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Rational curves on a smooth Hermitian surface

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Abstract. We study the set R of nonplanar rational curves of degree $d < q+2$ on a smooth Hermitian surface X of degree $q+1$ defined over an algebraically closed field of characteristic $p > 0$, where q is a power of p. We prove that R is the empty set when $d < q+1$. In the case where $d = q+1$, we count the number of elements of R by showing that the group of projective automorphisms of X acts transitively on R and by determining the stabilizer subgroup. In the special case where X is the Fermat surface, we present an element of R explicitly.

1. Introduction

Let q be a power of a prime p, and k an algebraic closure of the finite field $\mathbb{F}_q$. For a matrix m with entries in k, we denote by $m^{(q)}$ the matrix whose entries are the q-th power of those of m. We denote by a column vector $\boldsymbol{x} = {}^{\mathrm{t}}(x_0, x_1, x_2, x_3)$ a point in the k-projective space $\mathbb{P}^3$. Let A be a nonzero 4-by-4 matrix with entries in k. A k-Hermitian surface X_A is defined by

$$X_A := \{\boldsymbol{x} \in \mathbb{P}^3 \mid {}^{\mathrm{t}}\boldsymbol{x} A \boldsymbol{x}^{(q)} = 0\}.$$

If A is a Hermitian matrix, namely A has the entries in $\mathbb{F}_{q^2}$ and ${}^{\mathrm{t}}A = A^{(q)}$, the surface X_A is called a Hermitian surface. It is easily shown that X_A is smooth if and only if A is invertible.

The geometry of Hermitian varieties was systematically investigated by B. Segre in [8]. Especially, the number of linear spaces lying on a Hermitian variety and their configuration were considered. It was shown that the numbers of points and lines on a smooth Hermitian surface in $\mathbb{P}^3(\mathbb{F}_{q^2})$ are equal to $(q^3+1)(q^2+1)$ and $(q^3+1)(q+1)$ respectively, and no plane is contained. Further, the set of points and lines on a smooth Hermitian surface forms a block design, see also [3]. In recent years, the number of rational normal curves totally tangent to a smooth Hermitian variety X has been determined

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in [10] by considering the action of the automorphism group of X on the set of the curves. In [11], non-singular conics totally tangent to the smooth Hermitian curve of degree 6 in characteristic 5 were utilized for a geometric construction of strongly regular graphs. On the other hand, projective isomorphism classes of degenerate Hermitian varieties of corank 1 and the automorphism group of each isomorphism class have been determined in [7].

Let A be an invertible 4-by-4 matrix with entries in k. We will be concerned with rational curves of degree > 1 on a smooth k-Hermitian surface X_A. Let d be a positive integer and F a 4-by-$(d+1)$ matrix of $\operatorname{rank}(F) \geq 2$ with entries in k. A rational curve C_F of degree d in $\mathbb{P}^3$ is the image of a rational map

$$\mathbb{P}^1 \ni {}^{\mathrm{t}}(s,t) \mapsto F\ {}^{\mathrm{t}}(s^d, s^{d-1}t, \dots, st^{d-1}, t^d) \in \mathbb{P}^3. \tag{1}$$

We call $\operatorname{rank}(F)$ the rank of the curve C_F. If $\operatorname{rank}(F) = 2$, then C_F degenerates to a line. If $\operatorname{rank}(F) = 3$, then C_F degenerates to a plane curve of degree ≥ 2. When $\operatorname{rank}(F) = 4$, the curve C_F is nondegenerate and is a space curve of degree ≥ 3. Then C_F is said to be nonplanar, namely C_F is not contained in any plane. Thus the study of rational curves of rank 2 on X_A is reduced to that of lines on X_A. Further, an algebraic curve of rank 3 on X_A is a smooth k-Hermitian curve of degree $q+1$, which is of genus $q(q-1)/2 > 0$. Hence we may restrict ourselves to the case of rank 4.

Our results are as follows:

Theorem 1. *There is no nonplanar rational curve of degree $\leq q$ on a smooth k-Hermitian surface.*

Let R be the set of nonplanar rational curves of degree $q+1$ on a smooth k-Hermitian surface X_A. As will be seen later, the set R is nonempty and each element is projectively isomorphic over k to the smooth curve

$$C_0 := \{{}^{\mathrm{t}}(s^{q+1}, s^q t, st^q, t^{q+1}) \in \mathbb{P}^3 \mid {}^{\mathrm{t}}(s,t) \in \mathbb{P}^1\}.$$

We denote by $\operatorname{Aut}(X_A)$ the group of projective automorphisms of X_A. Let n be a positive integer. We deal with the group $\mathrm{PGU}_n(\mathbb{F}_{q^2})$ defined by

$$\{Q \in \mathrm{GL}_n(\mathbb{F}_{q^2}) \mid {}^{\mathrm{t}}QQ^{(q)} = I\}/\boldsymbol{\mu}_{q+1}I,$$

where $\boldsymbol{\mu}_{q+1}$ denotes the group of $(q+1)$-th roots of unity and I denotes the unit matrix. As is well-known, the group $\operatorname{Aut}(X_A)$ is isomorphic to $\mathrm{PGU}_4(\mathbb{F}_{q^2})$. Then we shall prove the following theorem.

Theorem 2. *The group* $\mathrm{Aut}(X_A)$ *acts transitively on the set* R, *and the stabilizer subgroup is isomorphic to* $\mathrm{PGU}_2(\mathbb{F}_{q^4})$.

By Theorem 2, the cardinality of R is equal to $|\mathrm{PGU}_4(\mathbb{F}_{q^2})|/|\mathrm{PGU}_2(\mathbb{F}_{q^4})|$. We know by [6, pp. 64–65] that

$$|\mathrm{PGU}_4(\mathbb{F}_{q^2})| = q^6(q^4-1)(q^3+1)(q^2-1) \quad \text{and} \quad |\mathrm{PGU}_2(\mathbb{F}_{q^4})| = q^2(q^4-1).$$

Thus we have the following.

