $\mathscr{F}=\mathscr{F}_1$ and $kern\ F=\Gamma$:

In this case, we have $\bar{H}_0=\{a^2\}$ and $\bar{\Gamma}_0=\Gamma$. If $G$ belongs to the type $H\divideontimes\Gamma$, then it must hold that $(e, c)\ (b, e)=(b\cdot l(b, c^{-1}, b), \sigma(c, b^{-1}, c)\cdot c)$. So, $b\cdot l(b,c^{-1}, b)$ must be $b$ or $ba^2$. But in $G$ it holds that $cb=a^2c$. It is a contradiction.

Case 3. $\mathscr{F}=\mathscr{F}_2$:

As $\mathscr{F}_2$ is the homomrphism of $H$ onto $\{I, \varphi\}$, the kernel of $\mathscr{F}_2$ is $\{a\}$. And as $K_{\mathscr{F}_2}=\{e\}$, $\bar{\Gamma}_0=\{e\}$. If the group $G$ belongs to the type $H\divideontimes\Gamma$, then it must hold that $(e, c)(a, e)=(F(c)a\cdot l(a, c^{-1}, a), c)$. But in $G$ it holds that $ca=a^3c^2$. It is a contradiction.

Thus, the group $G$ does not belong to the type $H\divideontimes\Gamma$.

Example 3. Let $R$ be the ring of the integers, and $R(2^2)$ be the ring of the residue classes mod $2^2$. And let $H=\Gamma=R(2^2)$, and consider them as the additive groups. Then the skew product of $H$ and $\Gamma$ is defined by

$$(a, \alpha)(b, \beta)=(a+b-2b^2\alpha,\ -2b\alpha^2+\alpha+\beta).$$

Then we define $F(\alpha)b=b$, $\alpha\mathscr{F}(b)=\alpha$, $l(a, \alpha, b)=2ab\alpha$ and $\sigma(\alpha, a, \beta)=2\alpha\beta a$. Then we have $H_0\neq\{e\}$ and $\Gamma_0\neq\{\varepsilon\}$. And we can easily show that $l(a, \alpha, b)$, $\sigma(\alpha, a, \beta)$, $F(\alpha)$ and $\mathscr{F}(b)$ satisfy the conditions (7.6), (7.7), (7.8), (7.9), (7.10); (7.14), (7.15), (7.16), (7.17), (7.18); (7.22) and (7.23). Therefore, $G$ belongs to the type $H\divideontimes\Gamma$.

Example 4. We show another group of type $H\divideontimes\Gamma$. Let $H$ and $\Gamma$ be two cyclic groups, and the order of $H$ be infinite and the order of $\Gamma$ be 4. And let $P$ and $\Pi$ be the generators of $H$ and $\Gamma$ respectively. Then we define the skew product $G$ of $H$ and $\Gamma$ as follows:

$$(P^i, \Pi^k)(P^j, \Pi^l)=(P^i \cdot P^{j(-1)^k}, \Pi^{k-2j\bar{k}} \cdot \Pi^l), \tag{7.24}$$

where

$$\bar{k}=\begin{cases} 0;\ 2 \mid k, \\ 1;\ 2 \nmid k, \end{cases}$$

here, $2 \mid k$ means that $k$ is divisible by 2, and $2 \nmid k$ means that $k$ is not divisible by 2.
(This is a special case of the skew product obtained by L. Rédei in[5]).

Now, we define

$$F(\Pi^k)P^j=P^{j(-1)^k},\ \Pi^k\mathfrak{S}(P^i)=\Pi^k,$$
$$l(P^i, \Pi^k, P^j)=e,\ \sigma(\Pi^k, P^j, \Pi^l)=\Pi^{j\bar{k}((-1)^l-1)}.$$

Then, $F(\Pi^k)$ is the automorphism of $H$ and it holds that $\Gamma\sim F(\Gamma)$. And, as $K_F=\{e\}$, $\bar{H}_0=\{e\}$. As all $\mathfrak{S}(P^i)$ are the identity automorphisms, it is clear that $H\sim\mathfrak{S}(H)$ and $K_{\mathfrak{S}}=\Gamma$.

