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Biharmonic homogeneous submanifolds in compact symmetric spaces and compact Lie groups

Shinji Ohno, Takashi Sakai and Hajime Urakawa

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Abstract. We give a necessary and sufficient condition for orbits of commutative Hermann actions and actions of the direct product of two symmetric subgroups on compact Lie groups to be biharmonic in terms of symmetric triad with multiplicities. By this criterion, we determine all the proper biharmonic orbits of these Lie group actions under some additional settings. As a consequence, we obtain many examples of proper biharmonic homogeneous submanifolds in compact symmetric spaces and compact Lie groups.

1. Introduction

Study of harmonic maps, which are critical points of the energy functional, is one of the central problems in differential geometry including minimal submanifolds. The Euler-Lagrange equation is given by the vanishing of the tension field. In 1983, J. Eells and L. Lemaire ([EL]) proposed to study biharmonic maps, which are critical points of the bienergy functional, by definition, half of the integral of square of the norm of tension field $\tau(\varphi)$ for a smooth map φ of a Riemannian manifold (M, g) into another Riemannian manifold (N, h). After a pioneering work of G. Y. Jiang [J], several geometers have studied biharmonic maps (see [CMP], [IIU1], [IIU2], [II], [LO], [MO], [OT2], [S], etc.). Notice that harmonic maps are always biharmonic. One of central problems is to ask whether the converse is true. *B.-Y. Chen's conjecture* is to ask whether every biharmonic submanifold of the Euclidean space $\mathbb{R}^n$ must be harmonic, i.e., minimal ([C]). There are many works supporting this

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conjecture ([D], [HV], [KU], [AM]). However, B.-Y. Chen's conjecture is still open in a general setting. Moreover, R. Caddeo, S. Montaldo, P. Piu ([CMP]) and C. Oniciuc ([On]) raised the *generalized B.-Y. Chen's conjecture* to ask whether each biharmonic submanifold in a Riemannian manifold (N, h) of non-positive sectional curvature must be harmonic (minimal). For the generalized Chen's conjecture, Y.-L. Ou and L. Tang gave ([OT1], [OT2]) a counter example in some Riemannian manifold of negative sectional curvature. However, it is also known (cf. [NU1], [NU2], [NUG]) that every biharmonic map of a complete Riemannian manifold into another Riemannian manifold of non-positive sectional curvature with finite energy and finite bienergy must be harmonic.

On the contrary, for the target Riemannian manifold (N, h) of non-negative sectional curvature, theories of biharmonic maps and/or biharmonic immersions seems to be quite different from the case (N, h) of non-positive sectional curvature. Indeed, there exist biharmonic submanifolds which are not harmonic in the sphere S^n.

In 2015, we characterized the biharmonic property of isometric immersions into Einstein manifolds whose tension fields are parallel with respect to the normal connections in terms of the second fundamental forms and the curvature tensors, and determined all the biharmonic hypersurfaces in irreducible symmetric spaces of compact type which are regular orbits of commutative Hermann actions of cohomogeneity one (cf. [OSU]). For this purpose, we used the description of second fundamental forms of orbits of commutative Hermann actions in terms of symmetric triad with multiplicities, which was given by O. Ikawa ([I1]). Recently, the first author ([Oh1]) applied the method of symmetric triad to study the geometry of orbits of actions of the direct product of two symmetric subgroups on compact Lie groups, which are associated to commutative Hermann actions.

In this paper, we characterize the biharmonic property of orbits of commutative Hermann actions and the actions of the direct product of two symmetric subgroups on compact Lie groups in terms of symmetric triad with multiplicities (cf. Theorems 6 and 7). By this characterization, we give a complete table of all the proper biharmonic singular orbits of commutative Hermann actions of cohomogeneity two (cf. Theorem 8), and also we give a complete list of all the proper biharmonic regular orbits of $(K_2 \times K_1)$-actions of cohomogeneity one on G for a commutative compact symmetric triad (G, K_1, K_2) (cf. Theorem 9). We note that recently J. Inoguchi and T. Sasahara ([IS]) also investigated biharmonic homogeneous hypersurfaces in compact symmetric spaces, and the first author studied biharmonic orbits of isotropy representations of symmetric spaces in the sphere ([Oh2]).

2. Biharmonic isometric immersions

We first recall the definition and fundamentals of harmonic maps and biharmonic maps. Let $\varphi : (M, g) \to (N, h)$ be a smooth map from an m-dimensional compact Riemannian manifold (M, g) into an n-dimensional Riemannian manifold (N, h). Then φ is said to be *harmonic* if it is a critical point of the *energy functional* defined by

$$E(\varphi) = \frac{1}{2}\int_M |d\varphi|^2 v_g.$$

That is, for any variation $\{\varphi_t\}$ of φ with $\varphi_0 = \varphi$,

$$\left.\frac{d}{dt}\right|_{t=0} E(\varphi_t) = -\int_M h(\tau(\varphi), V) v_g = 0. \tag{1}$$

Here $V \in \Gamma(\varphi^{-1}TN)$ is a variation vector field along φ which is given by $V(x) = \frac{d}{dt}\big|_{t=0}\varphi_t(x) \in T_{\varphi(x)}N$ $(x \in M)$, and $\tau(\varphi)$ is the *tension field* of φ which is given by $\tau(\varphi) = \sum_{i=1}^m B_\varphi(e_i, e_i) \in \Gamma(\varphi^{-1}TN)$, where $\{e_i\}_{i=1}^m$ is a locally defined orthonormal frame field on (M, g), and B_φ is the second fundamental form of φ defined by

$$\begin{aligned} B_\varphi(X, Y) &= (\tilde{\nabla}\, d\varphi)(X, Y) \\ &= (\tilde{\nabla}_X\, d\varphi)(Y) \\ &= \overline{\nabla}_X(d\varphi(Y)) - d\varphi(\nabla_X Y), \end{aligned}$$

for all vector fields $X, Y \in \mathfrak{X}(M)$. Here we denote by ∇ and ∇^h the Levi-Civita connections on TM, TN of (M, g), (N, h), and by $\overline{\nabla}$ and $\tilde{\nabla}$ the induced connections on $\varphi^{-1}TN$ and $T^*M \otimes \varphi^{-1}TN$, respectively. By (1), φ is harmonic if and only if $\tau(\varphi) = 0$. We note that if $\varphi : M \to N$ is an isometric immersion, then the tension field $\tau(\varphi)$ coincides with the mean curvature vector field of φ, hence φ is harmonic if and only if φ is a minimal immersion.

J. Eells and L. Lemaire [EL] proposed the notion of biharmonic maps, and G. Y. Jiang [J] studied the first and second variation formulas of biharmonic maps. Let us consider the *bienergy functional* defined by

$$E_2(\varphi) = \frac{1}{2}\int_M |\tau(\varphi)|^2 v_g,$$

where $|V|^2 = h(V, V)$ for $V \in \Gamma(\varphi^{-1}TN)$. The first variation formula of the bienergy functional is given by

$$\left.\frac{d}{dt}\right|_{t=0} E_2(\varphi_t) = -\int_M h(\tau_2(\varphi), V) v_g,$$

where $\tau_2(\varphi)$ is called the *bitension field* of φ which is defined by

$$\tau_2(\varphi) := J(\tau(\varphi)) = \bar{\Delta}(\tau(\varphi)) - \mathscr{R}(\tau(\varphi)).$$

Here J is the *Jacobi operator* acting on $\Gamma(\varphi^{-1}TN)$ given by

$$J(V) = \bar{\Delta}V - \mathscr{R}(V),$$

where $\bar{\Delta}V = \overline{\nabla}^*\overline{\nabla}V = -\sum_{i=1}^m \{\overline{\nabla}_{e_i}\overline{\nabla}_{e_i}V - \overline{\nabla}_{\nabla_{e_i}e_i}V\}$ is the rough Laplacian and $\mathscr{R}$ is the linear operator on $\Gamma(\varphi^{-1}TN)$ defined by $\mathscr{R}(V) = \sum_{i=1}^m R^h(V, d\varphi(e_i))d\varphi(e_i)$, where R^h is the curvature tensor of (N,h) given by $R^h(U,V)W = \nabla_U^h(\nabla_V^h W) - \nabla_V^h(\nabla_U^h W) - \nabla_{[U,V]}^h W$ for $U, V, W \in \mathfrak{X}(N)$. A smooth map φ of (M,g) into (N,h) is said to be *biharmonic* if $\tau_2(\varphi) = 0$. By definition, every harmonic map is biharmonic. We say that a smooth map $\varphi : (M,g) \to (N,h)$ is proper biharmonic if it is biharmonic but not harmonic.

Now we give a characterization theorem for an isometric immersion φ of a Riemannian manifold (M,g) into another Riemannian manifold (N,h) whose tension field $\tau(\varphi)$ satisfies $\overline{\nabla}_X^\perp \tau(\varphi) = 0$ for all $X \in \mathfrak{X}(M)$ to be biharmonic, where $\overline{\nabla}^\perp$ is the normal connection on the normal bundle $T^\perp M$. From Jiang's theorem ([J]), we showed the following theorem.

Theorem 1 ([OSU]). *Let $\varphi : (M,g) \to (N,h)$ be an isometric immersion which satisfies that $\overline{\nabla}_X^\perp \tau(\varphi) = 0$ for all $X \in \mathfrak{X}(M)$. Then φ is biharmonic if and only if*

$$\sum_{k=1}^m R^h(\tau(\varphi), d\varphi(e_k))d\varphi(e_k) = \sum_{j,k=1}^m h(\tau(\varphi), B_\varphi(e_j,e_k))B_\varphi(e_j,e_k) \tag{2}$$

holds.

The condition (2) is equivalent to the following equation.

$$\sum_{i=1}^m R^h(\tau(\varphi), d\varphi(e_i))d\varphi(e_i) = \sum_{i=1}^m B_\varphi(A_{\tau(\varphi)}e_i, e_i). \tag{3}$$

3. Hermann actions and associated $(K_2 \times K_1)$-actions

3.1. Hermann actions and symmetric triads. O. Ikawa ([I1]) introduced the notion of symmetric triad as a generalization of irreducible root system. He described the second fundamental forms of orbits of commutative Hermann actions in terms of symmetric triads with multiplicities, and studied geometric properties of the orbits as submanifolds in compact symmetric spaces. In this section, we review O. Ikawa's method, and we show that his method can be also applied to study geometric properties of orbits of actions of the direct product of two symmetric subgroups on compact Lie groups, which are associated to Hermann actions.

Let G be a compact connected semisimple Lie group, and K_1, K_2 closed subgroups of G. For each $i = 1, 2$, we assume that there exists an involutive automorphism θ_i of G which satisfies $(G_{\theta_i})_0 \subset K_i \subset G_{\theta_i}$, where G_{θ_i} is the set of fixed points of θ_i and $(G_i)_0$ is the identity component of G_{θ_i}. Then (G, K_1) and (G, K_2) are compact symmetric pairs, and the triple (G, K_1, K_2) is called a *compact symmetric triad*. We denote the Lie algebras of G, K_1 and K_2 by $\mathfrak{g}$, $\mathfrak{k}_1$ and $\mathfrak{k}_2$, respectively. The involutive automorphism of $\mathfrak{g}$ induced from θ_i will be also denoted by θ_i. Take an $\mathrm{Ad}(G)$-invariant inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$. Then the inner product $\langle \cdot, \cdot \rangle$ induces a bi-invariant Riemannian metric on G and G-invariant Riemannian metrics on the left coset manifold $N_1 := G/K_1$ and on the right coset manifold $N_2 := K_2 \backslash G$. We denote these Riemannian metrics on G, N_1 and N_2 by the same symbol $\langle \cdot, \cdot \rangle$. These Riemannian manifolds G, N_1 and N_2 are Riemannian symmetric spaces with respect to $\langle \cdot, \cdot \rangle$. The isometric action of K_2 on G/K_1 and that of K_1 on $K_2 \backslash G$ defined by

- $K_2 \curvearrowright N_1$: $\quad k_2 \cdot \pi_1(x) = \pi_1(k_2 x) \ \ (k_2 \in K_2,\ x \in G)$
- $K_1 \curvearrowright N_2$: $\quad k_1 \cdot \pi_2(x) = \pi_2(x k_1^{-1}) \ \ (k_1 \in K_1,\ x \in G)$

are called *Hermann actions*, where π_i denotes the natural projection from G onto N_i $(i = 1, 2)$. Under this setting, we can also consider the isometric action of $K_2 \times K_1$ on G defined by

- $K_2 \times K_1 \curvearrowright G$: $\quad (k_2, k_1) \cdot x = k_2 x k_1^{-1} \ \ (k_2 \in K_2,\ k_1 \in K_1,\ x \in G)$.

The three actions have the same orbit space, and in fact the following diagram is commutative:

$$\begin{tikzcd} & G \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ N_1 \arrow[dr, "\tilde{\pi}_1"'] & & N_2 \arrow[dl, "\tilde{\pi}_2"] \\ & K_2 \backslash G / K_1 & \end{tikzcd}$$

where $\tilde{\pi}_i$ is the natural projection from N_i onto the orbit space $K_2 \backslash G / K_1$. For $x \in G$, we denote the left (resp. right) transformation on G by L_x (resp. R_x). The isometry on N_1 (resp. N_2) induced by L_x (resp. R_x) will be also denoted by the same symbol L_x (resp. R_x).