Corollary 1. $|R| = q^4(q^3+1)(q^2-1)$.

The number $|R|$ is 432, 18144, 249600, 1890000, 39645312, 383162400, ... as $q = 2, 3, 4, 5, 7, 9, \ldots$.

In the special case where $A = I$, that is, where the surface X_A is the Fermat surface, we can explicitly give an element C_{F_J} of R such as

$$\{{}^{\mathrm{t}}(\eta^{-q}\xi^q s^{q+1} - \eta^{-q}t^{q+1}, s^q t, st^q, \omega\eta^{-1}\xi s^{q+1} + \omega\eta^{-1}t^{q+1}) \in \mathbb{P}^3 \mid {}^{\mathrm{t}}(s,t) \in \mathbb{P}^1\},$$

where ω, ξ, and η are elements of $\mathbb{F}_{q^2}$ satisfying $\omega^{q+1} = -1$, $\xi^{q+1} = 1$ with $\xi^2 \neq -1$, and $\eta^{q+1} = \xi^q + \xi$. Note that $\eta \neq 0$ because $\xi^2 \neq 0, -1$. The curve C_{F_J} is smooth since it is projectively isomorphic to the smooth curve C_0. On the other hand, a complete set of representatives for $\mathrm{Aut}(X_I)$ can be taken from $\mathrm{GL}_4(\mathbb{F}_{q^2})$ (see Lemma 4). Therefore we have the following.

Corollary 2. *All nonplanar rational curves of degree* $q+1$ *on* X_I *are projectively isomorphic over* $\mathbb{F}_{q^2}$ *to the smooth curve* C_{F_J}.

In the case where $q = 2$, we have $|X_I(\mathbb{F}_{q^2})| = 45$ where $X_I(\mathbb{F}_{q^2})$ denotes the set of $\mathbb{F}_{q^2}$-rational points of X_I, and $\mathrm{Aut}(X_I)$ is of order 25920. Then $|C_F(\mathbb{F}_{q^2})| = 5$ for each nonplanar cubic C_F on X_I. We can actually obtain by computation 432 nonplanar cubics on X_I and the stabilizer subgroup of $\mathrm{Aut}(X_I)$ fixing C_{F_J} of order 60. By restricting X_I to $X_I(\mathbb{F}_{q^2})$, we can verify that each cubic intersects 150 other cubics at a single point, 40 other cubics at two points and another cubic at five points. Here, when we say two cubics C_F, $C_{F'}$ intersect at n points we mean $|C_F(\mathbb{F}_{q^2}) \cap C_{F'}(\mathbb{F}_{q^2})| = n$. We can also verify that $\mathrm{Aut}(X_I)$ acts transitively on $X_I(\mathbb{F}_{q^2})$ and the stabilizer subgroup is of order 576, and furthermore, there are 48 cubics passing through each point of $X_I(\mathbb{F}_{q^2})$. These computational data files obtained by using GAP [4] are available upon request addressed to the author.

We give a brief outline of our paper. In the next section, we prove Theorem 1. By the same argument, we show directly that each irreducible conic, which is a rational curve of rank 3, is not contained in X_A. In section 3, we give a bijection between the set R and the quotient of certain sets consisting of invertible 4-by-4 matrices, by showing basic lemmas. In section

4, we first prove two lemmas which are necessary for our proof of Theorem 2. We prove Theorem 2 in the last of the section.

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2. Proof of Theorem 1

Proof (Proof of Theorem 1). Suppose that a nonplanar rational curve C_F defined by (1) is contained in a smooth k-Hermitian surface X_A. Denoting by $b_{i,j}$ the entries of the $(d+1)$-by-$(d+1)$ matrix ${}^{t}FAF^{(q)}$, one has the identity

$$\sum_{i,j=0}^{d} b_{i,j} s^{d-i+q(d-j)} t^{i+qj} \equiv 0. \tag{2}$$

Therefore if $d < q$, all the coefficients $b_{i,j}$ must vanish because the exponents $(i+qj)$'s are all different. This implies that ${}^{t}FAF^{(q)} = O$, but it is a contradiction. In fact, since $\operatorname{rank}(F) = 4$ by definition, we can take an invertible matrix F^* consisting of linearly independent 4 column vectors of F. Then, however, ${}^{t}F^*AF^{*(q)}$ must be O. If $d = q$, the coefficients $b_{i,j}$ must vanish except for $b_{q,l-1} = -b_{0,l}$ with $1 \le l \le q$. This implies that $\operatorname{rank}({}^{t}FAF^{(q)}) \le 2$, but it is a contradiction by the argument above. Hence we conclude that $C_F \not\subset X_A$. □

Remark 1. *We can similarly give a proof for the case of irreducible conics. In fact, since an irreducible conic C_F is of rank 3, we can make an invertible matrix F^* consisting of linearly independent 3 column vectors of F and a vector linearly independent to those vectors. Suppose that $C_F \subset X_A$. Since $d = 2 \le q$, one has $\operatorname{rank}({}^{t}FAF^{(q)}) \le 2$ in the same argument as the above proof. Therefore the 4-by-4 matrix ${}^{t}F^*AF^{*(q)}$ must be of rank 3 at the most, but ${}^{t}F^*AF^{*(q)}$ is of rank 4 by definition. This is a contradiction. As we have seen, this proof is valid for rational curves which are of rank ≥ 3 and degree $\le q$.*

3. Basic lemmas

In this section, we will prove some basic lemmas to prepare for our proof of Theorem 2. The following lemma gives a necessary and sufficient condition for a nonplanar rational curve of degree $q+1$ to be on a smooth k-Hermitian surface.