The function $\sigma(\Pi^k, P^j, \Pi^l)$ has the following properties:

$$\sigma(\Pi^k, P^j, \Pi^l)=\Pi^{j\bar{l}((-1)^k-1)}, \tag{7.25}$$

$$\sigma(\Pi^k, P^{-j}, \Pi^k)=\Pi^{-2j\bar{k}}. \tag{7.26}$$

For;

$$\sigma(\Pi^k, P^j, \Pi^l)=\begin{cases} \varepsilon \quad : 2 \mid k, \\ \varepsilon \quad : 2 \nmid k,\ 2 \mid l, \\ \Pi^{-2j}: 2 \nmid k,\ 2 \nmid l, \end{cases} \tag{7.27}$$
$$=\Pi^{j\bar{l}((-1)^k-1)}.$$

And,

$$\sigma(\Pi^k, P^{-j}, \Pi^k)=\begin{cases} \varepsilon \quad : 2 \mid k, \\ \Pi^{2j}: 2 \nmid k. \end{cases}$$

$$\Pi^{-2j\bar{k}}=\begin{cases} \varepsilon \quad : 2 \mid k, \\ \Pi^{-2j}=\Pi^{2j}: 2 \nmid k. \end{cases}$$

Further, it holds that $\sigma(\Pi^k, P^i, \Pi^l) \in \bar{\Gamma}_0$.

For, as $Z(\Gamma)=K_{\mathfrak{S}}=\Gamma$, we show that $\sigma(\Pi^k, P^i, \Pi^l) \in kern\ F$.

$$F(\sigma(\Pi^k, P^i, \Pi^l))P^j=\begin{cases} F(\varepsilon)P^j=P^j, \\ F(\Pi^{-2i})P^j=P^j. \end{cases} \qquad \text{(by (7.27))}$$

And from (7.27), it follows that $\{\sigma(\Pi^k, P^i, \Pi^l)\} \neq \{e\}$.

Next, we show that the function $\sigma(\Pi^k, P^i, \Pi^l)$ satisfies the conditions (7.14), (7.15), (7.16), (7.17) and (7.18).

$$\sigma(\Pi^k \cdot \Pi^m, P^i, \Pi^l) = \begin{cases} \varepsilon & : 2 \mid k,\ 2 \mid m, \\ \Pi^{i((-1)^l-1)} : & 2 \mid k,\ 2 \nmid m, \text{ or } 2 \nmid k,\ 2 \mid m, \\ \varepsilon & : 2 \nmid k,\ 2 \nmid m. \end{cases}$$

$$\sigma(\Pi^k, P^i, \Pi^l) \cdot \sigma(\Pi^m, P^i, \Pi^l) = \begin{cases} \varepsilon & : 2 \mid k,\ 2 \mid m, \\ \Pi^{i((-1)^l-1)} & : 2 \mid k,\ 2 \nmid m \text{ or } 2 \nmid k,\ 2 \mid m, \\ \Pi^{2i((-1)^l-1)} = \varepsilon : & 2 \nmid k,\ 2 \nmid m. \end{cases}$$

Therefore, (7.14) holds.

Similarly, we can show that (7.16) holds.

$$\sigma(\Pi^k, P^i \cdot P^j, \Pi^l) = \Pi^{(i+j)\bar{k}((-1)^l-1)} = \sigma(\Pi^k, P^i, \Pi^l) \cdot \sigma(\Pi^k, P^j, \Pi^l).$$

So, (7.15) holds.

$$\sigma(\sigma(\Pi^k, P^j, \Pi^m), P^i, \Pi^l) = \begin{cases} \sigma(\varepsilon, P^i, \Pi^l) = \varepsilon, \\ \sigma(\Pi^{-2j}, P^i, \Pi^l) = \varepsilon. \end{cases} \qquad \text{(by (7.27))}$$

Similarly, we have $\sigma(\Pi^k, P^i, \sigma(\Pi^m, P^j, \Pi^l)) = \varepsilon$.

And $\sigma(\Pi^k, l(P^i, \Pi^m, P^j), \Pi^l) = \sigma(\Pi^k, e, \Pi^l) = \varepsilon$. Therefore (7.17) holds.