For $i = 1, 2$, we set

$$\mathfrak{m}_i = \{ X \in \mathfrak{g} \mid \theta_i(X) = -X \}.$$

Then we have two orthogonal direct sum decompositions of $\mathfrak{g}$, that is the canonical decompositions:

$$\mathfrak{g} = \mathfrak{k}_1 \oplus \mathfrak{m}_1 = \mathfrak{k}_2 \oplus \mathfrak{m}_2.$$

The tangent space $T_{\pi_i(e)}N_i$ of N_i at the origin $\pi_i(e)$ is identified with $\mathfrak{m}_i$ in a natural way. We define a closed subgroup G_{12} of G by

$$G_{12} = \{x \in G \mid \theta_1(x) = \theta_2(x)\}.$$

Then θ_1 induces an involutive automorphism of the identity component $(G_{12})_0$ of G_{12}, hence $((G_{12})_0, K_{12})$ is a compact symmetric pair, where K_{12} is a closed subgroup of $(G_{12})_0$ defined by

$$K_{12} = \{k \in (G_{12})_0 \mid \theta_1(k) = k\}.$$

The canonical decomposition of $\mathfrak{g}_{12} = \mathrm{Lie}(G_{12})_0$ with respect to θ_1 is given by

$$\mathfrak{g}_{12} = (\mathfrak{k}_1 \cap \mathfrak{k}_2) \oplus (\mathfrak{m}_1 \cap \mathfrak{m}_2).$$

In general, an isometric action of a compact Lie group on a Riemannian manifold is said to be *hyperpolar* if there exists a closed connected submanifold which is flat in the induced metric and meets all orbits orthogonally. Such a submanifold is called a *section* of the Lie group action. In our setting, fix a maximal abelian subspace $\mathfrak{a}$ in $\mathfrak{m}_1 \cap \mathfrak{m}_2$. Then $\exp \mathfrak{a}$ is a torus subgroup in $(G_{12})_0$. Then $\exp \mathfrak{a}$, $\pi_1(\exp \mathfrak{a})$ and $\pi_2(\exp \mathfrak{a})$ are sections of the $(K_2 \times K_1)$-action on G, the K_2-action on N_1, and the K_1-action on N_2, respectively. Hence these three actions are hyperpolar, and their cohomogeneities are equal to $\dim \mathfrak{a}$. A. Kollross ([K]) classified hyperpolar actions on compact irreducible symmetric spaces. By the classification, we can see that a hyperpolar action on a compact irreducible symmetric space whose cohomogeneity is greater than or equal to two is orbit-equivalent to some Hermann action.

In order to describe the orbit spaces of the three actions, we consider an equivalent relation $\sim$ on $\mathfrak{a}$ defined as follows: for $H_1, H_2 \in \mathfrak{a}$, we define $H_1 \sim H_2$ if $(K_2 \times K_1) \cdot \exp(H_1) = (K_2 \times K_1) \cdot \exp(H_2)$. Clearly, we have $H_1 \sim H_2$ if and only if $K_2 \cdot \pi_1(\exp(H_1)) = K_2 \cdot \pi_1(\exp(H_2))$, and similarly, $H_1 \sim H_2$ if and only if $K_1 \cdot \pi_2(\exp(H_1)) = K_1 \cdot \pi_2(\exp(H_2))$. This implies that $\mathfrak{a}/\sim \cong K_2 \backslash G / K_1$. For a subgroup L of G, we define the normalizer $N_L(\mathfrak{a})$ and the centralizer $Z_L(\mathfrak{a})$ of $\mathfrak{a}$ in L by

$$N_L(\mathfrak{a}) = \{k \in L \mid \mathrm{Ad}(k)\mathfrak{a} = \mathfrak{a}\},$$

$$Z_L(\mathfrak{a}) = \{k \in L \mid \mathrm{Ad}(k)H = H \ \ (H \in \mathfrak{a})\}.$$

Then $Z_L(\mathfrak{a})$ is a normal subgroup of $N_L(\mathfrak{a})$. We define a group $\tilde{J}$ by

$$\tilde{J} = \{([s], Y) \in (N_{K_2}(\mathfrak{a})/Z_{K_1 \cap K_2}(\mathfrak{a})) \ltimes \mathfrak{a} \mid \exp(-Y)s \in K_1\}.$$

The group $\tilde{J}$ naturally acts on $\mathfrak{a}$ by the following manner:

$$([s], Y) \cdot H = \mathrm{Ad}(s)H + Y \qquad (([s], Y) \in \tilde{J},\ H \in \mathfrak{a}).$$

T. Matsuki ([M]) proved that

$$K_2 \backslash G / K_1 \cong \mathfrak{a}/\tilde{J}.$$

Hereafter, we suppose $\theta_1\theta_2 = \theta_2\theta_1$. In such a case, (G, K_1, K_2) is called a *commutative* compact symmetric triad, and the K_2-action on N_1 and the K_1-action on N_2 are called *commutative* Hermann actions. Then we have an orthogonal direct sum decomposition of $\mathfrak{g}$:

$$\mathfrak{g} = (\mathfrak{k}_1 \cap \mathfrak{k}_2) \oplus (\mathfrak{m}_1 \cap \mathfrak{m}_2) \oplus (\mathfrak{k}_1 \cap \mathfrak{m}_2) \oplus (\mathfrak{m}_1 \cap \mathfrak{k}_2).$$

We define subspaces of $\mathfrak{g}$ as follows:

$$\begin{aligned} \mathfrak{k}_0 &= \{X \in \mathfrak{k}_1 \cap \mathfrak{k}_2 \mid [\mathfrak{a}, X] = \{0\}\}, \\ V(\mathfrak{k}_1 \cap \mathfrak{m}_2) &= \{X \in \mathfrak{k}_1 \cap \mathfrak{m}_2 \mid [\mathfrak{a}, X] = \{0\}\}, \\ V(\mathfrak{m}_1 \cap \mathfrak{k}_2) &= \{X \in \mathfrak{m}_1 \cap \mathfrak{k}_2 \mid [\mathfrak{a}, X] = \{0\}\}. \end{aligned}$$

For $\lambda \in \mathfrak{a}$,

$$\begin{aligned} \mathfrak{k}_\lambda &= \{X \in \mathfrak{k}_1 \cap \mathfrak{k}_2 \mid [H, [H, X]] = -\langle \lambda, H \rangle^2 X \ \ (H \in \mathfrak{a})\}, \\ \mathfrak{m}_\lambda &= \{X \in \mathfrak{m}_1 \cap \mathfrak{m}_2 \mid [H, [H, X]] = -\langle \lambda, H \rangle^2 X \ \ (H \in \mathfrak{a})\}, \\ V_\lambda^\perp(\mathfrak{k}_1 \cap \mathfrak{m}_2) &= \{X \in \mathfrak{k}_1 \cap \mathfrak{m}_2 \mid [H, [H, X]] = -\langle \lambda, H \rangle^2 X \ \ (H \in \mathfrak{a})\}, \\ V_\lambda^\perp(\mathfrak{m}_1 \cap \mathfrak{k}_2) &= \{X \in \mathfrak{m}_1 \cap \mathfrak{k}_2 \mid [H, [H, X]] = -\langle \lambda, H \rangle^2 X \ \ (H \in \mathfrak{a})\}. \end{aligned}$$

We set

$$\begin{aligned} \Sigma &= \{\lambda \in \mathfrak{a}\backslash\{0\} \mid \mathfrak{k}_\lambda \neq \{0\}\}, \\ W &= \{\alpha \in \mathfrak{a}\backslash\{0\} \mid V_\alpha^\perp(\mathfrak{k}_1 \cap \mathfrak{m}_2) \neq \{0\}\}, \\ \tilde{\Sigma} &= \Sigma \cup W. \end{aligned}$$

It is known that $\dim \mathfrak{k}_\lambda = \dim \mathfrak{m}_\lambda$ and $\dim V_\lambda^\perp(\mathfrak{k}_1 \cap \mathfrak{m}_2) = \dim V_\lambda^\perp(\mathfrak{m}_1 \cap \mathfrak{k}_2)$ for each $\lambda \in \tilde{\Sigma}$. Thus we set $m(\lambda) := \dim \mathfrak{k}_\lambda$, $n(\lambda) := \dim V_\lambda^\perp(\mathfrak{k}_1 \cap \mathfrak{m}_2)$. Notice that Σ is the restricted root system of the symmetric pair $((G_{12})_0, K_{12})$, and $\tilde{\Sigma}$ becomes a root system of $\mathfrak{a}$ (see [I1]).

We define an open subset $\mathfrak{a}_r$ of $\mathfrak{a}$ by

$$\mathfrak{a}_r = \bigcap_{\lambda \in \Sigma, \alpha \in W} \left\{ H \in \mathfrak{a} \mid \langle \lambda, H \rangle \notin \pi\mathbb{Z},\ \langle \alpha, H \rangle \notin \frac{\pi}{2} + \pi\mathbb{Z} \right\}.$$

A connected component of $\mathfrak{a}_r$ is called a *cell*. The *affine Weyl group* $\tilde{W}(\tilde{\Sigma}, \Sigma, W)$ of $(\tilde{\Sigma}, \Sigma, W)$ is a subgroup of $\mathrm{Aff}(\mathfrak{a})$ generated by

$$\left\{ \left(s_\lambda, \frac{2n\pi}{\langle \lambda, \lambda \rangle} \lambda \right) \middle| \lambda \in \Sigma,\ n \in \mathbb{Z} \right\} \cup \left\{ \left(s_\alpha, \frac{(2n+1)\pi}{\langle \alpha, \alpha \rangle} \alpha \right) \middle| \alpha \in W,\ n \in \mathbb{Z} \right\},$$

where $\mathrm{Aff}(\mathfrak{a})$ is the affine group of $\mathfrak{a}$ which is expressed as the semidirect product $\mathrm{O}(\mathfrak{a}) \ltimes \mathfrak{a}$. The action of $(s_\lambda, (2n\pi/\langle \lambda, \lambda \rangle)\lambda)$ on $\mathfrak{a}$ is the reflection with respect to the hyperplane $\{H \in \mathfrak{a} \mid \langle \lambda, H \rangle = n\pi\}$, and the action of $(s_\alpha, ((2n+1)\pi/\langle \alpha, \alpha \rangle)\alpha)$ on $\mathfrak{a}$ is the reflection with respect to the hyperplane $\{H \in \mathfrak{a} \mid \langle \alpha, H \rangle = (n + 1/2)\pi\}$. The affine Weyl group $\tilde{W}(\tilde{\Sigma}, \Sigma, W)$ acts transitively on the set of all cells. More precisely, for each cell P, it holds that

$$\mathfrak{a} = \bigcup_{s \in \tilde{W}(\tilde{\Sigma}, \Sigma, W)} s\bar{P}.$$

For $x = \exp H$ $(H \in \mathfrak{a})$, the orbit $K_2 \cdot \pi_1(x)$ in N_1 is regular, so $(K_2 \times K_1) \cdot x$ in G is, if and only if $H \in \mathfrak{a}_r$. Here we call an orbit *regular* if it is an orbit of the maximal dimension. In [I1], it is proved that $\tilde{W}(\tilde{\Sigma}, \Sigma, W)$ is a subgroup of $\tilde{J}$. Moreover, if N_1 and N_2 are simply-connected, then $\tilde{W}(\tilde{\Sigma}, \Sigma, W) = \tilde{J}$, hence the orbit space $K_2 \backslash G / K_1$ of the actions can be identified with $\mathfrak{a}/\tilde{J} = \mathfrak{a}/\tilde{W}(\tilde{\Sigma}, \Sigma, W) \cong \bar{P}$. Indeed, for each orbit $K_2 \cdot \pi_1(x)$ in N_1 and $(K_2 \times K_1) \cdot x$ in G, there exists $H \in \bar{P}$ uniquely so that $K_2 \cdot \pi_1(x) = K_2 \cdot \pi_1(\exp H)$ in N_1 and $(K_2 \times K_1) \cdot x = (K_2 \times K_1) \cdot (\exp H)$ in G. An interior point H in $\bar{P}$ corresponds to a regular orbit, and a point H in the boundary of $\bar{P}$ corresponds to a singular orbit. In fact, $\bar{P}$ is a closed region in $\mathfrak{a}$, which is a direct product of some simplexes. Then the cell decomposition of $\bar{P}$ gives a stratification of orbit types of the action. We should note that, in general, the cell decomposition of $\bar{P}$ gives a stratification of *local* orbit types of the action. In this paper, we study the biharmonicity of the orbits, that is a local property of a submanifold. Therefore, without loss of generality, we may assume that N_1 and N_2 are simply-connected, hence $K_2 \backslash G / K_1 \cong \bar{P}$. In Sections 5 and 6, for some orbit types, we will examine the number of biharmonic orbits. If N_1 or N_2 are not simply-connected, then the number of biharmonic orbits does not increase compared with the cases of N_1 and N_2 are simply-connected.

O. Ikawa ([I1]) introduced the notion of *symmetric triad* with multiplicities as a generalization of irreducible root system. For the precise definition of symmetric triad with multiplicities, we refer to O. Ikawa's papers ([I1, I2, I3]). In general, the triad $(\tilde{\Sigma}, \Sigma, W)$, which is obtained from a compact symmetric triad (G, K_1, K_2), is not a symmetric triad with multiplicities in the sense of [I1]. However, we know the following theorem.