Lemma 1. *Let C_F be a nonplanar rational curve of degree $q+1$ defined by (1). The curve C_F is contained in a smooth k-Hermitian surface X_A if and only if the $(q+2)$-by-$(q+2)$ matrix ${}^{t}FAF^{(q)}$ is of the form*

$$\begin{pmatrix} 0 & b_{0,1} & 0,\dots,0 & 0 & b_{0,q+1} \\ 0 & b_{1,1} & 0,\dots,0 & 0 & b_{1,q+1} \\ 0 & 0 & 0,\dots,0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0,\dots,0 & 0 & 0 \\ -b_{0,1} & 0 & 0,\dots,0 & -b_{0,q+1} & 0 \\ -b_{1,1} & 0 & 0,\dots,0 & -b_{1,q+1} & 0 \end{pmatrix}.$$

If the above condition is satisfied, the matrix F is of the form

$$(\boldsymbol{f}_0, \boldsymbol{f}_1, \mathbf{0}, \dots, \mathbf{0}, \boldsymbol{f}_q, \boldsymbol{f}_{q+1}).$$

Proof. As was seen above, the curve C_F is contained in X_A if and only if one has (2). In the present case where $d = q+1$, if $C_F \subset X_A$ then the coefficients $b_{i,j}$ must vanish except for $b_{q,l-1} = -b_{0,l}$, $b_{q+1,l-1} = -b_{1,l}$ with $1 \le l \le q+1$. Since $\mathrm{rank}(F) = 4$, there are 4 column vectors $\boldsymbol{f}_x$, $\boldsymbol{f}_y$, $\boldsymbol{f}_z$, $\boldsymbol{f}_w$ of F with $0 \le x < y < z < w \le q+1$ such that the matrix $F^* := (\boldsymbol{f}_x, \boldsymbol{f}_y, \boldsymbol{f}_z, \boldsymbol{f}_w)$ is invertible. Then none of x, y, z, w is from 2 to $q-1$ because ${}^{\mathrm{t}}F^*AF^{*(q)}$ is also invertible, and thus $x = 0$, $y = 1$, $z = q$, $w = q+1$. Let $\boldsymbol{f}_i$ be the i-th column vector with $2 \le i \le q-1$ of F. Then one has

$${}^{\mathrm{t}}\boldsymbol{f}_i A F^{*(q)} = (b_{i,0}, b_{i,1}, b_{i,q}, b_{i,q+1}) = (0,0,0,0),$$

and thus $\boldsymbol{f}_i = \mathbf{0}$. Hence F and ${}^{\mathrm{t}}FAF^{(q)}$ are of the form described above. The converse is obvious since (2) holds automatically. □

A rational curve C_F defined by (1) is also obtained by replacing F by $\lambda F \varphi(g)$, where λ is an element of the multiplicative group $k^\times$ and φ is a homomorphism from $\mathrm{GL}_2(k)$ to $\mathrm{GL}_{d+1}(k)$ defined by the following: for each ${}^{\mathrm{t}}(s,t) \in k^2$ with ${}^{\mathrm{t}}(s,t) \ne {}^{\mathrm{t}}(0,0)$ and $g \in \mathrm{GL}_2(k)$, put ${}^{\mathrm{t}}(u,v) := g\ {}^{\mathrm{t}}(s,t)$, then

$$\begin{array}{cccc} \varphi: & \mathrm{GL}_2(k) & \to & \mathrm{GL}_{d+1}(k) \\ & \Psi & & \Psi \\ & (g : {}^{\mathrm{t}}(s,t) \mapsto {}^{\mathrm{t}}(u,v)) & \mapsto & (\varphi(g) : {}^{\mathrm{t}}(s^d, s^{d-1}t, \dots, t^d) \mapsto {}^{\mathrm{t}}(u^d, u^{d-1}v, \dots, v^d)). \end{array}$$

Indeed, it is obvious by definition that $\varphi(I) = I$. Putting ${}^{\mathrm{t}}(x,y) := h\ {}^{\mathrm{t}}(u,v)$ for each $h \in \mathrm{GL}_2(k)$, one has

$$\begin{aligned} \varphi(hg)\ {}^{\mathrm{t}}(s^d, s^{d-1}t, \dots, t^d) &= {}^{\mathrm{t}}(x^d, x^{d-1}y, \dots, y^d) \\ &= \varphi(h)\ {}^{\mathrm{t}}(u^d, u^{d-1}v, \dots, v^d) \\ &= \varphi(h)\varphi(g)\ {}^{\mathrm{t}}(s^d, s^{d-1}t, \dots, t^d). \end{aligned}$$

Hence $\varphi(hg) = \varphi(h)\varphi(g)$, and thus $\varphi(g) \in \mathrm{GL}_{d+1}(k)$.

Conversely if there is a matrix F' such that $C_F = C_{F'}$, then one has

$$F\ {}^t(s^d, s^{d-1}t, \dots, st^{d-1}, t^d) = F'\ {}^t(u^d, u^{d-1}v, \dots, uv^{d-1}, v^d) \in \mathbb{P}^3.$$

This implies that there are homogeneous polynomials f, f' of degree d such that $f(s,t) = f'(u,v)$. Therefore there is an element g of $\mathrm{GL}_2(k)$ such that ${}^t(s,t) = g\ {}^t(u,v) \in \mathbb{P}^1$, and thus $F' = \lambda F\varphi(g)$ for some $\lambda \in k^\times$. Hence, denoting by $\mathrm{Im}(\varphi)$ the image of φ, we see that the set $k^\times F\ \mathrm{Im}(\varphi)$ corresponds one-to-one with C_F.