Further, we can easily see that $\sigma(\Pi^k \mathfrak{F}(P^j), P^i, \Pi^l) = \sigma(\Pi^k, P^i, \Pi^l \mathfrak{F}(P^j)) = \sigma(\Pi^k, P^i, \Pi^l)$. Also,

$$\sigma(\Pi^k, F(\Pi^m)P^i, \Pi^l) = \begin{cases} \varepsilon & : 2 \mid k, \\ \varepsilon & : 2 \nmid k,\ 2 \mid l, \\ \Pi^{-2i} & : 2 \nmid k,\ 2 \nmid l,\ 2 \mid m, \\ \Pi^{2i} = \Pi^{-2i} : & 2 \nmid k,\ 2 \nmid l,\ 2 \nmid m, \end{cases}$$

and

$$\sigma(\Pi^k, P^i, \Pi^l) = \begin{cases} \varepsilon & : 2 \mid k, \\ \varepsilon & : 2 \nmid k,\ 2 \mid l, \\ \Pi^{-2i} : & 2 \nmid k,\ 2 \nmid l. \end{cases}$$

So, (7.18) holds.

Furthermore $F(\Pi^k \mathfrak{F}(P^i)) P^j = F(\Pi^k) P^j$. And $\Pi^k \mathfrak{F}(F(\Pi^l) P^i) = \Pi^k$, and $\Pi^k \mathfrak{F}(P^i) \cdot \sigma(\Pi^k, P^{-i}, \Pi^l) \cdot \sigma(\Pi^l, P^{-i}, \Pi^k) = \Pi^k$. So, (7.22) and (7.23) hold.

Therefore the group $G$ defined by (7.24) is a group of the type $H※\Gamma$.

## § 8. Some properties of group $H※\Gamma$.

In this section, we investigate the properties of the group $H※\Gamma$ which is defined in § 7.

In § 7, we have shown that:

$$H\divideontimes\Gamma/(H_0,\ \Gamma_0)\cong H/H_0 ⓩ \Gamma/\Gamma_0, \tag{8.1}$$

where $H_0=\{l(a,\alpha,b)\}$ and $\Gamma_0=\{\sigma(\alpha,a,\beta)\}$, and $H/H_0 ⓩ \Gamma/\Gamma_0$ is a group the product of which is defined by

$$(a_1,\ \alpha_1)(b_1,\ \beta_1)=(a_1\cdot F(\alpha_1)b_1,\ \alpha_1\mathscr{F}(b_1)\cdot\beta_1),\ a_1,\ b_1\in H/H_0,\ \alpha_1,\ \beta_1\in\Gamma/\Gamma_0.$$

From the above results, we have the properties:

(1) *If the group $H$ and $\Gamma$ are solvable, then the group is also solvable.*

(2) *The group $H\divideontimes\Gamma$ is not simple.*

Proof of (1).

We prove this, dividing into the two cases:

Case 1. $H_0=\{e\}$ and $\Gamma_0=\{\varepsilon\}$. In this case, by (8.1) we have $H\divideontimes\Gamma\cong H ⓩ \Gamma$. So, by Szép's Theorem, it holds that

$$\begin{cases}(H_1,\ \Gamma_1)\cong H_1\times\Gamma_1 & \text{(direct product)}\\ H ⓩ \Gamma/(H_1,\ \Gamma_1)\cong H/H_1\times\Gamma/\Gamma_1 & \text{(direct product)}\end{cases} \tag{8.2}$$

where $H_1=kern\ \mathscr{F}$ and $\Gamma_1=kern\ F$.

As the subgroup and factor group of the solvable group are solvable, and as the direct product of two solvable groups is also solvable, $H_1$, $\Gamma_1$, $H_1\times\Gamma_1$, $H/H_1$, $\Gamma/\Gamma_1$ and $H/H_1\times\Gamma/\Gamma_1$ are all solvable groups. Therefore the group $H\divideontimes\Gamma$ is solvable, because the extension of a solvable group by a solvable group is solvable.

Case 2. The case where at least one of $H_0$ and $\Gamma_0$ is not constituted of only one element.

By case 1, the group $H/H_0 ⓩ \Gamma/\Gamma_0$ is solvable. So, by (8.1) we can show that the group $H\divideontimes\Gamma$ is solvable.

Proof of (2).

It follows easily from (8.1) and (8.2), so we omit it.