Theorem 2 ([I2] Theorem 3.1, [I3] Theorem 1.14). *Let (G, K_1, K_2) be a compact symmetric triad which satisfies one of the following conditions.*

(A) *G is simple and $\theta_1 \not\sim \theta_2$, i.e. θ_1 and θ_2 can not be transformed each other by an inner automorphism of $\mathfrak{g}$.*

(B) *There exists a compact connected simple Lie group U and a symmetric subgroup $\bar{K}$ of U such that*

$$G = U \times U, \qquad K_1 = \Delta G = \{(u,u) \mid u \in U\}, \qquad K_2 = \bar{K} \times \bar{K}.$$

(C) *There exists a compact connected simple Lie group U and an involutive outer automorphism σ such that*

$$G = U \times U, \qquad K_1 = \Delta G = \{(u,u) \mid u \in U\},$$
$$K_2 = \{(u_1, u_2) \mid (\sigma(u_2), \sigma(u_1)) = (u_1, u_2)\}.$$

Then the triple $(\tilde{\Sigma}, \Sigma, W)$ defined as above is a symmetric triad of $\mathfrak{a}$, moreover $m(\lambda)$ and $n(\alpha)$ are multiplicities of $\lambda \in \Sigma$ and $\alpha \in W$. Conversely every symmetric triad is obtained in this way.

When (G, K_1, K_2) satisfies (A), (B) or (C) in Theorem 2, hence $(\tilde{\Sigma}, \Sigma, W)$ is a symmetric triad, we take a fundamental system $\tilde{\Pi}$ of $\tilde{\Sigma}$. We denote by $\tilde{\Sigma}^+$ the set of positive roots in $\tilde{\Sigma}$. Set $\Sigma^+ = \tilde{\Sigma}^+ \cap \Sigma$ and $W^+ = \tilde{\Sigma}^+ \cap W$. Denote by Π the set of simple roots of Σ. We set

$$W_0 = \{\alpha \in W^+ \mid \alpha + \lambda \notin W \ (\lambda \in \Pi)\}.$$

From the classification of symmetric triads, we have that W_0 consists of the only one element, denoted by $\tilde{\alpha}$. We define an open subset P_0 of $\mathfrak{a}$ by

$$P_0 = \left\{ H \in \mathfrak{a} \,\middle|\, \langle \tilde{\alpha}, H \rangle < \frac{\pi}{2},\ \langle \lambda, H \rangle > 0 \ (\lambda \in \Pi) \right\}. \tag{4}$$

Then P_0 is a cell. For a nonempty subset $\Delta \subset \Pi \cup \{\tilde{\alpha}\}$, set

$$P_0^{\Delta} = \left\{ H \in \bar{P}_0 \,\middle|\, \begin{array}{c} \langle \lambda, H \rangle > 0 \ (\lambda \in \Delta \cap \Pi) \\ \langle \mu, H \rangle = 0 \ (\mu \in \Pi \backslash \Delta) \\ \langle \tilde{\alpha}, H \rangle \begin{cases} < (\pi/2) & (\text{if } \tilde{\alpha} \in \Delta) \\ = (\pi/2) & (\text{if } \tilde{\alpha} \notin \Delta) \end{cases} \end{array} \right\}.$$

Then

$$\bar{P}_0 = \bigcup_{\Delta \subset \Pi \cup \{\tilde{\alpha}\}} P_0^{\Delta} \quad \text{(disjoint union)}. \tag{5}$$

When (G, K_1, K_2) satisfies $\theta_1 \sim \theta_2$, i.e. θ_1 and θ_2 are transformed each other by an inner automorphism of $\mathfrak{g}$, the Hermann action of K_2 on the compact symmetric space G/K_1 is equivalent to the action of the isotropy group K_1 on G/K_1. Hence we may assume that $\theta_1 = \theta_2$, and so $K_1 = K_2$. Since $W = \emptyset$, then $(\tilde{\Sigma}, \Sigma, W)$ is not a symmetric triad, and $\tilde{\Sigma} = \Sigma$ is the restricted root system of (G, K_1). In this case, we can describe the orbit space of K_1-action on G/K_1 in terms of the restricted root system Σ of (G, K_1). For simplicity, here we assume that G is simple. We take a fundamental system Π of Σ, and denote the set of positive roots by Σ^+ and the highest root by δ. We define an open subset P_0 of $\mathfrak{a}$ by

$$P_0 = \{H \in \mathfrak{a} \mid \langle \delta, H \rangle < \pi,\ \langle \lambda, H \rangle > 0 \ \ (\lambda \in \Pi)\}. \tag{6}$$

Then P_0 is a cell. For a nonempty subset $\Delta \subset \Pi \cup \{\delta\}$, set

$$P_0^{\Delta} = \left\{ H \in \bar{P}_0 \ \middle|\ \begin{array}{l} \langle \lambda, H \rangle > 0 \ \ (\lambda \in \Delta \cap \Pi) \\ \langle \mu, H \rangle = 0 \ \ (\mu \in \Pi \backslash \Delta) \\ \langle \delta, H \rangle \begin{cases} < \pi & (\text{if } \delta \in \Delta) \\ = \pi & (\text{if } \delta \notin \Delta) \end{cases} \end{array} \right\}.$$

Then

$$\bar{P}_0 = \bigcup_{\Delta \subset \Pi \cup \{\delta\}} P_0^{\Delta} \quad \text{(disjoint union)}. \tag{7}$$

3.2. Second fundamental forms of orbits. We express the second fundamental forms and mean curvature vector fields of orbits of commutative Hermann actions and their associated $K_2 \times K_1$-actions.

Here, let G be a compact connected semisimple Lie group and (G, K_1, K_2) a commutative compact symmetric triad. Fix a maximal abelian subspace $\mathfrak{a}$ in $\mathfrak{m}_1 \cap \mathfrak{m}_2$, then we have a triad $(\tilde{\Sigma}, \Sigma, W)$. We take a fundamental system $\tilde{\Pi}$ of $\tilde{\Sigma}$, and set

$$\tilde{\Sigma}^+ = \{\lambda \in \tilde{\Sigma} \mid \lambda > 0\}, \qquad \Sigma^+ = \Sigma \cap \tilde{\Sigma}^+, \qquad W^+ = W \cap \tilde{\Sigma}^+.$$

Then we have an orthogonal direct sum decomposition of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{k}_0 \oplus \sum_{\lambda \in \Sigma^+} \mathfrak{k}_\lambda \oplus \mathfrak{a} \oplus \sum_{\lambda \in \Sigma^+} \mathfrak{m}_\lambda \oplus V(\mathfrak{k}_1 \cap \mathfrak{m}_2) \oplus \sum_{\alpha \in W^+} V_\alpha^{\perp}(\mathfrak{k}_1 \cap \mathfrak{m}_2)$$
$$\oplus V(\mathfrak{m}_1 \cap \mathfrak{k}_2) \oplus \sum_{\alpha \in W^+} V_\alpha^{\perp}(\mathfrak{m}_1 \cap \mathfrak{k}_2).$$

According to the above orthogonal direct sum decomposition of $\mathfrak{g}$, we have the following lemma.

LEMMA 1 ([I1] Lemmas 4.3 and 4.16). (1) *For each* $\lambda \in \Sigma^+$, *there exist orthonormal bases* $\{S_{\lambda,i}\}_{i=1}^{m(\lambda)}$ *and* $\{T_{\lambda,i}\}_{i=1}^{m(\lambda)}$ *of* $\mathfrak{k}_\lambda$ *and* $\mathfrak{m}_\lambda$ *respectively such that for any* $H \in \mathfrak{a}$,

$$[H, S_{\lambda,i}] = \langle \lambda, H \rangle T_{\lambda,i}, \qquad [H, T_{\lambda,i}] = -\langle \lambda, H \rangle S_{\lambda,i}, \qquad [S_{\lambda,i}, T_{\lambda,i}] = \lambda,$$

$$\mathrm{Ad}(\exp H) S_{\lambda,i} = \cos\langle \lambda, H \rangle S_{\lambda,i} + \sin\langle \lambda, H \rangle T_{\lambda,i},$$

$$\mathrm{Ad}(\exp H) T_{\lambda,i} = -\sin\langle \lambda, H \rangle S_{\lambda,i} + \cos\langle \lambda, H \rangle T_{\lambda,i}.$$

(2) *For each* $\alpha \in W^+$, *there exist orthonormal bases* $\{X_{\alpha,j}\}_{j=1}^{n(\alpha)}$ *and* $\{Y_{\alpha,j}\}_{j=1}^{n(\alpha)}$ *of* $V_\alpha^\perp(\mathfrak{k}_1 \cap \mathfrak{m}_2)$ *and* $V_\alpha^\perp(\mathfrak{m}_1 \cap \mathfrak{k}_2)$ *respectively such that for any* $H \in \mathfrak{a}$

$$[H, X_{\alpha,j}] = \langle \alpha, H \rangle Y_{\alpha,j}, \qquad [H, Y_{\alpha,j}] = -\langle \alpha, H \rangle X_{\alpha,j}, \qquad [X_{\alpha,j}, Y_{\alpha,j}] = \alpha,$$

$$\mathrm{Ad}(\exp H) X_{\alpha,j} = \cos\langle \alpha, H \rangle X_{\alpha,j} + \sin\langle \alpha, H \rangle Y_{\alpha,j},$$

$$\mathrm{Ad}(\exp H) Y_{\alpha,j} = -\sin\langle \alpha, H \rangle X_{\alpha,j} + \cos\langle \alpha, H \rangle Y_{\alpha,j}.$$

First, for $x \in G$, we consider an orbit $K_2 \cdot \pi_1(x)$ of the commutative Hermann action of K_2 on N_1. Without loss of generality we can assume that $x = \exp H$ where $H \in \mathfrak{a}$, since $\pi_1(\exp \mathfrak{a})$ is a section of the action. For $H \in \mathfrak{a}$, we set

$$\Sigma_H = \{\lambda \in \Sigma \mid \langle \lambda, H \rangle \in \pi\mathbb{Z}\}, \qquad W_H = \{\alpha \in W \mid \langle \alpha, H \rangle \in (\pi/2) + \pi\mathbb{Z}\},$$

$$\tilde{\Sigma}_H = \Sigma_H \cup W_H, \qquad \Sigma_H^+ = \Sigma^+ \cap \Sigma_H, \qquad W_H^+ = W^+ \cap W_H, \qquad \tilde{\Sigma}_H^+ = \Sigma_H^+ \cup W_H^+.$$

Then, for $H \in \mathfrak{a}$, H is in $\mathfrak{a}_r$ if and only if $\Sigma_H = \emptyset$ and $W_H = \emptyset$. In such case, the tangent space

$$\begin{aligned} T_{\pi_1(x)}(N_1) &= dL_x(\mathfrak{m}_1) = dL_x((\mathfrak{m}_1 \cap \mathfrak{m}_2) \oplus (\mathfrak{m}_1 \cap \mathfrak{k}_2)) \\ &= dL_x\left(\mathfrak{a} \oplus \sum_{\lambda \in \Sigma^+} \mathfrak{m}_\lambda \oplus V(\mathfrak{m}_1 \cap \mathfrak{k}_2) \oplus \sum_{\alpha \in W^+} V_\alpha^\perp(\mathfrak{m}_1 \cap \mathfrak{k}_2)\right) \end{aligned}$$

of N_1 at $\pi_1(x)$ is decomposed to the tangent space $T_{\pi_1(x)}(K_2 \cdot \pi_1(x))$ and the normal space $T_{\pi_1(x)}^\perp(K_2 \cdot \pi_1(x))$ of the orbit $K_2 \cdot \pi_1(x)$ as follows.

$$\begin{aligned} T_{\pi_1(x)}(K_2 \cdot \pi_1(x)) &= \left\{ \frac{d}{dt} \exp(tX_2) \cdot \pi_1(x) \Big|_{t=0} \,\Big|\, X_2 \in \mathfrak{k}_2 \right\} \\ &= dL_x(d\pi_1(\mathrm{Ad}(x)^{-1}\mathfrak{k}_2)) \\ &= dL_x\left(\sum_{\lambda \in \Sigma^+ \setminus \Sigma_H} \mathfrak{m}_\lambda \oplus V(\mathfrak{m}_1 \cap \mathfrak{k}_2) \oplus \sum_{\alpha \in W^+ \setminus W_H} V_\alpha^\perp(\mathfrak{m}_1 \cap \mathfrak{k}_2) \right), \end{aligned}$$

$$T^{\perp}_{\pi_1(x)}(K_2 \cdot \pi_1(x)) = dL_x((\mathrm{Ad}(x)^{-1}\mathfrak{m}_2) \cap \mathfrak{m}_1)$$
$$= dL_x\left(\mathfrak{a} \oplus \sum_{\lambda \in \Sigma_H^+} \mathfrak{m}_\lambda \oplus \sum_{\alpha \in W_H^+} V_\alpha^{\perp}(\mathfrak{m}_1 \cap \mathfrak{k}_2)\right).$$

From Lemma 1, O. Ikawa proved the following theorems.