Let S be the set of matrices F such that ${}^tFAF^{(q)}$ satisfies the condition of Lemma 1. Then by Lemma 1, for each $F \in S$ the set $k^\times F\ \mathrm{Im}(\varphi)$ corresponds one-to-one with the nonplanar rational curve C_F on X_A. Therefore one has the following bijection

$$k^\times \backslash S/\mathrm{Im}(\varphi) \ni k^\times F\ \mathrm{Im}(\varphi) \mapsto C_F \in R. \tag{3}$$

By Lemma 1, we define the map

$${}^*\colon S \ni F = (\boldsymbol{f}_0, \boldsymbol{f}_1, \boldsymbol{0}, \dots, \boldsymbol{0}, \boldsymbol{f}_q, \boldsymbol{f}_{q+1}) \mapsto F^* = (\boldsymbol{f}_0, \boldsymbol{f}_1, \boldsymbol{f}_q, \boldsymbol{f}_{q+1}) \in S^*,$$

where S^* is written as

$$S^* = \{F^* \in \mathrm{GL}_4(k) \mid {}^tF^*AF^{*(q)} = D_B,\ B \in \mathrm{GL}_2(k)\},$$

and D_B is a matrix defined by

$$D_B := \begin{pmatrix} \boldsymbol{0} & \boldsymbol{b}_1 & \boldsymbol{0} & \boldsymbol{b}_2 \\ -\boldsymbol{b}_1 & \boldsymbol{0} & -\boldsymbol{b}_2 & \boldsymbol{0} \end{pmatrix} \in \mathrm{GL}_4(k) \qquad \text{for } B = (\boldsymbol{b}_1, \boldsymbol{b}_2) \in \mathrm{GL}_2(k).$$

Further, we define the map $_*$ from $\mathrm{Im}(\varphi) \subset \mathrm{GL}_{q+2}(k)$ to $\mathrm{Im}(\varphi)_* \subset \mathrm{GL}_4(k)$ as follows:

$$\text{for every } g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{GL}_2(k),$$

$$\varphi(g) = \begin{pmatrix} \alpha^{q+1} & \alpha^q\beta & ,\dots, & \alpha\beta^q & \beta^{q+1} \\ \alpha^q\gamma & \alpha^q\delta & ,\dots, & \gamma\beta^q & \delta\beta^q \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha\gamma^q & \beta\gamma^q & ,\dots, & \alpha\delta^q & \beta\delta^q \\ \gamma^{q+1} & \delta\gamma^q & ,\dots, & \gamma\delta^q & \delta^{q+1} \end{pmatrix}$$

$$\mapsto \varphi(g)_* = \begin{pmatrix} \alpha^{q+1} & \alpha^q\beta & \alpha\beta^q & \beta^{q+1} \\ \alpha^q\gamma & \alpha^q\delta & \gamma\beta^q & \delta\beta^q \\ \alpha\gamma^q & \beta\gamma^q & \alpha\delta^q & \beta\delta^q \\ \gamma^{q+1} & \delta\gamma^q & \gamma\delta^q & \delta^{q+1} \end{pmatrix},$$

where $\mathrm{Im}(\varphi)_*$ is written as

$$\mathrm{Im}(\varphi)_* = \left\{ \begin{pmatrix} \alpha^q g & \beta^q g \\ \gamma^q g & \delta^q g \end{pmatrix} \in \mathrm{GL}_4(k) \,\middle|\, g \in \mathrm{GL}_2(k) \right\}.$$

Indeed, it is easy to see that $\det(\varphi(g)_*) = \det(g)^{2q+2}$ for every $g \in \mathrm{GL}_2(k)$, and thus $\varphi(g)_* \in \mathrm{GL}_4(k)$.

We denote by φ_* the composition of φ and $_*$, namely $\varphi_*(g) = \varphi(g)_*$ for every $g \in \mathrm{GL}_2(k)$.

Lemma 2. *The map φ_* is a homomorphism from $\mathrm{GL}_2(k)$ to $\mathrm{GL}_4(k)$. There is the following natural bijection*

$$k^\times \backslash S/\mathrm{Im}(\varphi) \to k^\times \backslash S^*/\mathrm{Im}(\varphi)_*.$$

Proof. For each

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \qquad h = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in \mathrm{GL}_2(k),$$

one has

$$gh = \begin{pmatrix} \alpha x + \beta z & \alpha y + \beta w \\ \gamma x + \delta z & \gamma y + \delta w \end{pmatrix}.$$

Therefore

$$\varphi_*(gh) = \begin{pmatrix} (\alpha x + \beta z)^q gh & (\alpha y + \beta w)^q gh \\ (\gamma x + \delta z)^q gh & (\gamma y + \delta w)^q gh \end{pmatrix}.$$

On the other hand,

$$\begin{aligned} \varphi_*(g)\varphi_*(h) &= \begin{pmatrix} \alpha^q g & \beta^q g \\ \gamma^q g & \delta^q g \end{pmatrix} \begin{pmatrix} x^q h & y^q h \\ z^q h & w^q h \end{pmatrix} \\ &= \begin{pmatrix} \alpha^q x^q gh + \beta^q z^q gh & \alpha^q y^q gh + \beta^q w^q gh \\ \gamma^q x^q gh + \delta^q z^q gh & \gamma^q y^q gh + \delta^q w^q gh \end{pmatrix} \\ &= \begin{pmatrix} (\alpha^q x^q + \beta^q z^q) gh & (\alpha^q y^q + \beta^q w^q) gh \\ (\gamma^q x^q + \delta^q z^q) gh & (\gamma^q y^q + \delta^q w^q) gh \end{pmatrix}. \end{aligned}$$

Since the q-th power is an automorphism of k, one has $\varphi_*(gh) = \varphi_*(g)\varphi_*(h)$ and thus φ_* is a homomorphism from $\mathrm{GL}_2(k)$ to $\mathrm{GL}_4(k)$.