Further the group $H\divideontimes\Gamma$ has the following property:

(3) *If the groups $H$ and $\Gamma$ are a torsion group, then the group $H\divideontimes\Gamma$ is also a torsion group.*

Proof. For $(a,\alpha)\in H\divideontimes\Gamma$ and a positive integer $i$, it holds that

$$(a,\ \alpha)^i=(a\cdot F(\alpha)a\cdot\cdots\cdot F(\alpha^{i-1})a\cdot(l(a,\ \alpha^{-1},\ a))^{1^2+2^2+\cdots+(i-1)^2},$$
$$(\sigma(\alpha,\ a^{-1},\ \alpha))^{1^2+2^2+\cdots+(i-1)^2}\cdot\alpha\mathscr{F}(a^{i-1})\cdot\cdots\cdot\alpha\mathscr{F}(a)\cdot\alpha).$$

Let $m$, $n$, $p$ and $q$ be the orders of elements $a$, $\alpha$, $a\cdot F(\alpha)a\cdot\cdots\cdot F(\alpha^{n-1})\,a$ and $\alpha\mathscr{F}(a^{m-1})\cdot\cdots\cdot\mathscr{F}(a)\cdot\alpha$ respectively. Then follows

$$(a,\ \alpha)^{mnpq}=((l(a,\ \alpha^{-1},\ a))^{1^2+2^2+\cdots+(mnpq-1)^2},\ (\sigma(\alpha,\ ^{-1},\ \alpha))^{1^2+2^2+\cdots+(mnpq-1)^2}).$$

Further, let $s$ and $t$ be the orders of the elements $(l(a, \alpha^{-1}, a))^{1^2+2^2+\cdots+(mnpq-1)^2}$ and $((\sigma(\alpha, a^{-1}, \alpha))^{1^2+2^2+\cdots+(mnpq-1)^2}$ respectively. Then we have $(a, \alpha)^N=(e, \varepsilon)$ where $N=mnpqst$. Therefore, $H\divideontimes\Gamma$ is a torsion group.

Let $H_S$ be the subgroup of $H$ generated by $r_0(a, \alpha, b\ \beta)$, where $r_0(a, \alpha, b, \beta)$ $=a\cdot F(\alpha)b\cdot F(\alpha\cdot\beta\cdot\alpha^{-1})a^{-1}\cdot F(\alpha\cdot\beta\cdot\alpha^{-1}\cdot\beta^{-1})b^{-1}\cdot l(a, \beta^{-1}, a)\cdot l(b, \alpha, b)\cdot$ $l(a, \alpha\cdot\beta^{-1}, b)\cdot l(b, \alpha\cdot\beta^{-1}, a)$. The subgroup $H_S$ contains the commutator subgroup of $H$, so it is a normal subgroup of $H$. Similary let $\Gamma_S$ be a subgroup of $\Gamma$ generated by $\rho_0(a, \alpha, b, \beta)$, where $\rho_0(a, \alpha, b,\beta)=\sigma(\alpha, a\cdot b^{-1},\beta)\cdot\sigma(\beta, a\cdot b^{-1}, \alpha)\cdot$ $\sigma(\beta, a, \beta)\cdot\sigma(\alpha, b^{-1}, \alpha)\cdot\beta^{-1}\mathfrak{F}(b^{-1}\cdot a^{-1}\cdot b\cdot a)\cdot\alpha^{-1}\mathfrak{F}(a^{-1}\cdot b\cdot a)\cdot\beta\mathfrak{F}(a)\cdot\alpha$. Then $\Gamma_s$ is a normal subgroup of $\Gamma$. Thus, we have the property:

(4) *$(H_S, \Gamma_S)$ is a normal subgroup of the group $H\divideontimes\Gamma$, and it holds that*

$$H\divideontimes\Gamma/(H_S, \Gamma_S)\cong H/H_S\oplus\Gamma/\Gamma_S \qquad \textit{(direct sum)}. \tag{8.3}$$

PROOF. First, we show that $(H_S, \Gamma_S)$ is a subgroup of $H\divideontimes\Gamma$. $(H_S, \Gamma_S)$ has $(e, \varepsilon)$ as the unit element. And it holds that

$$(r_0(a, \alpha, b, \beta))^{-1}=r_0(F([\alpha, \beta])b, [\alpha, \beta]\cdot\beta\cdot([\alpha, \beta])^{-1}, F([\alpha, \beta])a, [\alpha, \beta]\cdot\alpha\cdot([\alpha, \beta])^{-1}), \tag{8.4}$$

$$r_0(a, \varepsilon, e, \beta)=a\cdot F(\beta)a^{-1}\cdot l(a, \beta^{-1}, a), \tag{8.5}$$

where $[\alpha, \beta]=\alpha\cdot\beta\cdot\alpha^{-1}\cdot\beta^{-1}$. For $r_s\in H_s$, $\rho_s\in\Gamma_s$, we have

$$(r_s. \rho_s)^{-1}=(F(\rho_s^{-1})r_s^{-1}\cdot l(r_s, \rho_s, r_s), \sigma(\rho_s, r_s, \rho_s)\cdot\rho_s^{-1}\mathfrak{F}(r_s^{-1})).$$