THEOREM 3 ([I1] Lemma 4.22). *For $x = \exp H$ $(H \in \mathfrak{a})$, we denote the second fundamental forms of the orbits $K_2 \cdot \pi_1(x)$ in N_1 by B_H^1. Then we have*
(1) $dL_x^{-1} B_H^1(dL_x(T_{\lambda,i}), dL_x(T_{\mu,j})) = \cot(\langle \mu, H\rangle)[T_{\lambda,i}, S_{\mu,j}]^{\perp}$,
(2) $dL_x^{-1} B_H^1(dL_x(Y_{\alpha,i}), dL_x(Y_{\beta,j})) = -\tan(\langle \beta, H\rangle)[Y_{\alpha,i}, X_{\beta,j}]^{\perp}$,
(3) $B_H^1(dL_x(Y_1), dL_x(Y_2)) = 0$,
(4) $B_H^1(dL_x(T_{\lambda,i}), dL_x(Y_2)) = 0$,
(5) $B_H^1(dL_x(Y_{\alpha,i}), dL_x(Y_2)) = 0$,
(6) $dL_x^{-1} B_H^1(dL_x(T_{\lambda,i}), dL_x(Y_{\beta,j})) = -\tan(\langle \beta, H\rangle)[T_{\lambda,i}, X_{\beta,j}]^{\perp}$,
for

$$\lambda, \mu \in \Sigma^+ \text{ with } \langle \lambda, H\rangle, \langle \mu, H\rangle \notin \pi\mathbb{Z},\ 1 \le i \le m(\lambda),\ 1 \le j \le m(\mu),$$
$$\alpha, \beta \in W^+ \text{ with } \langle \alpha, H\rangle, \langle \beta, H\rangle \notin \frac{\pi}{2} + \pi\mathbb{Z},\ 1 \le i \le n(\alpha),\ 1 \le j \le n(\beta),$$
$$Y_1, Y_2 \in V(\mathfrak{m}_1 \cap \mathfrak{k}_2).$$

Here $X^{\perp}$ is the normal component, i.e. $(\mathrm{Ad}(x^{-1})\mathfrak{m}_2) \cap \mathfrak{m}_1$-component, of a tangent vector $X \in \mathfrak{m}_1$.

THEOREM 4 ([I1] Corollaries 4.23, 4.29, 4.24, and [GT] Theorem 5.3). *For $x = \exp H$ $(H \in \mathfrak{a})$, we denote the mean curvature vector field of $K_2 \cdot \pi_1(x)$ in N_1 by τ_H^1. Then we have*

$$dL_x^{-1}(\tau_H^1)_{\pi_1(x)} = -\sum_{\lambda \in \Sigma^+ \setminus \Sigma_H} m(\lambda) \cot\langle \lambda, H\rangle \lambda + \sum_{\alpha \in W^+ \setminus W_H} n(\alpha) \tan\langle \alpha, H\rangle \alpha.$$

We can also apply Theorem 4 for the orbit $K_1 \cdot \pi_2(x)$ in N_2. Thus we have the following corollary.

COROLLARY 1 ([I1] Corollary 4.30). *The orbit $K_2 \cdot \pi_1(x)$ is minimal if and only if $K_1 \cdot \pi_2(x)$ is minimal.*

Next we consider the second fundamental forms of orbits of the $(K_2 \times K_1)$-action on G. For $x = \exp H$ $(H \in \mathfrak{a})$, the tangent space $T_x((K_2 \times K_1) \cdot x)$ and the normal space $T_x^{\perp}((K_2 \times K_1) \cdot x)$ of the orbit $(K_2 \times K_1) \cdot x$ at x are given as follows.

$$
\begin{aligned}
&T_x((K_2 \times K_1) \cdot x) \\
&\quad = \left\{ \frac{d}{dt} \exp(tX_2) x \exp(-tX_1) \Big|_{t=0} \;\Big|\; X_1 \in \mathfrak{k}_1,\ X_2 \in \mathfrak{k}_2 \right\} \\
&\quad = dL_x((\mathrm{Ad}(x)^{-1}\mathfrak{k}_2) + \mathfrak{k}_1) \\
&\quad = dL_x \Bigg(\mathfrak{k}_0 \oplus \sum_{\lambda \in \Sigma^+} \mathfrak{k}_\lambda \oplus \sum_{\lambda \in \Sigma^+ \setminus \Sigma_H} \mathfrak{m}_\lambda \oplus V(\mathfrak{k}_1 \cap \mathfrak{m}_2) \\
&\qquad\qquad \oplus \sum_{\alpha \in W^+} V_\alpha^\perp(\mathfrak{k}_1 \cap \mathfrak{m}_2) \oplus V(\mathfrak{m}_1 \cap \mathfrak{k}_2) \oplus \sum_{\alpha \in W^+ \setminus W_H} V_\alpha^\perp(\mathfrak{m}_1 \cap \mathfrak{k}_2) \Bigg), \quad (8)
\end{aligned}
$$

$$
\begin{aligned}
&T_x^\perp((K_2 \times K_1) \cdot x) \\
&\quad = dL_x((\mathrm{Ad}(x)^{-1}\mathfrak{m}_2) \cap \mathfrak{m}_1) \\
&\quad = dL_x \Bigg(\mathfrak{a} \oplus \sum_{\lambda \in \Sigma_H^+} \mathfrak{m}_\lambda \oplus \sum_{\alpha \in W_H^+} V_\alpha^\perp(\mathfrak{m}_1 \cap \mathfrak{k}_2) \Bigg). \quad (9)
\end{aligned}
$$

For $X = (X_2, X_1) \in \mathfrak{g} \times \mathfrak{g}$, we define a Killing vector field X^* on G by

$$
(X^*)_y = \frac{d}{dt} \exp(tX_2) y \exp(-tX_1) \Big|_{t=0} \qquad (y \in G).
$$

Then

$$
(X^*)_y = (dL_y)(\mathrm{Ad}(y)^{-1}X_2 - X_1)
$$

holds. If $X_2 = 0$, then X^* is a left invariant vector field. Denote by ∇ the Levi-Civita connection on G. By Koszul's formula, we have the following.

Lemma 2 ([Oh1] Lemma 3). *Let $y \in G$, $X = (X_2, X_1)$, $Y = (Y_2, Y_1) \in \mathfrak{g} \times \mathfrak{g}$. Then we have*

$$
(\nabla_{X^*} Y^*)_y = -\frac{1}{2} dL_y[\mathrm{Ad}(y)^{-1}X_2 - X_1, \mathrm{Ad}(y)^{-1}Y_2 + Y_1].
$$

Here, $\mathfrak{g} \times \mathfrak{g}$ denotes the Lie algebra of $G \times G$.

By Lemma 2, the first author proved the following theorems.

Theorem 5 ([Oh1] Theorem 3). *For $x = \exp H$ $(H \in \mathfrak{a})$, we denote the second fundamental form of the orbit $(K_2 \times K_1) \cdot x$ in G by B_H. We define subspaces V_1 and V_2 of $dL_x^{-1}(T_x(K_2 \times K_1) \cdot x)$ by*

$$V_1 = \sum_{\lambda \in \Sigma^+ \setminus \Sigma_H} \mathfrak{m}_\lambda \oplus \sum_{\alpha \in W^+ \setminus W_H} V_\alpha^\perp(\mathfrak{m}_1 \cap \mathfrak{k}_2),$$

$$V_2 = \sum_{\lambda \in \Sigma^+} \mathfrak{k}_\lambda \oplus \sum_{\alpha \in W^+} V_\alpha^\perp(\mathfrak{k}_1 \cap \mathfrak{m}_2).$$

Then we have

(1) *For* $X \in \mathfrak{k}_0$, $B_H(dL_x(X), Y) = 0$ *where* $Y \in T_x((K_2 \times K_1) \cdot x)$.

(2) *For* $X \in V(\mathfrak{k}_1 \cap \mathfrak{m}_2)$,

$$dL_x^{-1} B_H(dL_x(X), dL_x(Y)) = \begin{cases} 0 & (Y \in \mathfrak{k}_1 \oplus V(\mathfrak{m}_1 \cap \mathfrak{k}_2)) \\ -\dfrac{1}{2}[X, Y]^\perp & (Y \in V_1). \end{cases}$$

(3) *For* $X \in V(\mathfrak{m}_1 \cap \mathfrak{k}_2)$,

$$dL_x^{-1} B_H(dL_x(X), dL_x(Y)) = \begin{cases} 0 & (Y \in V(\mathfrak{m}_1 \cap \mathfrak{k}_2) \oplus V_1) \\ \dfrac{1}{2}[X, Y]^\perp & (Y \in V_2). \end{cases}$$

(4) *For* $S_{\lambda,i}$ $(\lambda \in \Sigma^+,\ 1 \leq i \leq m(\lambda))$,

$$dL_x^{-1} B_H(dL_x(S_{\lambda,i}), dL_x(Y)) = \begin{cases} 0 & (Y \in V_2) \\ -\dfrac{1}{2}[S_{\lambda,i}, Y]^\perp & (Y \in V_1). \end{cases}$$

(5) *For* $X_{\alpha,i}$ $(\alpha \in W^+,\ 1 \leq i \leq n(\alpha))$,

$$dL_x^{-1} B_H(dL_x(X_{\alpha,i}), dL_x(Y)) = \begin{cases} 0 & (Y \in V_2) \\ -\dfrac{1}{2}[X_{\alpha,i}, Y]^\perp & (Y \in V_1). \end{cases}$$

(6) *For* $T_{\lambda,i}$ $(\lambda \in \Sigma^+ \setminus \Sigma_H,\ 1 \leq i \leq m(\lambda))$,

- $dL_x^{-1} B_H(dL_x(T_{\lambda,i}), dL_x(T_{\mu,j})) = \cot\langle \mu, H \rangle [T_{\lambda,i}, S_{\mu,j}]^\perp$ *where* $\mu \in \Sigma^+ \setminus \Sigma_H$, $1 \leq j \leq m(\mu)$.
- $dL_x^{-1} B_H(dL_x(T_{\lambda,i}), dL_x(Y_{\beta,j})) = -\tan\langle \beta, H \rangle [T_{\lambda,i}, X_{\beta,j}]^\perp$ *where* $\beta \in W^+ \setminus W_H$, $1 \leq j \leq n(\beta)$.

(7) *For* $Y_{\alpha,i}$ $(\alpha \in W^+ \setminus W_H,\ 1 \leq i \leq n(\alpha))$,

$$dL_x^{-1} B_H(dL_x(Y_{\alpha,i}), dL_x(Y_{\beta,j})) = -\tan\langle \beta, H \rangle [Y_{\alpha,i}, X_{\beta,j}]^\perp$$

where $\beta \in W^+ \setminus W_H$, $1 \leq j \leq n(\beta)$.

Here, $X^\perp$ *denotes the normal component, i.e. the* $((\mathrm{Ad}(x)^{-1}\mathfrak{m}_2) \cap \mathfrak{m}_1)$-*component, of a tangent vector* $X \in \mathfrak{g}$.

By Theorems 4 and 5, we obtain the following corollary.

Corollary 2 ([Oh1] Corollary 2). *For $x = \exp H$ $(H \in \mathfrak{a})$, we denote the mean curvature vector field of the orbit $(K_2 \times K_1) \cdot x$ in G by τ_H. Then,*

$$dL_x^{-1}(\tau_H)_x = -\sum_{\lambda \in \Sigma^+ \setminus \Sigma_H} m(\lambda) \cot\langle \lambda, H \rangle \lambda + \sum_{\alpha \in W^+ \setminus W_H} n(\alpha) \tan\langle \alpha, H \rangle \alpha.$$

Moreover, $dL_x^{-1}(\tau_H)_x = dL_x^{-1}(\tau_H^1)_{\pi_1(x)}$ holds. Hence, the orbit $(K_2 \times K_1) \cdot x$ in G is minimal if and only if $K_2 \cdot \pi_1(x)$ in N_1 is minimal.

We show the following properties of the mean curvature vector field τ_H of $(K_2 \times K_1) \cdot x$ in G and τ_H^1 of $K_2 \cdot \pi_1(x)$ in N_1.

Proposition 1. *For $H \in \mathfrak{a}$ and $\sigma = ([s], Y) \in \tilde{J}$, we set $H' = \sigma \cdot H \in \mathfrak{a}$, $x = \exp(H)$ and $x' = \exp(H')$. Then $(K_2 \times K_1) \cdot x = (K_2 \times K_1) \cdot x'$ and*

$$dL_{x'}^{-1}(\tau_H)_{x'} = [s] \cdot dL_x^{-1}(\tau_H)_x.$$

Proof. By the definition of $\tilde{J}$, there exists $s \in N_{K_2}(\mathfrak{a})$ such that $\mathrm{Ad}(s)|_{\mathfrak{a}} = [s]$ and $(s, \exp(-Y)s) \in K_2 \times K_1$. Then we have

$$(s, \exp(-Y)s) \cdot \exp(H) = s \exp(H) s^{-1} \exp(Y) = \exp(\mathrm{Ad}(s)H + Y) = \exp(H').$$

Thus, $(K_2 \times K_1) \cdot x = (K_2 \times K_1) \cdot x'$. Since $L_s \circ R_{s^{-1} \exp(Y)}$ is an isometry, we have

$$\begin{aligned}
(\tau_H)_{x'} &= (\tau_H)_{L_s \circ R_{s^{-1} \exp(Y)}(x)} \\
&= dL_s \circ dR_{s^{-1} \exp(Y)}((\tau_H)_x) \\
&= \frac{d}{dt} s \exp(H) \exp(t\, dL_x^{-1}(\tau_H)_x) s^{-1} \exp(Y) \Big|_{t=0} \\
&= \frac{d}{dt} \exp(\mathrm{Ad}(s)(t\, dL_x^{-1}(\tau_H)_x + H)) \exp(Y) \Big|_{t=0} \\
&= \frac{d}{dt} \exp(\mathrm{Ad}(s)H + Y) \exp(t\, \mathrm{Ad}(s)(dL_x^{-1}(\tau_H)_x)) \Big|_{t=0} \\
&= dL_{x'}(\mathrm{Ad}(s) dL_x^{-1}(\tau_H)_x) \\
&= dL_{x'}([s] \cdot dL_x^{-1}(\tau_H)_x).
\end{aligned}$$

By Lemmas 4.4 and 4.21 in [I1], we have that $\tilde{W}(\tilde{\Sigma}, \Sigma, W)$ is a subgroup of $\tilde{J}$. Then we have the following Lemma.