For each $F \in S$, $g \in \mathrm{GL}_2(k)$, denoting by $a_{i,j}$ the entries of $\varphi(g)$, we can write the j-th column vector $\boldsymbol{g}_j$ with $j \in \{0, 1, q, q+1\}$ of $F\varphi(g)$ as

$$\boldsymbol{g}_j = \sum_{i \in \{0,1,q,q+1\}} a_{i,j} \boldsymbol{f}_i,$$

since $\boldsymbol{f}_i = \mathbf{0}$ for $2 \le i \le q-1$. Then it is immediate from definition that

$$F^*\varphi_*(g) = (\boldsymbol{g}_0, \boldsymbol{g}_1, \boldsymbol{g}_q, \boldsymbol{g}_{q+1}),$$

and thus $(F\varphi(g))^* = F^*\varphi_*(g)$. This implies that there is the natural map from $k^\times \backslash S/\mathrm{Im}(\varphi)$ to $k^\times \backslash S^*/\mathrm{Im}(\varphi)_*$. The bijectivity is obvious since by definition the map $S \to S^*$ is bijective. □

By (3) and Lemma 2, one has the bijection

$$k^\times \backslash S^*/\mathrm{Im}(\varphi)_* \ni k^\times F^* \,\mathrm{Im}(\varphi)_* \mapsto C_F \in R. \tag{4}$$

The following well-known proposition is useful. The readers may find a proof for example in [2] and [9, Proposition 2.5.].

Proposition 1. *For each element A of $\mathrm{GL}_n(k)$, there is an element B of $\mathrm{GL}_n(k)$ such that $A = {}^{\mathrm{t}}BB^{(q)}$. If A is a Hermitian matrix, then the matrix B can be taken from $\mathrm{GL}_n(\mathbb{F}_{q^2})$.*

By Proposition 1, it follows immediately that a smooth k-Hermitian (resp. Hermitian) surface is projectively isomorphic over k (resp. $\mathbb{F}_{q^2}$) to the Fermat surface X_I.

We define the set

$$M := \left\{ D_B := \begin{pmatrix} \mathbf{0} & \boldsymbol{b}_1 & \mathbf{0} & \boldsymbol{b}_2 \\ -\boldsymbol{b}_1 & \mathbf{0} & -\boldsymbol{b}_2 & \mathbf{0} \end{pmatrix} \in \mathrm{GL}_4(k) \,\middle|\, B = (\boldsymbol{b}_1 \quad \boldsymbol{b}_2) \in \mathrm{GL}_2(k) \right\}.$$

Then the following map is surjective:

$$S^* \ni F^* \mapsto {}^{\mathrm{t}}F^*AF^{*(q)} \in M. \tag{5}$$

In fact, by Proposition 1 there is an element D of $\mathrm{GL}_4(k)$ such that $D_B = {}^{\mathrm{t}}DD^{(q)}$ for each $D_B \in M$. Similarly there is an element A' of $\mathrm{GL}_4(k)$ such that $A = {}^{\mathrm{t}}A'A'^{(q)}$. Hence putting $F^* := A'^{-1}D$, one has ${}^{\mathrm{t}}F^*AF^{*(q)} = D_B$, and thus $F^* \in S^*$.

Lemma 3. *The set R is nonempty, and each element of R is projectively isomorphic over k to the smooth curve*

$$C_0 := \{{}^{\mathrm{t}}(s^{q+1}, s^q t, st^q, t^{q+1}) \in \mathbb{P}^3 \mid {}^{\mathrm{t}}(s,t) \in \mathbb{P}^1\}.$$

Proof. The set S^* is nonempty by the surjectivity of the map (5). Hence by (4) the set R is nonempty. For each element C_F of R, it is obvious by definition that

$$F^{*-1}F = (\boldsymbol{e}_1, \boldsymbol{e}_2, \mathbf{0}, \ldots, \mathbf{0}, \boldsymbol{e}_3, \boldsymbol{e}_4) \qquad \text{with } (\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4) = I.$$

This implies that C_F is projectively isomorphic over k to C_0. Then by definition, the curve C_0 is smooth clearly. □

Remark 2. *It is known that each nonplanar nonreflexive curve of degree* $q+1$ *is projectively isomorphic to the curve* C_0 (cf. [1, Theorem 2]). *For nonreflexive curves, see also* [5]. *Hence by Lemma* 3, *each element of* R *is projectively isomorphic to each nonplanar nonreflexive curve of degree* $q+1$.

Remark 3. *In the case where* $A = I$, *we can find an element of* R. *We put*

$$J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Then the matrix D_J *is a Hermitian matrix. Hence by Proposition* 1, *there is an element* F_J^* *of* $\mathrm{GL}_4(\mathbb{F}_{q^2})$ *such that* ${}^{\mathrm{t}}F_J^* F_J^{*(q)} = D_J$. *Actually taking* F_J^* *such as*

$$\begin{pmatrix} \eta^{-q}\xi^{q} & 0 & 0 & -\eta^{-q} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \omega\eta^{-1}\xi & 0 & 0 & \omega\eta^{-1} \end{pmatrix}$$

for ω, ξ *and* η *as mentioned in Introduction, one has by* (4) *the corresponding curve* C_{F_J} *lying on* X_I.

4. Proof of Theorem 2

The group $\mathrm{Aut}(X_A)$ of projective automorphisms of X_A is equal to

$$\{Q \in \mathrm{GL}_4(k) \mid {}^{\mathrm{t}}QAQ^{(q)} = \lambda A, \lambda \in k^{\times}\}/k^{\times}I.$$

By Proposition 1, the group $\mathrm{Aut}(X_A)$ is conjugate to $\mathrm{Aut}(X_I)$ in $\mathrm{PGL}_4(k)$.

We prove the following lemma on matrix groups of arbitrary rank because we need the lemma to our proof of Theorem 2.