And from (8.5), follows $F(\rho_s^{-1})r_s^{-1}\cdot l(r_s, \rho_s, r_s)=r_s^{-1}\cdot r_0(r_s, \varepsilon, e, \rho_s^{-1})$. From this and (8.4), $F(\rho_s^{-1})r_s^{-1}\cdot l(r_s, \rho_s, r_s)\in H_s$. Similarly, $\sigma(\rho_s, r_s, \rho_s)\cdot\rho_s^{-1}\mathfrak{F}(r_s^{-1})\in\Gamma_s$. Consequently, we have $(r_s, \rho_s)^{-1}\in(H_s, \Gamma_s)$. For $(r_s, \rho_s)$, $('r_s, '\rho_s)\in(H_s, \Gamma_s)$, we have $(r_s, \rho_s)('r_s, '\rho_s)=(r_s\cdot F(\rho_s)'r_s\cdot l('r_s, \rho_s^{-1}, 'r_s), \sigma(\rho_s, 'r_s^{-1}, \rho_s)\cdot\rho_s\mathfrak{F}('r_s)\cdot'\rho_s)$. By (8.5), $F(\rho_s)'r_s\cdot l('r_s, \rho_s^{-1}, 'r_s)='r_s\cdot r_o('r_s^{-1}, \varepsilon, e, \rho_s)$, so follows $r_s\cdot F(\rho_s)'r_s\cdot l('r_s, \rho_s^{-1}, 'r_s)\in H_s$. Similarly $\sigma(\rho_s, 'r_s^{-1}, \rho_s)\cdot\rho_s\mathfrak{F}('r_s)\cdot'\rho_s\in\Gamma_s$. Consequently, we have $(r_s, \rho_s)('r_s, '\rho_s)$ $\in(H_s, \Gamma_s)$. Therefore $(H_s, \Gamma_s)$ is a subgroup of $H\divideontimes\Gamma$.

Next, we show that $(H_s, \Gamma_s)$ is a normal subgroup of $H\divideontimes\Gamma$. In order to prove this, it is sufficient to prove that $(H_s, \Gamma_s)$ contains the commutator subgroup of $H\divideontimes\Gamma$. For $(a, \alpha), (b, \beta)\in H\divideontimes\Gamma$, $(a,\alpha)(b, \beta)(a, \alpha)^{-1}(b, \beta)^{-1}=(r_o(a, \alpha, b, \beta),$ $\rho_o(F(\beta^{-1})b^{-1}, \beta^{-1}\mathfrak{F}(b^{-1}), F(\alpha^{-1})a^{-1}, \alpha^{-1}\mathfrak{F}(a^{-1})))$, so $[(a, \alpha), (b, \beta)]\in(H_s, \Gamma_s)$. Therefore, $(H_s, \Gamma_s)$ contains the commutator subgroup of $H\divideontimes\Gamma$. So it is a normal subgroup.

We proceed to prove (8.3). First, we have $F(\gamma)r_0(a, \alpha, b, \beta)=r_0(F(\gamma)a,$ $\gamma\cdot\alpha\cdot\gamma^{-1}, F(\gamma)b, \gamma\cdot\beta\cdot\gamma^{-1})$, that is, $F(\gamma)H_s\subset H_s$ for every $\gamma\in\Gamma$. Similarly, it holds $\Gamma_s\mathfrak{F}(c)\subset\Gamma_s$, for every $c\in H$. Next, we define a mapping $\psi$ of $H\divideontimes\Gamma$ into $H/H_s\oplus\Gamma/\Gamma_s$ by $\psi(a,\alpha)=(aH_s, \alpha\Gamma_s)$. Then $\psi$ is a homomorphism of $H\divideontimes\Gamma$ onto $H/H_s\oplus\Gamma/\Gamma_s$. For, it is clear that $\psi$ is onto. And from the definition of $\rho_o$ (a, $\alpha$, $b$, $\beta$), it holds

$$\rho_o(e, \alpha, b, \varepsilon)=\sigma(\alpha, b^{-1}, \alpha)\cdot\alpha^{-1}\mathfrak{F}(b)\cdot\alpha. \tag{8.6}$$