Lemma 3. *For $x = \exp H$ $(H \in \mathfrak{a})$, we have*

$$\langle \lambda, dL_x^{-1}(\tau_H)_x \rangle = 0 \qquad (\lambda \in \tilde{\Sigma}_H).$$

PROOF. When $(\tau_H)_x = 0$, it is trivial. Thus we assume $(\tau_H)_x \neq 0$. Since $\lambda \in \tilde{\Sigma}_H$, we have

$$\left(s_\lambda, 2\frac{\langle \lambda, H\rangle}{\langle \lambda, \lambda\rangle}\lambda\right) \in \tilde{W}(\tilde{\Sigma}, \Sigma, W).$$

Then,

$$\left(s_\lambda, 2\frac{\langle \lambda, H\rangle}{\langle \lambda, \lambda\rangle}\lambda\right) H = s_\lambda(H) + 2\frac{\langle \lambda, H\rangle}{\langle \lambda, \lambda\rangle}\lambda = H.$$

By Proposition 1, we have

$$dL_x^{-1}(\tau_H)_x = s_\lambda(dL_x^{-1}(\tau_H)_x) = dL_x^{-1}(\tau_H)_x - 2\frac{\langle \lambda, dL_x^{-1}(\tau_H)_x\rangle}{\langle \lambda, \lambda\rangle}\lambda.$$

Therefore, we obtain $\langle \lambda, dL_x^{-1}(\tau_H)_x\rangle = 0$.

PROPOSITION 2. *For* $x = \exp H$ $(H \in \mathfrak{a})$, *we have*

$$\nabla_X^\perp \tau_H = 0 \qquad (X \in \mathfrak{X}((K_2 \times K_1) \cdot x)).$$

PROOF. Since the orbit $(K_2 \times K_1) \cdot x$ is a homogeneous submanifold in G, it is sufficient to prove that $\nabla_X^\perp \tau_H = 0$ at one point x in G.

Let $X \in T_x((K_2 \times K_1) \cdot x)$. Then there exists $(X_2, X_1) \in \mathfrak{k}_2 \times \mathfrak{k}_1$ such that $X = (X_2, X_1)_x^*$. For $k_2 \in K_2$, we have

$$\begin{aligned}
(0, -dL_x^{-1}(\tau_H)_x)_{k_2 x}^* &= \frac{d}{dt} k_2 x \exp(t\, dL_x^{-1}(\tau_H)_x)\Big|_{t=0} \\
&= dL_{k_2} dL_x \frac{d}{dt} \exp(t\, dL_x^{-1}(\tau_H)_x)\Big|_{t=0} \\
&= dL_{k_2}(\tau_H)_x \\
&= (\tau_H)_{k_2 x}.
\end{aligned}$$

Since H and $dL_x^{-1}(\tau_H)_x$ are in $\mathfrak{a}$ from Corollary 2, we have, for $k_1 \in K_1$,

$$\begin{aligned}
(dL_x^{-1}(\tau_H)_x, 0)_{x k_1^{-1}}^* &= \frac{d}{dt} \exp(t\, dL_x^{-1}(\tau_H)_x) x k_1^{-1}\Big|_{t=0} \\
&= \frac{d}{dt} x \exp(t\, dL_x^{-1}(\tau_H)_x) k_1^{-1}\Big|_{t=0} \\
&= dR_{k_1}^{-1} dL_x \frac{d}{dt} \exp(t\, dL_x^{-1}(\tau_H)_x)\Big|_{t=0}
\end{aligned}$$

$$= dR_{k_1}^{-1}(\tau_H)_x$$
$$= (\tau_H)_{xk_1^{-1}}.$$

In particular, $\tau_H = (0, -dL_x^{-1}(\tau_H)_x)^*$ on the curve $\exp(tX_2)x$ for $X_2 \in \mathfrak{k}_2$ and $\tau_H = (dL_x^{-1}(\tau_H)_x, 0)^*$ on the curve $x\exp(tX_1)$ for $X_1 \in \mathfrak{k}_1$.

Since H and $dL_x^{-1}(\tau_H)_x$ are in $\mathfrak{a}$,

$$\mathrm{Ad}(x)^{-1}\, dL_x^{-1}(\tau_H)_x = dL_x^{-1}(\tau_H)_x.$$

Hence, by Lemma 2, we have

$$\begin{aligned}
(\nabla_X^{\perp}\tau_H)_x &= (\nabla_{(X_2, X_1)^*}\tau_H)_x^{\perp} \\
&= (\nabla_{(X_2, 0)^*}\tau_H)_x^{\perp} + (\nabla_{(0, X_1)^*}\tau_H)_x^{\perp} \\
&= (\nabla_{(X_2, 0)^*}(0, -dL_x^{-1}(\tau_H)_x)^*)_x^{\perp} + (\nabla_{(0, X_1)^*}(dL_x^{-1}(\tau_H)_x, 0)^*)_x^{\perp} \\
&= \left(-\frac{1}{2}\, dL_x[\mathrm{Ad}(x)^{-1}X_2, -dL_x^{-1}(\tau_H)_x]\right)^{\perp} + \left(-\frac{1}{2}\, dL_x[-X_1, dL_x^{-1}(\tau_H)_x]\right)^{\perp} \\
&= \left(\frac{1}{2}\, dL_x[\mathrm{Ad}(x)^{-1}X_2 + X_1, dL_x^{-1}(\tau_H)_x]\right)^{\perp}.
\end{aligned}$$

Therefore, in order to prove $\nabla_X^{\perp}\tau_H = 0$, it is sufficient to show that

$$[(\mathrm{Ad}(x)^{-1}\mathfrak{k}_2) + \mathfrak{k}_1, dL_x^{-1}(\tau_H)_x] \subset (\mathrm{Ad}(x)^{-1}\mathfrak{k}_2) + \mathfrak{k}_1.$$

From (8), we have

$$\begin{aligned}
&(\mathrm{Ad}(x)^{-1}\mathfrak{k}_2) + \mathfrak{k}_1 \\
&\quad = \Bigg(\mathfrak{k}_0 \oplus \sum_{\lambda\in\Sigma^+}\mathfrak{k}_\lambda \oplus \sum_{\lambda\in\Sigma^+\setminus\Sigma_H}\mathfrak{m}_\lambda \oplus V(\mathfrak{k}_1\cap\mathfrak{m}_2) \oplus \sum_{\alpha\in W^+} V_\alpha^{\perp}(\mathfrak{k}_1\cap\mathfrak{m}_2) \\
&\qquad \oplus V(\mathfrak{m}_1\cap\mathfrak{k}_2) \oplus \sum_{\alpha\in W^+\setminus W_H} V_\alpha^{\perp}(\mathfrak{m}_1\cap\mathfrak{k}_2)\Bigg).
\end{aligned}$$

Since $dL_x^{-1}(\tau_H)_x \in \mathfrak{a}$ and Lemma 1, we have

$$[\mathfrak{k}_0 \oplus V(\mathfrak{m}_1\cap\mathfrak{k}_2) \oplus V(\mathfrak{k}_1\cap\mathfrak{m}_2), dL_x^{-1}(\tau_H)_x] = \{0\},$$

$$[\mathfrak{k}_\lambda \oplus \mathfrak{m}_\lambda, dL_x^{-1}(\tau_H)_x] \subset \mathfrak{k}_\lambda \oplus \mathfrak{m}_\lambda,$$

$$[V_\alpha^{\perp}(\mathfrak{m}_1\cap\mathfrak{k}_2) \oplus V_\alpha^{\perp}(\mathfrak{k}_1\cap\mathfrak{m}_2), dL_x^{-1}(\tau_H)_x] \subset V_\alpha^{\perp}(\mathfrak{m}_1\cap\mathfrak{k}_2) \oplus V_\alpha^{\perp}(\mathfrak{k}_1\cap\mathfrak{m}_2)$$

for $\lambda \in \Sigma^+ \backslash \Sigma_H$ and $\alpha \in W^+ \backslash W_H$. By Lemmas 1 and 3, we also have

$$[\mathfrak{k}_\lambda, dL_x(\tau_H)_x] = \{0\}, \qquad [V_\alpha^\perp(\mathfrak{k}_1 \cap \mathfrak{m}_2), dL_x(\tau_H)_x] = \{0\}$$

for $\lambda \in \Sigma_H^+$ and $\alpha \in W_H^+$. Therefore we have the consequence.

In Theorem 4, we described the tension field of the orbit of commutative Hermann actions of K_2 on N_1. From this expression, we can verify the existence of minimal orbits. In the case of isotropy actions of compact symmetric spaces, the orbit space can be identified with a cell P_0 as in (6). D. Hirohashi, H. Tasaki, H. J. Song and R. Takagi, proved that, according to the stratification of orbit types, there exists a unique minimal orbit in each orbit type (cf. [HTST, Theorem 3.1]). For commutative Hermann actions which satisfy one of (A), (B) and (C) in Theorem 2, O. Ikawa obtained the same result (cf. [I1, Theorem 2.24]).

4. Characterizations of biharmonic orbits

In the previous section, we described the second fundamental forms of orbits of the Hermann action of K_2 on N_1 and the $(K_2 \times K_1)$-action on G. In this section, we give a necessary and sufficient condition for an orbit to be a biharmonic submanifold.

4.1. Characterization of biharmonic orbits of commutative Hermann actions. First, we consider orbits of commutative Hermann actions. Since all orbits of Hermann actions satisfy $\nabla_X^\perp \tau_H^1 = 0$ (see [IST1]), we can apply Theorem 1.

THEOREM 6. *Let (G, K_1, K_2) be a commutative compact symmetric triad. For $x = \exp H$ $(H \in \mathfrak{a})$, the orbit $K_2 \cdot \pi_1(x)$ is biharmonic in N_1 if and only if*

$$\sum_{\lambda \in \Sigma^+ \backslash \Sigma_H} m(\lambda) \langle dL_x^{-1}(\tau_H^1)_{\pi_1(x)}, \lambda \rangle (1 - (\cot\langle \lambda, H \rangle)^2) \lambda$$
$$+ \sum_{\alpha \in W^+ \backslash W_H} n(\alpha) \langle dL_x^{-1}(\tau_H^1)_{\pi_1(x)}, \alpha \rangle (1 - (\tan\langle \alpha, H \rangle)^2) \alpha = 0 \qquad (10)$$

holds.

PROOF. The curvature tensor $R^{\langle,\rangle}$ of the Riemannian symmetric space $(N_1, \langle\,,\rangle)$ is given by

$$R^{\langle,\rangle}(dL_x(X), dL_x(Y))dL_x(Z) = -dL_x[[X, Y], Z] \qquad (X, Y, Z \in \mathfrak{m}_1).$$

Since $dL_x^{-1}(\tau_H^1)_{\pi_1(x)} \in \mathfrak{a}$, we have

- for $\lambda \in \Sigma^+ \backslash \Sigma_H$, $1 \le i \le m(\lambda)$,

$$\begin{aligned} R^{\langle,\rangle}((\tau_H^1)_{\pi_1(x)}, dL_x(T_{\lambda,i}))dL_x(T_{\lambda,i}) &= -dL_x[[dL_x^{-1}(\tau_H^1)_{\pi_1(x)}, T_{\lambda,i}], T_{\lambda,i}] \\ &= \langle dL_x^{-1}(\tau_H^1)_{\pi_1(x)}, \lambda\rangle dL_x[S_{\lambda,i}, T_{\lambda,i}] \\ &= \langle dL_x^{-1}(\tau_H^1)_{\pi_1(x)}, \lambda\rangle dL_x(\lambda), \end{aligned}$$

- for $\alpha \in W^+ \backslash W_H$, $1 \le j \le m(\alpha)$,

$$R^{\langle,\rangle}((\tau_H^1)_{\pi_1(x)}, dL_x(Y_{\alpha,j}))dL_x(Y_{\alpha,j}) = \langle dL_x^{-1}(\tau_H^1)_{\pi_1(x)}, \alpha\rangle dL_x(\alpha),$$

- for $X \in V(\mathfrak{m}_1 \cap \mathfrak{k}_2)$,

$$R^{\langle,\rangle}((\tau_H^1)_{\pi_1(x)}, dL_x(X))dL_x(X) = 0.$$

Since $dL_x^{-1}(\tau_H^1)_{\pi_1(x)}$ is in $\mathfrak{a}$, by Theorem 3, we have

- for $\lambda \in \Sigma^+ \backslash \Sigma_H$, $1 \le i \le m(\lambda)$,

$$A_{(\tau_H^1)_{\pi_1(x)}}\, dL_x(T_{\lambda,i}) = -\langle dL_x^{-1}(\tau_H^1)_{\pi_1(x)}, \lambda\rangle(\cot\langle\lambda, H\rangle)dL_x(T_{\lambda,i}),$$

$$\begin{aligned} &B_H^1(A_{(\tau_H^1)_{\pi_1(x)}}\, dL_x(T_{\lambda,i}), dL_x(T_{\lambda,i})) \\ &\quad = -\langle dL_x^{-1}(\tau_H^1)_{\pi_1(x)}, \lambda\rangle(\cot\langle\lambda, H\rangle)B_H^1(dL_x(T_{\lambda,i}), dL_x(T_{\lambda,i})) \\ &\quad = \langle dL_x^{-1}(\tau_H^1)_{\pi_1(x)}, \lambda\rangle(\cot\langle\lambda, H\rangle)^2 dL_x(\lambda), \end{aligned}$$

- for $\alpha \in W^+ \backslash W_H$, $1 \le j \le n(\alpha)$,

$$\begin{aligned} &B_H^1(A_{(\tau_H^1)_{\pi_1(x)}}\, dL_x(Y_{\alpha,j}), dL_x(Y_{\alpha,j})) \\ &\quad = -\langle dL_x^{-1}(\tau_H^1)_{\pi_1(x)}, \alpha\rangle(\tan\langle\alpha, H\rangle)B_H^1(dL_x(Y_{\alpha,j}), dL_x(Y_{\alpha,j})) \\ &\quad = \langle dL_x^{-1}(\tau_H^1)_{\pi_1(x)}, \alpha\rangle(\tan\langle\alpha, H\rangle)^2 dL_x(\alpha), \end{aligned}$$

- for $X \in V(\mathfrak{m}_1 \cap \mathfrak{k}_2)$,

$$B_H^1(A_{(\tau_H^1)_{\pi_1(x)}}\, dL_x(X), dL_x(X)) = 0.$$

Therefore, by Theorem 1, we have the consequence.