Lemma 4. *Let* n *be a positive integer. The group* $\mathrm{PGU}_n(\mathbb{F}_{q^2})$ *is isomorphic to*

$$G := \{Q \in \mathrm{GL}_n(k) \mid {}^{\mathrm{t}}QQ^{(q)} = \lambda I, \lambda \in k^{\times}\}/k^{\times}I.$$

Proof. We consider the map

$$G \ni Qk^{\times} \mapsto \xi_{\lambda}Q\boldsymbol{\mu}_{q+1} \in \mathrm{PGU}_n(\mathbb{F}_{q^2}),$$

where λ is the element of $k^{\times}$ satisfying ${}^{\mathrm{t}}QQ^{(q)} = \lambda I$ and ξ_{λ} is an element of $k^{\times}$ satisfying $\xi_{\lambda}^{q+1} = \lambda^{-1}$. Then the map is well-defined. In fact, it is obvious that ${}^{\mathrm{t}}(\xi_{\lambda}Q)(\xi_{\lambda}Q)^{(q)} = I$, and the matrix $\xi_{\lambda}Q$ has the entries in $\mathbb{F}_{q^2}$ because I is a Hermitian matrix. Hence $\xi_{\lambda}Q\boldsymbol{\mu}_{q+1}$ belongs to $\mathrm{PGU}_n(\mathbb{F}_{q^2})$. Further, putting $P := \alpha Q$ for each $\alpha \in k^{\times}$, one has ${}^{\mathrm{t}}PP^{(q)} = \alpha^{q+1}\lambda I$. It is easily shown by definition that

$$\xi_{\alpha^{q+1}\lambda}\boldsymbol{\mu}_{q+1} = \xi_{\alpha^{q+1}}\xi_{\lambda}\boldsymbol{\mu}_{q+1} \qquad \text{and} \qquad \alpha\xi_{\alpha^{q+1}}\boldsymbol{\mu}_{q+1} = \boldsymbol{\mu}_{q+1}.$$

Therefore we conclude that

$$\xi_{\alpha^{q+1}\lambda}P\boldsymbol{\mu}_{q+1} = \xi_{\lambda}Q\boldsymbol{\mu}_{q+1}.$$

Thus the map is independent of the choice of representatives for G.

Let $Q'k^{\times}$ be an element of G with ${}^{\mathrm{t}}Q'Q'^{(q)} = \eta I$ for some $\eta \in k^{\times}$. Then one has

$$(\xi_{\eta}Q'\boldsymbol{\mu}_{q+1})(\xi_{\lambda}Q\boldsymbol{\mu}_{q+1}) = \xi_{\eta\lambda}Q'Q\boldsymbol{\mu}_{q+1},$$

since $\xi_{\eta}\xi_{\lambda}\boldsymbol{\mu}_{q+1} = \xi_{\eta\lambda}\boldsymbol{\mu}_{q+1}$. Hence the map is a homomorphism from G to $\mathrm{PGU}_n(\mathbb{F}_{q^2})$. The injectivity and the surjectivity are immediate from definition. □

By Lemma 4, the group $\mathrm{Aut}(X_A)$ isomorphic to $\mathrm{PGU}_4(\mathbb{F}_{q^2})$.

The following lemma is a key ingredient in our proof of Theorem 2.

Lemma 5. *For every g, $B \in \mathrm{GL}_2(k)$, one has*

$${}^{\mathrm{t}}\varphi_*(g)D_B\varphi_*(g)^{(q)} = \det(g)^q D_{{}^{\mathrm{t}}gBg^{(q^2)}}.$$

Proof. The proof is due to straightforward computation. We put

$$g := \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \qquad B := (\boldsymbol{b}_1, \boldsymbol{b}_2).$$

Then one has

$$\begin{aligned}
&{}^{\mathrm{t}}\varphi_*(g)D_B\varphi_*(g)^{(q)} \\
&= \begin{pmatrix} \alpha^q\,{}^{\mathrm{t}}g & \gamma^q\,{}^{\mathrm{t}}g \\ \beta^q\,{}^{\mathrm{t}}g & \delta^q\,{}^{\mathrm{t}}g \end{pmatrix}\begin{pmatrix} \mathbf{0} & \boldsymbol{b}_1 & \mathbf{0} & \boldsymbol{b}_2 \\ -\boldsymbol{b}_1 & \mathbf{0} & -\boldsymbol{b}_2 & \mathbf{0} \end{pmatrix}\begin{pmatrix} \alpha^{q^2}g^{(q)} & \beta^{q^2}g^{(q)} \\ \gamma^{q^2}g^{(q)} & \delta^{q^2}g^{(q)} \end{pmatrix} \\
&= \begin{pmatrix} -\gamma^q\,{}^{\mathrm{t}}g\boldsymbol{b}_1 & \alpha^q\,{}^{\mathrm{t}}g\boldsymbol{b}_1 & -\gamma^q\,{}^{\mathrm{t}}g\boldsymbol{b}_2 & \alpha^q\,{}^{\mathrm{t}}g\boldsymbol{b}_2 \\ -\delta^q\,{}^{\mathrm{t}}g\boldsymbol{b}_1 & \beta^q\,{}^{\mathrm{t}}g\boldsymbol{b}_1 & -\delta^q\,{}^{\mathrm{t}}g\boldsymbol{b}_2 & \beta^q\,{}^{\mathrm{t}}g\boldsymbol{b}_2 \end{pmatrix} \\
&\quad \times \begin{pmatrix} \alpha^{q^2+q} & \alpha^{q^2}\beta^q & \alpha^q\beta^{q^2} & \beta^{q^2+q} \\ \alpha^{q^2}\gamma^q & \alpha^{q^2}\delta^q & \gamma^q\beta^{q^2} & \delta^q\beta^{q^2} \\ \alpha^q\gamma^{q^2} & \beta^q\gamma^{q^2} & \alpha^q\delta^{q^2} & \beta^q\delta^{q^2} \\ \gamma^{q^2+q} & \delta^q\gamma^{q^2} & \gamma^q\delta^{q^2} & \delta^{q^2+q} \end{pmatrix}.
\end{aligned}$$