Further,

$$
\begin{aligned}
&\psi(a, \alpha)(b, \beta)=\psi(a\cdot F(\alpha)b \cdot l(b, \alpha^{-1}, b), \sigma(\alpha, b^{-1}, \alpha) \cdot \alpha\mathscr{F}(b) \cdot \beta) \\
&=(a \cdot F(\alpha)b \cdot l(b, \alpha^{-1}, b)H_s, \sigma(\alpha, b^{-1}\alpha) \cdot \alpha\mathscr{F}(b) \cdot \beta\Gamma_s) \\
&=(a \cdot bH_s, \alpha \cdot \beta\Gamma_s) \qquad \text{(by (8.5), (8.6))} \\
&=(aH_s+bH_s, \alpha\Gamma_s+\beta\Gamma_s)
\end{aligned}
$$

And,

$$
\begin{aligned}
\psi(a, \alpha)\psi(b, \beta)&=(aH_s, \alpha\Gamma_s)+(bH_s, \beta\Gamma_s) \\
&=(aH_S+bH_S, \alpha\Gamma_S+\beta\Gamma_S)
\end{aligned}
$$

Therefore,

$$
\psi(a, \alpha)(b, \beta)=\psi(a, \alpha)\psi(b, \beta)=(aH_S+bH_S, \alpha\Gamma_S+\beta\Gamma_S).
$$

Moreover, $(a, \alpha) \in kern\psi$ if and only if $a \in H_S$ and $\alpha \in \Gamma_S$. Thus, we have (8.3).

Finally, we prove the property:

(5) *Let $H$ and $\Gamma$ be two abelian groups. Then the group $H \divideontimes \Gamma$ is abelian if and only if $H \divideontimes \Gamma = H \times \Gamma$ (direct product).*

Proof. Suppose that for $(a, \alpha), (b, \beta) \in H \divideontimes \Gamma$, $(a, \alpha)(b, \beta)=(b, \beta)(a, \alpha)$. Then we have $a \cdot F(\alpha)b \cdot l(b, \alpha^{-1}, b) = b \cdot F(\beta)a \cdot l(a, \beta^{-1}, a)$ and $\sigma(\alpha, b^{-1}, \alpha) \cdot \alpha\mathscr{F}(b) \cdot \beta = \sigma(\beta, a^{-1}, \beta) \cdot \beta\mathscr{F}(a) \cdot \alpha$. From the former for $a=e$ it follows that $F(\alpha)b \cdot l(b, \alpha^{-1}, b)=b$, and from the latter for $\beta=\varepsilon$ follows $\sigma(\alpha, b^{-1}, \alpha) \cdot \alpha\mathscr{F}(b)=\alpha$. Therefore, in this case the product of $H \divideontimes \Gamma$ must be $(a, \alpha)(b, \beta) = (a \cdot b, \alpha \cdot \beta)$. And the sufficiency of this condition is obvious.

## § 9. Final remarks.

In this section, we study the special cases of the skew product $H \divideontimes \Gamma$ the product of which is defined by

$$
\text{(9.1)} \qquad (a, \alpha)(b, \beta)=(aR(\beta) \cdot L(\alpha)b, \alpha\mathscr{R}(b) \cdot \mathscr{L}(a)\beta), \\
(aR(\varepsilon)=L(\varepsilon)a=a, \alpha\mathscr{R}(e)=\mathscr{L}(e)\alpha=\alpha).
$$

And further we clarify the relation between the results which we have obtained in § 2 and § 3 and those obtained by L. Rédei and A. Stöhr.

We call $H \divideontimes \Gamma$, $k$-fach degenerated, when just $k$ of the following four conditions

$$
aR(\beta)=a, \; L(\alpha)b=b, \; \alpha\mathscr{R}(b)=\alpha, \; \mathscr{L}(a)\beta=\beta
$$

is identically satisfied.

For the sake of the symmetry of (9.1), there are two different types of the 1-fach degenerated case of $H \divideontimes \Gamma$, that is,

1–1 type: $(a, \alpha)(b, \beta)=(a \cdot L(\alpha)b, \alpha\mathcal{R}(b) \cdot \mathcal{L}(a)\beta)$,
1–2 type: $(a, \alpha)(b, \beta)=(aR(\beta) \cdot b, \alpha\mathcal{R}(b) \cdot \mathcal{L}(a)\beta)$.