Corollary 3. *Let (G, K_1, K_2) be a commutative compact symmetric triad which satisfies* $\dim \mathfrak{a} = 1$, *i.e.* $\tilde{\Sigma} \subset \{\alpha, 2\alpha\}$. *For* $x = \exp H$ $(H \in \mathfrak{a})$, *suppose that the orbit $K_2 \cdot \pi_1(x)$ is a regular orbit. Then $K_2 \cdot \pi_1(x)$ is biharmonic in N_1 if and only if*

$$\langle dL_x^{-1}(\tau_H^1)_{\pi_1(x)}, \alpha\rangle(m(\alpha)\{1-(\cot\langle\alpha,H\rangle)^2\}+4m(2\alpha)\{1-(\cot\langle 2\alpha,H\rangle)^2\}$$
$$+n(\alpha)\{1-(\tan\langle\alpha,H\rangle)^2\}+4n(2\alpha)\{1-(\tan\langle 2\alpha,H\rangle)^2\})=0$$

holds. Here, for $\lambda \in \mathfrak{a}$, if $\lambda \notin \Sigma$ (resp. $\lambda \notin W$), then $m(\lambda)=0$ (resp. $n(\lambda)=0$).

4.2. Characterization of biharmonic orbits of $(K_2 \times K_1)$-actions. Next, we consider orbits of the $(K_2 \times K_1)$-action on G. By Proposition 2, we can apply Theorem 1.

THEOREM 7. *Let (G, K_1, K_2) be a commutative compact symmetric triad. For $x = \exp H$ $(H \in \mathfrak{a})$, the orbit $(K_2 \times K_1)\cdot x$ is biharmonic in G if and only if*

$$\sum_{\lambda\in\Sigma^+\setminus\Sigma_H} m(\lambda)\langle dL_x^{-1}(\tau_H)_x, \lambda\rangle\left(\frac{3}{2}-(\cot\langle\lambda,H\rangle)^2\right)\lambda$$
$$+\sum_{\alpha\in W^+\setminus W_H} n(\alpha)\langle dL_x^{-1}(\tau_H)_x, \alpha\rangle\left(\frac{3}{2}-(\tan\langle\alpha,H\rangle)^2\right)\alpha$$
$$+\sum_{\mu\in\Sigma_H^+} m(\mu)\langle dL_x^{-1}(\tau_H)_x, \mu\rangle\mu+\sum_{\beta\in W_H^+} n(\beta)\langle dL_x^{-1}(\tau_H)_x, \beta\rangle\beta=0$$

holds.

PROOF. Since $(G, \langle\ ,\ \rangle)$ is a Riemannian symmetric space, the curvature tensor $R^{\langle,\rangle}$ of $(G, \langle\ ,\ \rangle)$ is given by

$$R^{\langle,\rangle}(dL_x(X), dL_x(Y))dL_x(Z) = -dL_x[[X,Y],Z] \qquad (X,Y,Z\in\mathfrak{g}).$$

Hence, we have

- for $\lambda\in\Sigma^+\setminus\Sigma_H$, $1\le i\le m(\lambda)$,
$$R^{\langle,\rangle}((\tau_H)_x, dL_x(T_{\lambda,i}))dL_x(T_{\lambda,i}) = \langle dL_x^{-1}(\tau_H)_x, \lambda\rangle dL_x(\lambda),$$
- for $\lambda\in\Sigma^+$, $1\le i\le m(\lambda)$,
$$R^{\langle,\rangle}((\tau_H)_x, dL_x(S_{\lambda,i}))dL_x(S_{\lambda,i}) = \langle dL_x^{-1}(\tau_H)_x, \lambda\rangle dL_x(\lambda),$$
- for $\alpha\in W^+\setminus W_H$, $1\le j\le n(\alpha)$,
$$R^{\langle,\rangle}((\tau_H)_x, dL_x(Y_{\alpha,j}))dL_x(Y_{\alpha,j}) = \langle dL_x^{-1}(\tau_H)_x, \alpha\rangle dL_x(\alpha),$$
- for $\alpha\in W^+$, $1\le j\le n(\alpha)$,
$$R^{\langle,\rangle}((\tau_H)_x, dL_x(X_{\alpha,j}))dL_x(X_{\alpha,j}) = \langle dL_x^{-1}(\tau_H)_x, \alpha\rangle dL_x(\alpha),$$
- for $X\in\mathfrak{k}_0\oplus V(\mathfrak{k}_1\cap\mathfrak{m}_2)\oplus V(\mathfrak{m}_1\cap\mathfrak{k}_2)$,
$$R^{\langle,\rangle}((\tau_H)_x, dL_x(X))dL_x(X)=0.$$

On the other hand, for each $\lambda \in \Sigma^+ \backslash \Sigma_H$, $1 \le i \le m(\lambda)$ and $X \in \mathrm{Ad}(x)^{-1}(\mathfrak{k}_2) + \mathfrak{k}_1$, by Theorem 5, we have

$$
\begin{aligned}
&\langle A_{(\tau_H)_x}\, dL_x(T_{\lambda,i}), dL_x(X)\rangle \\
&\quad = \langle B_H(dL_x(T_{\lambda,i}), dL_x(X)), (\tau_H)_x\rangle \\
&\quad = \begin{cases}
0 & (X \in \mathfrak{k}_0 \oplus V(\mathfrak{m}_1 \cap \mathfrak{k}_2)) \\
-\frac{1}{2}\langle dL_x[X, T_{\lambda,i}]^{\perp}, (\tau_H)_x\rangle & (X \in V(\mathfrak{k}_1 \cap \mathfrak{m}_2) \oplus \sum_{\mu \in \Sigma^+} \mathfrak{k}_\mu \\
& \quad \oplus \sum_{\alpha \in W^+} V_\alpha^{\perp}(\mathfrak{k}_1 \cap \mathfrak{m}_2)) \\
\cot\langle \mu, H\rangle \langle dL_x[T_{\lambda,i}, S_{\mu,j}]^{\perp}, (\tau_H)_x\rangle & (X = T_{\mu,j} \text{ for } \mu \in \Sigma^+ \backslash \Sigma_H, \\
& \quad 1 \le j \le m(\lambda)) \\
-\tan\langle \alpha, H\rangle \langle dL_x[T_{\lambda,i}, X_{\alpha,j}]^{\perp}, (\tau_H)_x\rangle & (X = Y_{\alpha,j} \text{ for } \alpha \in W^+ \backslash W_H, \\
& \quad 1 \le j \le n(\alpha))
\end{cases} \\
&\quad = \begin{cases}
-\frac{1}{2}\langle dL_x^{-1}(\tau_H)_x, \lambda\rangle & (X = S_{\lambda,i}) \\
-\cot\langle \lambda, H\rangle \langle dL_x^{-1}(\tau_H)_x, \lambda\rangle & (X = T_{\lambda,i}) \\
0 & (\text{if } \langle X, S_{\lambda,i}\rangle = \langle X, T_{\lambda,i}\rangle = 0).
\end{cases}
\end{aligned}
$$

Thus, we have

$$
A_{(\tau_H)_x}\, dL_x(T_{\lambda,i}) = -\frac{1}{2}\langle dL_x^{-1}(\tau_H)_x, \lambda\rangle dL_x S_{\lambda,i} - \cot\langle \lambda, H\rangle \langle dL_x^{-1}(\tau_H)_x, \lambda\rangle dL_x T_{\lambda,i}.
$$

Therefore, we obtain

$$
\begin{aligned}
&B_H(A_{(\tau_H)_x}\, dL_x(T_{\lambda,i}), dL_x(T_{\lambda,i})) \\
&\quad = -\frac{1}{2}\langle dL_x^{-1}(\tau_H)_x, \lambda\rangle B_H(dL_x(S_{\lambda,i}), dL_x(T_{\lambda,i})) \\
&\qquad - \langle dL_x^{-1}(\tau_H)_x, \lambda\rangle \cot(\langle \lambda, H\rangle) B_H(dL_x(T_{\lambda,i}), dL_x(T_{\lambda,i})) \\
&\quad = \langle dL_x^{-1}(\tau_H)_x, \lambda\rangle \left(\frac{1}{4} + (\cot\langle \lambda, H\rangle)^2\right) dL_x(\lambda)
\end{aligned}
$$

for $\lambda \in \Sigma^+ \backslash \Sigma_H$, $1 \le i \le m(\lambda)$. Similarly, we have

- for $\lambda \in \Sigma^+ \backslash \Sigma_H$, $1 \le i \le m(\lambda)$,

$$
\begin{aligned}
&B_H(A_{(\tau_H)_x}\, dL_x(S_{\lambda,i}), dL_x(S_{\lambda,i})) \\
&\quad = -\frac{1}{2}\langle dL_x^{-1}(\tau_H)_x, \lambda\rangle B_H(dL_x(T_{\lambda,i}), dL_x(S_{\lambda,i})) \\
&\quad = \frac{1}{4}\langle dL_x^{-1}(\tau_H)_x, \lambda\rangle dL_x(\lambda),
\end{aligned}
$$

- for $\alpha \in W^+ \backslash W_H,\ 1 \le j \le n(\alpha)$,

$$\begin{aligned} &B_H(A_{(\tau_H)_x}\, dL_x(Y_{\alpha,j}), dL_x(Y_{\alpha,j})) \\ &\quad = -\frac{1}{2}\langle dL_x^{-1}(\tau_H)_x, \alpha\rangle B_H(dL_x(X_{\alpha,j}), dL_x(Y_{\alpha,j})) \\ &\qquad + \langle dL_x^{-1}(\tau_H)_x, \alpha\rangle \tan(\langle \alpha, H\rangle) B_H(dL_x(Y_{\alpha,j}), dL_x(Y_{\alpha,j})) \\ &\quad = \langle dL_x^{-1}(\tau_H)_x, \alpha\rangle \left(\frac{1}{4} + (\tan\langle \alpha, H\rangle)^2\right) dL_x(\alpha), \end{aligned}$$

- for $\alpha \in W^+ \backslash W_H,\ 1 \le j \le n(\alpha)$,

$$\begin{aligned} &B_H(A_{(\tau_H)_x}\, dL_x(X_{\alpha,j}), dL_x(X_{\alpha,j})) \\ &\quad = -\frac{1}{2}\langle dL_x^{-1}(\tau_H)_x, \alpha\rangle B_H(dL_x(Y_{\alpha,j}), dL_x(X_{\alpha,j})) \\ &\quad = \frac{1}{4}\langle dL_x^{-1}(\tau_H)_x, \alpha\rangle dL_x(\alpha), \end{aligned}$$

- for $X \in \mathfrak{k}_0 \oplus V(\mathfrak{k}_1 \cap \mathfrak{m}_2) \oplus V(\mathfrak{m}_1 \cap \mathfrak{k}_2) \oplus \sum_{\lambda \in \Sigma_H^+} \mathfrak{k}_\lambda \oplus \sum_{\alpha \in W_H^+} V_\alpha^\perp(\mathfrak{k}_1 \cap \mathfrak{m}_2)$,

$$B_H(A_{(\tau_H)_x}\, dL_x(X), dL_x(X)) = 0.$$

Therefore, by Theorem 1, we have the consequence.

When $\dim \mathfrak{a} = 1$, we have the following corollary.

COROLLARY 4. *Let (G, K_1, K_2) be a commutative compact symmetric triad which satisfies $\dim \mathfrak{a} = 1$, i.e. $\tilde{\Sigma} \subset \{\alpha, 2\alpha\}$. For $x = \exp H$ $(H \in \mathfrak{a})$, suppose that $(K_2 \times K_1) \cdot x$ is a regular orbit. Then the orbit $(K_2 \times K_1) \cdot x$ is biharmonic in G if and only if*

$$\begin{aligned} \langle dL_x^{-1}(\tau_H)_x, \alpha\rangle \Bigg(& m(\alpha)\left\{\frac{3}{2} - (\cot\langle \alpha, H\rangle)^2\right\} + 4m(2\alpha)\left\{\frac{3}{2} - (\cot\langle 2\alpha, H\rangle)^2\right\} \\ & + n(\alpha)\left\{\frac{3}{2} - (\tan\langle \alpha, H\rangle)^2\right\} + 4n(2\alpha)\left\{\frac{3}{2} - (\tan\langle 2\alpha, H\rangle)^2\right\}\Bigg) = 0 \end{aligned}$$

holds. Here, for $\lambda \in \mathfrak{a}$, if $\lambda \notin \Sigma$ (resp. $\lambda \notin W$), then $m(\lambda) = 0$ (resp. $n(\lambda) = 0$).