Putting

$${}^{\mathrm{t}}\varphi_*(g)D_B\varphi_*(g)^{(q)} := \begin{pmatrix} \boldsymbol{c}_1 & \boldsymbol{c}_2 & \boldsymbol{c}_3 & \boldsymbol{c}_4 \\ \boldsymbol{c}_5 & \boldsymbol{c}_6 & \boldsymbol{c}_7 & \boldsymbol{c}_8 \end{pmatrix},$$

one has

$$\begin{aligned}
\boldsymbol{c}_1 &= -\alpha^{q^2+q}\gamma^q\ {}^{\mathrm{t}}g\boldsymbol{b}_1 + \alpha^{q^2}\gamma^q\alpha^q\ {}^{\mathrm{t}}g\boldsymbol{b}_1 - \alpha^q\gamma^{q^2}\gamma^q\ {}^{\mathrm{t}}g\boldsymbol{b}_2 + \gamma^{q^2+q}\alpha^q\ {}^{\mathrm{t}}g\boldsymbol{b}_2\\
&= \boldsymbol{0},\\
\boldsymbol{c}_2 &= -\alpha^{q^2}\beta^q\gamma^q\ {}^{\mathrm{t}}g\boldsymbol{b}_1 + \alpha^{q^2}\delta^q\alpha^q\ {}^{\mathrm{t}}g\boldsymbol{b}_1 - \beta^q\gamma^{q^2}\gamma^q\ {}^{\mathrm{t}}g\boldsymbol{b}_2 + \delta^q\gamma^{q^2}\alpha^q\ {}^{\mathrm{t}}g\boldsymbol{b}_2\\
&= \det(g)^q(\alpha^{q^2}\ {}^{\mathrm{t}}g\boldsymbol{b}_1 + \gamma^{q^2}\ {}^{\mathrm{t}}g\boldsymbol{b}_2)\\
&= \det(g)^q\ {}^{\mathrm{t}}g(\boldsymbol{b}_1,\boldsymbol{b}_2)\ {}^{\mathrm{t}}(\alpha^{q^2},\gamma^{q^2}),\\
\boldsymbol{c}_3 &= -\alpha^q\beta^{q^2}\gamma^q\ {}^{\mathrm{t}}g\boldsymbol{b}_1 + \gamma^q\beta^{q^2}\alpha^q\ {}^{\mathrm{t}}g\boldsymbol{b}_1 - \alpha^q\delta^{q^2}\gamma^q\ {}^{\mathrm{t}}g\boldsymbol{b}_2 + \gamma^q\delta^{q^2}\alpha^q\ {}^{\mathrm{t}}g\boldsymbol{b}_2\\
&= \boldsymbol{0},\\
\boldsymbol{c}_4 &= -\beta^{q^2+q}\gamma^q\ {}^{\mathrm{t}}g\boldsymbol{b}_1 + \delta^q\beta^{q^2}\alpha^q\ {}^{\mathrm{t}}g\boldsymbol{b}_1 - \beta^q\delta^{q^2}\gamma^q\ {}^{\mathrm{t}}g\boldsymbol{b}_2 + \delta^{q^2+q}\alpha^q\ {}^{\mathrm{t}}g\boldsymbol{b}_2\\
&= \det(g)^q(\beta^{q^2}\ {}^{\mathrm{t}}g\boldsymbol{b}_1 + \delta^{q^2}\ {}^{\mathrm{t}}g\boldsymbol{b}_2)\\
&= \det(g)^q\ {}^{\mathrm{t}}g(\boldsymbol{b}_1,\boldsymbol{b}_2)\ {}^{\mathrm{t}}(\beta^{q^2},\delta^{q^2}),\\
\boldsymbol{c}_5 &= -\alpha^{q^2+q}\delta^q\ {}^{\mathrm{t}}g\boldsymbol{b}_1 + \alpha^{q^2}\gamma^q\beta^q\ {}^{\mathrm{t}}g\boldsymbol{b}_1 - \alpha^q\gamma^{q^2}\delta^q\ {}^{\mathrm{t}}g\boldsymbol{b}_2 + \gamma^{q^2+q}\beta^q\ {}^{\mathrm{t}}g\boldsymbol{b}_2\\
&= -\det(g)^q(\alpha^{q^2}\ {}^{\mathrm{t}}g\boldsymbol{b}_1 + \gamma^{q^2}\ {}^{\mathrm{t}}g\boldsymbol{b}_2)\\
&= -\det(g)^q\ {}^{\mathrm{t}}g(\boldsymbol{b}_1,\boldsymbol{b}_2)\ {}^{\mathrm{t}}(\alpha^{q^2},\gamma^{q^2}),\\
\boldsymbol{c}_6 &= -\alpha^{q^2}\beta^q\delta^q\ {}^{\mathrm{t}}g\boldsymbol{b}_1 + \alpha^{q^2}\delta^q\beta^q\ {}^{\mathrm{t}}g\boldsymbol{b}_1 - \beta^q\gamma^{q^2}\delta^q\ {}^{\mathrm{t}}g\boldsymbol{b}_2 + \delta^q\gamma^{q^2}\beta^q\ {}^{\mathrm{t}}g\boldsymbol{b}_2\\
&= \boldsymbol{0},\\
\boldsymbol{c}_7 &= -\alpha^q\beta^{q^2}\delta^q\ {}^{\mathrm{t}}g\boldsymbol{b}_1 + \gamma^q\beta^{q^2}\beta^q\ {}^{\mathrm{t}}g\boldsymbol{b}_1 - \alpha^q\delta^{q^2}\delta^q\ {}^{\mathrm{t}}g\boldsymbol{b}_2 + \gamma^q\delta^{q^2}\beta^q\ {}^{\mathrm{t}}g\boldsymbol{b}_2\\
&= -\det(g)^q(\beta^{q^2}\ {}^{\mathrm{t}}g\boldsymbol{b}_1 + \delta^{q^2}\ {}^{\mathrm{t}}g\boldsymbol{b}_2)\\
&= -\det(g)^q\ {}^{\mathrm{t}}g(\boldsymbol{b}_1,\boldsymbol{b}_2)\ {}^{\mathrm{t}}(\beta^{q^2},\delta^{q^2}),\\
\boldsymbol{c}_8 &= -\beta^{q^2+q}\delta^q\ {}^{\mathrm{t}}g\boldsymbol{b}_1 + \delta^q\beta^{q^2}\beta^q\ {}^{\mathrm{t}}g\boldsymbol{b}_1 - \beta^q\delta^{q^2}\delta^q\ {}^{\mathrm{t}}g\boldsymbol{b}_2 + \delta^{q^2+q}\beta^q\ {}^{\mathrm{t}}g\boldsymbol{b}_2\\
&= \boldsymbol{0}.
\end{aligned}$$