There are four different types of the 2–fach degenerated case, that is,

2–1 type: $(a, \alpha)(b, \beta)=(a \cdot b, \alpha\mathcal{R}(b) \cdot \mathcal{L}(a)\beta)$,
2–2 type: $(a, \alpha)(b, \beta)=(a \cdot L(\alpha)b, \alpha\mathcal{R}(b) \cdot \beta)$,
2–3 type: $(a, \alpha)(b, \beta)=(aR(\beta) \cdot b, \alpha\mathcal{R}(b) \cdot \beta)$,
2–4 type: $(a, \alpha)(b, \beta)=(aR(\beta) \cdot b, \alpha \cdot \mathcal{L}(a)\beta)$.

There are two different types of the 3–fach degenerated case, that is,

3–1 type: $(a, \alpha)(b, \beta)=(a \cdot b, \alpha \cdot \mathcal{L}(a)\beta)$,
3–2 type: $(a, \alpha)(b, \beta)=(a \cdot b, \alpha\mathcal{R}(b) \cdot \beta)$.

And the 4–fach degenerated case of $H \divideontimes \Gamma$ is the direct product of $H$ and $\Gamma$.

Here, we restrict ourselves to the types 2–1, 2–2, and 2–3. In each case, by Theorem 1 we investigate the condition in order that the skew product may be a group, and inquire the form of the product of $G_1$ which is isomorphic to the above groups, by using (3.10). About the other special types of $H \divideontimes \Gamma$, we can obtain the results similar to the above types, by using Theorem 1 and (3.10).

(1) 2–1 type:

The skew product of 2–1 type is a group with the unit element $(e, \varepsilon)$ if and only if the following conditions are satisfied:

$$(9.2)\quad \begin{cases} ① \ \varepsilon\mathcal{R}(a)=\mathcal{L}(a)\varepsilon=\varepsilon, & ④ \ \mathcal{L}(a \cdot b)\beta=\mathcal{L}(a)(\mathcal{L}(b)\beta), \\ ② \ (\mathcal{L}(a)\beta)\mathcal{R}(b)=\mathcal{L}(a)(\beta\mathcal{R}(b)), & ⑤ \ (\beta \cdot \gamma)\mathcal{R}(a)=\beta\mathcal{R}(a) \cdot \gamma\mathcal{R}(a), \\ ③ \ \beta\mathcal{R}(a \cdot b)=(\beta\mathcal{R}(a))\mathcal{R}(b), & ⑥ \ \mathcal{L}(a)(\beta \cdot \gamma)=\mathcal{L}(a)\beta \cdot \mathcal{L}(a)\gamma. \end{cases}$$

From the conditions ⑤ and ⑥ of (9.2), $\mathcal{R}(a)$ and $\mathcal{L}(a)$ are the automorphisms of $\Gamma$. Consequently, the product of the group $G_1$ which is isomorphic to 2–1 type of group $H \divideontimes \Gamma$, is as follows:

$$(9.3)\qquad (a, \alpha)(b, \beta)=(a{\cdot}b, \alpha\mathcal{S}(b) \cdot \beta),$$

where $\alpha\mathcal{S}(b)=\mathcal{L}(b^{-1})\alpha\mathcal{R}(b)$ is an automorphism of $\Gamma$.

Therefore, from (9.3), the group of 2–1 type is isomorphic to the semi-direct product of the groups $H$ and $\Gamma$.

Note 1. The groups of 3–1 type are the special cases of the group of 2–1 type. So, from the above results we can see that the groups of 3–1 type are isomorphic to the semi-direct product of $H$ and $\Gamma$.