5. Biharmonic homogeneous submanifolds in compact symmetric spaces

In the previous section, we characterized the biharmonic property of orbits of commutative Hermann actions and the actions of the direct product of two symmetric subgroups on compact Lie groups in terms of symmetric triad with multiplicities. In this section, we study proper biharmonic orbits of commutative Hermann actions by using Theorems 4 and 6. In [OSU], we classified proper biharmonic hypersurfaces in irreducible compact symmetric spaces which are orbits of commutative Hermann actions. In order to obtain proper biharmonic orbits of higher codimension, we have to consider the cases of $\dim \mathfrak{a} \geq 2$. Here we determine the biharmonic properties of singular orbits of commutative Hermann actions in the cases of $\dim \mathfrak{a} = 2$. Then all cohomogeneity two isotropy actions and commutative Hermann actions satisfying the condition (A), (B), or (C) in Theorem 2 are classified as follows:

Cases of $\theta_1 = \theta_2$ (isotropy actions). Since $K_1 = K_2$, we show the list of irreducible symmetric pairs of compact type of rank two.

- Type A_2
  - $(\mathrm{SU}(3), \mathrm{SO}(3))$,
  - $(\mathrm{SU}(3) \times \mathrm{SU}(3), \mathrm{SU}(3))$,
  - $(\mathrm{SU}(6), \mathrm{Sp}(3))$,
  - (E_6, F_4),
- Type B_2
  - $(\mathrm{SO}(5) \times \mathrm{SO}(5), \mathrm{SO}(5))$,
  - $(\mathrm{SO}(4+n), \mathrm{SO}(2) \times \mathrm{SO}(2+n))$,
- Type C_2
  - $(\mathrm{Sp}(2), \mathrm{U}(2))$,
  - $(\mathrm{Sp}(2) \times \mathrm{Sp}(2), \mathrm{Sp}(2))$,
  - $(\mathrm{Sp}(4), \mathrm{Sp}(2) \times \mathrm{Sp}(2))$,
  - $(\mathrm{SU}(4), \mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(2)))$,
  - $(\mathrm{SO}(8), \mathrm{U}(4))$,
- Type BC_2
  - $(\mathrm{SU}(4+n), \mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(2+n)))$,
  - $(\mathrm{SO}(10), \mathrm{U}(5))$,
  - $(\mathrm{Sp}(4+n), \mathrm{Sp}(2) \times \mathrm{Sp}(2+n))$,
  - $(E_6, \mathrm{T}^1 \cdot \mathrm{Spin}(10))$,
- Type G_2
  - $(G_2, \mathrm{SO}(4))$,
  - $(G_2 \times G_2, G_2)$,

Cases of $\theta_1 \not\sim \theta_2$. The following classification is due to O. Ikawa [I2].

- Type I-B_2
  - $(\mathrm{SO}(2+s+t), \mathrm{SO}(2+s) \times \mathrm{SO}(t), \mathrm{SO}(2) \times \mathrm{SO}(s+t))$ $(2 < t,\ 1 \leq s)$,
  - $(\mathrm{SO}(6) \times \mathrm{SO}(6), \Delta(\mathrm{SO}(6) \times \mathrm{SO}(6)), K_2)$ (condition (C)).
    Here $K_2 = \{(u_1, u_2) \in \mathrm{SO}(6) \times \mathrm{SO}(6) \mid (\sigma(u_2), \sigma(u_1)) = (u_1, u_2)\}$ and σ is an involutive outer automorphism on $\mathrm{SO}(6)$. Then $(G_\sigma)_0 \cong \mathrm{SO}(3) \times \mathrm{SO}(3)$ or $\mathrm{SO}(5)$.

- Type I-C_2
  - $(\mathrm{SO}(8), \mathrm{SO}(4) \times \mathrm{SO}(4), \mathrm{U}(4))$,
  - $(\mathrm{SU}(4), \mathrm{SO}(4), \mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(2)))$,
  - $(\mathrm{SU}(4) \times \mathrm{SU}(4), \Delta(\mathrm{SU}(4) \times \mathrm{SU}(4)), K_2)$ (condition (C)).
    Here $K_2 = \{(u_1, u_2) \in \mathrm{SU}(4) \times \mathrm{SU}(4) \mid (\sigma(u_2), \sigma(u_1)) = (u_1, u_2)\}$ and σ is an involutive outer automorphism on $\mathrm{SU}(4)$. Then $(G_\sigma)_0 \cong \mathrm{SO}(4)$.
  - $(\mathrm{SU}(4) \times \mathrm{SU}(4), \Delta(\mathrm{SU}(4) \times \mathrm{SU}(4)), K_2)$ (condition (C)).
    Here $K_2 = \{(u_1, u_2) \in \mathrm{SU}(4) \times \mathrm{SU}(4) \mid (\sigma(u_2), \sigma(u_1)) = (u_1, u_2)\}$ and σ is an involutive outer automorphism on $\mathrm{SU}(4)$. Then $(G_\sigma)_0 \cong \mathrm{Sp}(2)$.
- Type I-BC_2-A_1^2
  - $(\mathrm{SU}(2+s+t), \mathrm{S}(\mathrm{U}(2+s) \times \mathrm{U}(t)), \mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(s+t)))$ $(2 < t,\ 1 \leq s)$,
  - $(\mathrm{Sp}(2+s+t), \mathrm{Sp}(2+s) \times \mathrm{Sp}(t), \mathrm{Sp}(2) \times \mathrm{Sp}(s+t))$ $(2 < t,\ 1 \leq s)$,
  - $(\mathrm{SO}(12), \mathrm{U}(6), \mathrm{U}(6)')$. Here, we define $\mathrm{U}(6)' = \{g \in \mathrm{SO}(12) \mid JgJ^{-1} = g\}$, where

$$J = \left[\begin{array}{cc|cc} & & I_5 & \\ & & & -1 \\ \hline -I_5 & & & \\ & 1 & & \end{array}\right]$$

    and I_l denotes the identity matrix of $l \times l$.
- Type I-BC_2-B_2
  - $(\mathrm{SO}(4+2s), \mathrm{SO}(4) \times \mathrm{SO}(2s), \mathrm{U}(2+s))$ $(2 < s)$,
  - $(E_6, \mathrm{SU}(6) \cdot \mathrm{SU}(2), \mathrm{SO}(10) \cdot \mathrm{U}(1))$,
  - $(E_7, \mathrm{SO}(12) \cdot \mathrm{SU}(2), E_6 \cdot \mathrm{U}(1))$,
- Type II-BC_2
  - $(\mathrm{SU}(2+s), \mathrm{SO}(2+s), \mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(s)))$ $(2 < s)$,
  - $(\mathrm{SO}(10), \mathrm{SO}(5) \times \mathrm{SO}(5), \mathrm{U}(5))$,
  - $(E_6, \mathrm{Sp}(4), \mathrm{SO}(10) \cdot \mathrm{U}(1))$,
  - $(\mathrm{SU}(5) \times \mathrm{SU}(5), \Delta(\mathrm{SU}(5) \times \mathrm{SU}(5)), K_2)$ (condition (C)).
    Here $K_2 = \{(u_1, u_2) \in \mathrm{SU}(5) \times \mathrm{SU}(5) \mid (\sigma(u_2), \sigma(u_1)) = (u_1, u_2)\}$ and σ is an involutive outer automorphism on $\mathrm{SU}(5)$. Then $(G_\sigma)_0 \cong \mathrm{SO}(5)$.
- Type III-A_2
  - $(\mathrm{SU}(6), \mathrm{Sp}(3), \mathrm{SO}(6))$,
  - $(E_6, \mathrm{Sp}(4), F_4)$,
  - $(U \times U, \Delta(U \times U), \bar{K} \times \bar{K})$ (condition (B)).
    Here $(U, \bar{K})$ is a compact symmetric pair of type A_2.
- Type III-B_2

– $(U \times U, \Delta(U \times U), \overline{K} \times \overline{K})$ (condition (B)).
Here $(U, \overline{K})$ is a compact symmetric pair of type B_2.

- Type III-C_2
  – $(\mathrm{SU}(8), \mathrm{S}(\mathrm{U}(4) \times \mathrm{U}(4)), \mathrm{Sp}(4))$,
  – $(\mathrm{Sp}(4), \mathrm{U}(4), \mathrm{Sp}(2) \times \mathrm{Sp}(2))$,
  – $(U \times U, \Delta(U \times U), \overline{K} \times \overline{K})$ (condition (B)).
  Here $(U, \overline{K})$ is a compact symmetric pair of type C_2.
- Type III-BC_2
  – $(\mathrm{SU}(4+2s), \mathrm{S}(\mathrm{U}(4) \times \mathrm{U}(2s)), \mathrm{Sp}(2+s))$ $(2 < s)$,
  – $(\mathrm{SU}(10), \mathrm{S}(\mathrm{U}(5) \times \mathrm{U}(5)), \mathrm{Sp}(5))$,
  – $(\mathrm{Sp}(2+s), \mathrm{U}(2+s), \mathrm{Sp}(2) \times \mathrm{Sp}(s))$ $(2 < s)$,
  – $(U \times U, \Delta(U \times U), \overline{K} \times \overline{K})$ (condition (B)).
  Here $(U, \overline{K})$ is a compact symmetric pair of type BC_2.
- Type III-G_2
  – $(U \times U, \Delta(U \times U), \overline{K} \times \overline{K})$ (condition (B)).
  Here $(U, \overline{K})$ is a compact symmetric pair of type G_2.

In the following, we consider the biharmonic properties of singular orbits of Hermann actions for each compact symmetric triad in the above list. For $H \in \mathfrak{a}$, we set $x = \exp(H)$ and consider the orbit $K_2 \cdot \pi_1(x)$ of the K_2-action on N_1 through $\pi_1(x)$. For simplicity, we denote the tension field $dL_x^{-1}(\tau_H^1)_{\pi_1(x)}$ by τ_H. According to the classification of types of symmetric triads, we examine biharmonic property of orbits individually. Hereafter, we assume that N_1 and N_2 are simply-connected.

Cases of $\theta_1 \sim \theta_2$. First, we examine isotropy actions of compact symmetric spaces. When $\theta_1 \sim \theta_2$, Hermann actions are orbit equivalent to isotropy actions of compact symmetric spaces. We set a basis $\{H_\alpha\}_{\alpha \in \Pi}$ of $\mathfrak{a}$ as follows;

$$\langle H_\alpha, \beta \rangle = 0 \quad (\alpha \neq \beta,\ \alpha, \beta \in \Pi), \qquad \langle H_\alpha, \delta \rangle = \pi,$$

where δ is the highest root of Σ. Then we have

$$P_0 = \left\{ \sum_{\alpha \in \Pi} t_\alpha H_\alpha \,\middle|\, t_\alpha > 0 \ (\alpha \in \Pi), \sum_{\alpha \in \Pi} t_\alpha < 1 \right\}.$$

From (6) and (7), the orbit space of an isotropy action is described as $\overline{P_0} = \bigcup_{\Delta \subset \Pi \cup \{\delta\}} P_0^\Delta$. Since $\dim \mathfrak{a} = 2$, $\Pi = \{\alpha_1, \alpha_2\}$ and P_0 is a triangle region in $\mathfrak{a}$. We apply Theorem 6 to the following three cases;

(1) $H \in P_0^{\{\alpha_1, \delta\}} = \{tH_{\alpha_1} \mid 0 < t < 1\}$,
(2) $H \in P_0^{\{\alpha_2, \delta\}} = \{tH_{\alpha_2} \mid 0 < t < 1\}$,
(3) $H \in P_0^{\{\alpha_1, \alpha_2\}} = \{tH_{\alpha_1} + (1-t)H_{\alpha_2} \mid 0 < t < 1\}$.

These three cases correspond to three edges of $\overline{P_0}$. In these cases, we can solve the equation (10) in Theorem 6 concretely in most cases. In the following, we compute the equation (10) in Theorem 6 for each root type.

5.1. Type A$_2$. We set

$$\mathfrak{a} = \{\xi_1 e_1 + \xi_2 e_2 + \xi_3 e_3 \,|\, \xi_1 + \xi_3 + \xi_3 = 0\}.$$

Then, we have

$$\Sigma^+ = \{\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3, \alpha_1 + \alpha_2\}, \qquad W^+ = \emptyset,$$
$$m = m(\alpha) \qquad (\alpha \in \Sigma).$$

(1) When $H \in P_0^{\{\alpha_1,\delta\}} = \{tH_{\alpha_1} \,|\, 0 < t < 1\}$, we have $\Sigma_H^+ = \{\alpha_2\}$. Hence we obtain

$$\tau_H = m\cot\langle\alpha_1, H\rangle\alpha_1 + m\cot\langle\alpha_1 + \alpha_2, H\rangle(\alpha_1 + \alpha_2) = m\cot\langle\alpha_1, H\rangle(2\alpha_1 + \alpha_2).$$

Thus the orbit $K_2 \cdot \pi_1(x)$ is harmonic if and only if $\langle\alpha_1, H\rangle = \pi/2$. By Theorem 6, the orbit $K_2 \cdot \pi_1(x)$ is biharmonic if and only if

$$\begin{aligned} 0 &= m\langle\tau_H, \alpha_1\rangle(1 - (\cot\langle\alpha_1, H\rangle)^2)\alpha_1 \\ &\quad + m\langle\tau_H, \alpha_1 + \alpha_2\rangle(1 - (\cot\langle\alpha_1 + \alpha_2, H\rangle)^2)(\alpha_1 + \alpha_2) \\ &= m\langle\tau_H, \alpha_1\rangle(1 - (\cot\langle\alpha_1, H\rangle)^2)(2\alpha_1 + \alpha_2). \end{aligned}$$

Thus we have $\tau_H = 0$ or $\langle\alpha_1, H\rangle = (1/4)\pi$, $(3/4)\pi$. Therefore, the orbit $K_2 \cdot \pi_1(x)$ is proper biharmonic if and only if $\langle\alpha_1, H\rangle = (1/4)\pi$ or $(3/4)\pi$. *In this case, there exist exactly two proper biharmonic orbits.* By the same argument, we have the followings:

(2) The orbit $K_2 \cdot \pi_1(x)$ is proper biharmonic if and only if $\langle\alpha_2, H\rangle = (1/4)\pi$, $(3/4)\pi$ for $H = tH_{\alpha_2}$ $(0 < t < 1)$.