Hence one has

$$(\boldsymbol{c}_2,\boldsymbol{c}_4) = \det(g)^q\ {}^{\mathrm{t}}gBg^{(q^2)} = -(\boldsymbol{c}_5,\boldsymbol{c}_7), \qquad \boldsymbol{c}_1 = \boldsymbol{c}_3 = \boldsymbol{c}_6 = \boldsymbol{c}_8 = \boldsymbol{0}.$$

This completes the proof. □

Proof (Proof of Theorem 2). We define an equivalence relation $\sim$ on the set M as follows: $D_B \sim D_{B'}$ for $D_B, D_{B'} \in M$ if there is an element $g \in \mathrm{GL}_2(k)$ such that $D_{B'} = {}^{\mathrm{t}}\varphi_*(g) D_B \varphi_*(g)^{(q)}$. We denote by $D_B^{\varphi_*}$ an equivalence class containing D_B. On the other hand, the group $\mathrm{Aut}(X_A)$ acts on $k^\times \backslash S^*/\mathrm{Im}(\varphi)_*$ by multiplication from the left. Then the following map is bijective:

$$\begin{array}{ccc} \mathrm{Aut}(X_A) k^\times \backslash S^*/\mathrm{Im}(\varphi)_* & \to & k^\times \backslash M/\sim \\ \Downarrow & & \Downarrow \\ \mathrm{Aut}(X_A) k^\times F^* \, \mathrm{Im}(\varphi)_* & \mapsto & k^\times ({}^{\mathrm{t}}F^* A F^{*(q)})^{\varphi_*}. \end{array}$$

Indeed, the surjectivity is obvious since the map (5) is surjective. If we assume that $k^\times ({}^{\mathrm{t}}F^* A F^{*(q)})^{\varphi_*} = k^\times ({}^{\mathrm{t}}F_1^* A F_1^{*(q)})^{\varphi_*}$ for some $F_1^* \in S^*$, then we have

$${}^{\mathrm{t}}(F_1^* \varphi_*(g) F^{*-1}) A (F_1^* \varphi_*(g) F^{*-1})^{(q)} = \lambda A$$

for some $g \in \mathrm{GL}_2(k)$ and $\lambda \in k^\times$. Therefore $k^\times F_1^* \varphi_*(g) F^{*-1}$ belongs to $\mathrm{Aut}(X_A)$. This implies the injectivity, and thus bijectivity. By Proposition 1, there is an element B' of $\mathrm{GL}_2(k)$ such that $B = {}^{\mathrm{t}}B' B'^{(q^2)}$ for each $D_B \in M$. Then by Lemma 5, one has

$${}^{\mathrm{t}}\varphi_*(B'^{-1}) D_B \varphi_*(B'^{-1})^{(q)} = \det(B'^{-1})^q D_I.$$

This implies that $k^\times D_B^{\varphi_*} = k^\times D_I^{\varphi_*}$. Hence $|k^\times \backslash M/\sim| = 1$ and thus $|\mathrm{Aut}(X_A) k^\times \backslash S^*/\mathrm{Im}(\varphi)_*| = 1$, and by (4) one has $|\mathrm{Aut}(X_A) \backslash R| = 1$. This proves half of our theorem.

Let $\Gamma/k^\times I$ be the stabilizer subgroup of $\mathrm{Aut}(X_A)$ fixing the element $k^\times F_I^* \, \mathrm{Im}(\varphi)_*$ of $k^\times \backslash S^*/\mathrm{Im}(\varphi)_*$ such that ${}^{\mathrm{t}}F_I^* A F_I^{*(q)} = D_I$. Then it follows immediately that

$$\Gamma = F_I^* \, \mathrm{Im}(\varphi)_* F_I^{*-1} \cap \{ Q \in \mathrm{GL}_4(k) \mid {}^{\mathrm{t}}Q A Q^{(q)} = \lambda A, \lambda \in k^\times \}.$$

Hence each element of Γ can be written as $F_I^* \varphi_*(g) F_I^{*-1}$ for some element g of $\mathrm{GL}_2(k)$ satisfying

$${}^{\mathrm{t}}(F_I^* \varphi_*(g) F_I^{*-1}) A (F_I^* \varphi_*(g) F_I^{*-1})^{(q)} = \lambda A \qquad \text{for } \lambda \in k^\times,$$

or equivalently,

$${}^{\mathrm{t}}\varphi_*(g) D_I \varphi_*(g)^{(q)} = \lambda D_I \qquad \text{for } \lambda \in k^\times.$$

By Lemma 5, this equality is equivalent to ${}^{\mathrm{t}}g g^{(q^2)} = \lambda I$ for $\lambda \in k^\times$. Consequently, one has the following isomorphism:

$$\begin{array}{ccc} \{ g \in \mathrm{GL}_2(k) \mid {}^{\mathrm{t}}g g^{(q^2)} = \lambda I, \lambda \in k^\times \}/k^\times I & \to & \Gamma/k^\times I \\ \Downarrow & & \Downarrow \\ g k^\times & \mapsto & F_I^* \varphi_*(g) F_I^{*-1} k^\times. \end{array}$$

By Lemma 4, we conclude that $\mathrm{PGU}_2(\mathbb{F}_{q^4}) \simeq \Gamma/k^\times I$. □

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