(2) 2–2 type:

The skew product of this type is $H_2\Gamma$ defined by L. Rédei in [4] (cf. Remark 1). From Theorem 1, we can obtain the condition in order that the skew product of 2–2 type may be a group with the unit element $(e, \varepsilon)$, that is,

$$(9.4)\quad \begin{cases} ①_1\ L(\alpha)e=e, & ①_2\ \varepsilon\mathcal{R}(a)=\varepsilon, \\ ②_1\ L(\alpha\cdot\beta)b=L(\alpha)(L(\beta)b), & ②_2\ \beta\mathcal{R}(a\cdot b)=(\beta\mathcal{R}(a))\mathcal{R}(b), \\ ③_1\ L(\alpha)(b\cdot c)=L(\alpha)b\cdot L(\alpha\mathcal{R}(b))c, & ③_2\ (\beta\cdot\gamma)\mathcal{R}(a)=\beta\mathcal{R}(L(\gamma)a)\cdot\gamma\mathcal{R}(a). \end{cases}$$

NOTE 2. In this type, the assumptions $L(\varepsilon)a=a$ and $\alpha\mathcal{R}(e)=\alpha$ written in (9.1) are contained in the necessary conditions.

This system (9.4) of the conditions coincides with the results of L. Rédei (cf. [4] Theorem 6).

(3) 2–3 type:

The skew product of this type is $H^{2'}\Gamma$ defined by L. Rédei and A. Stöhr in [6] (cf. Remark 1).

The condition in order that the skew product of 2–3 type may be a group with the unit element $(e, \varepsilon)$ is as follows:

$$(9.5)\quad \begin{cases} ①_1\ eR(\alpha)=e, & ①_2\ \varepsilon\mathcal{R}(a)=\varepsilon, \\ ②_1\ bR(\alpha\cdot\beta)=(bR(\alpha))R(\beta), & ②_2\ \beta R(a\cdot b)=(\beta\mathcal{R}(a))\mathcal{R}(b), \\ ③_1\ (b\cdot c)R(\alpha)=bR(\alpha)\cdot cR(\alpha), & ③_2\ (\beta\cdot\gamma)\mathcal{R}(a)=\beta\mathcal{R}(a)\cdot\gamma\mathcal{R}(a), \\ ④_1\ aR(\beta\mathcal{R}(c))=aR(\beta), & ④_2\ \alpha\mathcal{R}(bR(\gamma))=\alpha\mathcal{R}(b). \end{cases}$$

NOTE 3. In this type, the assumptions $aR(\varepsilon)=a$ and $\alpha\mathcal{R}(e)=\alpha$ written in (9.1) are contained in the necessary conditions.

By $③_1$ and $③_2$, $R(\alpha)$, $\mathcal{R}(a)$ in (9.5) are the automorphisms of $H$ and $\Gamma$ respectively. And the group $G_1$ which is isomorphic to the group of 2–3 type is as follows:

$$(9.6)\qquad (a, \alpha)(b, \beta)=(a\cdot bR(\alpha^{-1}),\ \alpha\mathcal{R}(b)\cdot\beta).$$

(9.5) and (9.6) are the results obtained by L. Rédei and A. Stöhr in [6].

In conclusion, I wish to express my hearty thanks to Prof. K. Morinaga for his encouragemet and kind guidance in the course of the present work.

## References


[1] W. Burnside, *Theory of groups of finite order*, 2nd. ed. Cambrige, 1911.

[2] F. Haimo, *Semi-direct products with ample homomorphisms*, Trans. Amer. Math. Soc., **84** (1957), 401-425.

[3] R. Kochendörffer, *Zur Theorie der Rédeischen schiefen produkte*, J. reine angew. Math., **192** (1953), 96-101.

[4] L. Rédei, *Die Anwendung des schiefen produktes in Gruppentheorie*, J. reine angew. Math., **188** (1950), 201-227.

[5] ———, *Zur Theorie der factorisierbaren Gruppen*, I., Acta Math. Acad. Sci. Hungaricae, **1** (1950), 74-98.

[6] L. Rédei und A. Stöhr, *Über ein spezille schiefes Produkt in Gruppentheorie*, Acta Sci. Math., **15** (1953), 7-11.

[7] R. Rühs, *Über ein spezielle Rédeisches schiefes Produkt in der Gruppentheorie*, Acta Sci. Math., **16** (1955), 160-164.

[8] ———, *Über die einfach ausgearteten Rédeischen schiefen Produkte*, J. reine angew. Math., **198** (1957), 81-86.

[9] ———, *Über das allgemeine Rédeische schiefe Produkt*, J. reine angew. Math., **200** (1958), 99-111.

[10] J. Szép, *On the structure of groups which can be represented as the product of two subgroups*, Acta Sci. Math., **12**A (1950), 57-61.

*Shinonome Bunkô,*
*Hiroshima University.*