(3) The orbit $K_2 \cdot \pi_1(x)$ is proper biharmonic if and only if $\langle\alpha_1, H\rangle = (1/4)\pi$, $(3/4)\pi$ for $H = tH_{\alpha_1} + (1-t)H_{\alpha_2}$ $(0 < t < 1)$.

5.2. Type B$_2$ **and** C$_2$. We set

$$\Sigma^+ = \{\alpha_1 = e_1 - e_2, \alpha_2 = e_2, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2\}, \qquad W^+ = \emptyset,$$
$$\delta = \alpha_1 + 2\alpha_2 = e_1 + e_2,$$

and

$$m_1 = m(e_1), \qquad m_2 = m(e_1 - e_2).$$

(1) When $H \in P_0^{\{\alpha_1,\delta\}} = \{tH_{\alpha_1} \mid 0 < t < 1\}$, we have $\Sigma_H^+ = \{e_2\}$. By Theorem 4, we have

$$\begin{aligned}\tau_H &= -m_2 \cot\langle\alpha_1, H\rangle\alpha_1 - m_1 \cot\langle\alpha_1 + \alpha_2, H\rangle(\alpha_1 + \alpha_2)\\ &\quad - m_2 \cot\langle\alpha_1 + 2\alpha_2, H\rangle(\alpha_1 + 2\alpha_2)\\ &= -(2m_2 + m_1) \cot\langle\alpha_1, H\rangle(\alpha_1 + \alpha_2).\end{aligned}$$

Hence, $\tau_H = 0$ if and only if $\langle\alpha_1, H\rangle = \pi/2$. By Theorem 6, $K_2 \cdot \pi_1(x)$ is biharmonic if and only if

$$\begin{aligned}0 &= m_2\langle\tau_H, \alpha_1\rangle(1 - (\cot\langle\alpha_1, H\rangle)^2)\alpha_1\\ &\quad + m_1\langle\tau_H, \alpha_1 + \alpha_2\rangle(1 - (\cot\langle\alpha_1, H\rangle)^2)(\alpha_1 + \alpha_2)\\ &\quad + m_2\langle\tau_H, \alpha_1 + 2\alpha_2\rangle(1 - (\cot\langle\alpha_1, H\rangle)^2)(\alpha_1 + 2\alpha_2)\\ &= \langle\tau_H, \alpha_1\rangle(2m_2 + m_1)(1 - (\cot\langle\alpha_1, H\rangle)^2)(\alpha_1 + \alpha_2).\end{aligned}$$

Therefore, the orbit $K_2 \cdot \pi_1(x)$ is biharmonic if and only if $\tau_H = 0$ or $\langle\alpha_1, H\rangle = \pi/4$, $(3/4)\pi$. In particular, $K_2 \cdot \pi_1(x)$ is proper biharmonic if and only if $\langle\alpha_1, H\rangle = \pi/4$ or $(3/4)\pi$. *In this case, there exist exactly two proper biharmonic orbits.*

(2) When $H \in P_0^{\{\alpha_2,\delta\}} = \{tH_{\alpha_2} \mid 0 < t < 1\}$, we have $\Sigma_H^+ = \{e_1 - e_2\}$. By Theorem 4, we have

$$\begin{aligned}\tau_H &= -m_1 \cot\langle\alpha_2, H\rangle\alpha_2 - m_1 \cot\langle\alpha_1 + \alpha_2, H\rangle(\alpha_1 + \alpha_2)\\ &\quad - m_2 \cot\langle\alpha_1 + 2\alpha_2, H\rangle(\alpha_1 + 2\alpha_2)\\ &= -\frac{1}{2}\{(2m_1 + m_2) \cot\langle\alpha_2, H\rangle - m_2 \tan\langle\alpha_2, H\rangle\}(\alpha_1 + 2\alpha_2).\end{aligned}$$

Hence, $\tau_H = 0$ if and only if

$$(\cot\langle\alpha_2, H\rangle)^2 = \frac{m_2}{2m_1 + m_2}.$$

By Theorem 6, the orbit $K_2 \cdot \pi_1(x)$ is biharmonic if and only if

$$\begin{aligned}0 &= m_1\langle\tau_H, \alpha_2\rangle(1 - (\cot\langle\alpha_2, H\rangle)^2)\alpha_2\\ &\quad + m_1\langle\tau_H, \alpha_1 + \alpha_2\rangle(1 - (\cot\langle\alpha_2, H\rangle)^2)(\alpha_1 + \alpha_2)\\ &\quad + m_2\langle\tau_H, \alpha_1 + 2\alpha_2\rangle(1 - (\cot\langle 2\alpha_2, H\rangle)^2)(\alpha_1 + 2\alpha_2)\\ &= \frac{1}{2}\langle\tau_H, \alpha_2\rangle\{(2m_1 + m_2)(1 - (\cot\langle\alpha_2, H\rangle)^2)\\ &\quad + m_2(1 - (\tan\langle\alpha_2, H\rangle)^2) + 4m_2\}(\alpha_1 + 2\alpha_2).\end{aligned}$$

Therefore, the orbit $K_2 \cdot \pi_1(x)$ is biharmonic if and only if $\tau_H = 0$ or

$$(2m_1 + m_2)(1 - (\cot\langle\alpha_2, H\rangle)^2) + m_2(1 - (\tan\langle\alpha_2, H\rangle)^2) + 4m_2 = 0$$

holds. The last equation is equivalent to

$$((2m_1 + m_2)(\cot\langle\alpha_2, H\rangle)^2 - m_2)((\cot\langle\alpha_2, H\rangle)^2 - 1) = 4m_2(\cot\langle\alpha_2, H\rangle)^2.$$

Since $m_2 > 0$, the solutions of the equation are not harmonic. Hence the orbit $K_2 \cdot \pi_1(x)$ is proper biharmonic if and only if

$$(\cot\langle\alpha_2, H\rangle)^2 = \frac{m_1 + 3m_2 \pm \sqrt{m_1^2 + 4m_1m_2 + 8m_2^2}}{2m_1 + m_2}.$$

In this case, there exist exactly two proper biharmonic orbits.

(3) When $H \in P_0^{\{\alpha_1, \alpha_2\}} = \{tH_{\alpha_1} + (1-t)H_{\alpha_2} \mid 0 < t < 1\}$, we have $\Sigma_H^+ = \{e_1 + e_2\}$ and $\langle\alpha_2, H\rangle = (\pi/2) - \langle\alpha_1, H\rangle/2$. By Theorem 4, we have

$$\begin{aligned}\tau_H &= -m_2 \cot\langle\alpha_1, H\rangle\alpha_1 - m_1 \cot\langle\alpha_2, H\rangle\alpha_2 - m_1 \cot\langle\alpha_1 + \alpha_2, H\rangle(\alpha_1 + \alpha_2) \\ &= \frac{1}{2}\left\{-m_2 \cot\left(\frac{\langle\alpha_1, H\rangle}{2}\right) + (2m_1 + m_2) \tan\left(\frac{\langle\alpha_1, H\rangle}{2}\right)\right\}\alpha_1.\end{aligned}$$

Hence, $\tau_H = 0$ if and only if

$$\left(\cot\frac{\langle\alpha_1, H\rangle}{2}\right)^2 = \frac{2m_1 + m_2}{m_2}.$$

By Theorem 6, the orbit $K_2 \cdot \pi_1(x)$ is biharmonic if and only if

$$\begin{aligned}0 &= m_2\langle\tau_H, \alpha_1\rangle(1 - (\cot\langle\alpha_1, H\rangle)^2)\alpha_1 + m_1\langle\tau_H, \alpha_2\rangle(1 - (\cot\langle\alpha_2, H\rangle)^2)\alpha_2 \\ &\quad + m_1\langle\tau_H, \alpha_1 + \alpha_2\rangle(1 - (\cot\langle\alpha_1 + \alpha_2, H\rangle)^2)(\alpha_1 + \alpha_2) \\ &= \frac{1}{4}\langle\tau_H, \alpha_1\rangle\left\{4m_2 + m_2\left(1 - \left(\cot\frac{\langle\alpha_1, H\rangle}{2}\right)^2\right)\right. \\ &\quad \left. + (2m_1 + m_2)\left(1 - \left(\tan\frac{\langle\alpha_1, H\rangle}{2}\right)^2\right)\right\}\alpha_1.\end{aligned}$$

Therefore, the orbit $K_2 \cdot \pi_1(x)$ is biharmonic if and only if $\tau_H = 0$ or

$$4m_2 + m_2\left(1 - \left(\cot\frac{\langle\alpha_1, H\rangle}{2}\right)^2\right) + (2m_1 + m_2)\left(1 - \left(\tan\frac{\langle\alpha_1, H\rangle}{2}\right)^2\right) = 0$$

holds. The last equation is equivalent to

$$\left(m_2\left(\cot\frac{\langle\alpha_1,H\rangle}{2}\right)^2-(2m_1+m_2)\right)\left(\left(\cot\frac{\langle\alpha_1,H\rangle}{2}\right)^2-1\right)=4m_2\left(\cot\frac{\langle\alpha_1,H\rangle}{2}\right)^2.$$

Since $m_2>0$, the solutions of the equation are not harmonic. Hence the orbit $K_2\cdot\pi_1(x)$ is proper biharmonic if and only if

$$\left(\cot\frac{\langle\alpha_1,H\rangle}{2}\right)^2=\frac{m_1+3m_2\pm\sqrt{m_1^2+4m_1m_2+8m_2^2}}{m_2}$$

holds. *In this case, there exist exactly two proper biharmonic orbits.*

5.3. Type BC_2**.** We set

$$\Sigma^+=\{\alpha_1=e_1-e_2,\alpha_2=e_2,\alpha_1+\alpha_2,\alpha_1+2\alpha_2,2\alpha_2,2\alpha_1+2\alpha_2\},$$
$$W^+=\varnothing,\qquad \delta=2\alpha_1+2\alpha_2,$$

and

$$m_1=m(e_1),\qquad m_2=m(e_1-e_2),\qquad m_3=m(2e_1).$$

(1) When $H\in P_0^{\{\alpha_1,\delta\}}=\{tH_{\alpha_1}\,|\,0<t<1\}$, we have $\Sigma_H^+=\{e_2,2e_2\}$. By Theorem 4, we have

$$\tau_H=\{-(m_1+2m_2+m_3)\cot\langle\alpha_1,H\rangle+m_3\tan\langle\alpha_1,H\rangle\}(\alpha_1+\alpha_2).$$

Hence, $\tau_H=0$ if and only if

$$(\cot\langle\alpha_1,H\rangle)^2=\frac{m_3}{m_1+2m_2+m_3}$$

holds. By Theorem 6, $K_2\cdot\pi_1(x)$ is biharmonic if and only if

$$0=\langle\tau_H,\alpha_1\rangle\{(2m_2+m_1+m_3)(1-(\cot\langle\alpha_1,H\rangle)^2)$$
$$+m_3(1-(\tan\langle\alpha_1,H\rangle)^2)+4m_3\}(\alpha_1+\alpha_2).$$

Therefore, the orbit $K_2\cdot\pi_1(x)$ is biharmonic if and only if $\tau_H=0$ or

$$(2m_2+m_1+m_3)(1-(\cot\langle\alpha_1,H\rangle)^2)+m_3(1-(\tan\langle\alpha_1,H\rangle)^2)+4m_3=0$$

holds. The last equation is equivalent to

$$((2m_2+m_1+m_3)(\cot\langle\alpha_1,H\rangle)^2-m_3)((\cot\langle\alpha_1,H\rangle)^2-1)=4m_3(\cot\langle\alpha_1,H\rangle)^2.$$

Since $m_3>0$, the solutions of the equation are not harmonic. Hence the orbit $K_2\cdot\pi_1(x)$ is proper biharmonic if and only if

$$(\cot\langle\alpha_1, H\rangle)^2$$
$$= \frac{m_1 + 2m_2 + 6m_3 \pm \sqrt{(m_1 + 2m_2 + 6m_3)^2 - 4(m_1 + 2m_2 + m_3)m_3}}{2(m_1 + 2m_2 + m_3)}.$$

In this case, there exist exactly two proper biharmonic orbits.

(2) When $H \in P_0^{\{\alpha_2,\delta\}} = \{tH_{\alpha_2} \mid 0 < t < 1\}$, we have $\Sigma_H^+ = \{e_1 - e_2\}$. By Theorem 4, we have

$$\tau_H = -\frac{1}{2}\{(2m_1 + m_2 + 2m_3)\cot\langle\alpha_2, H\rangle - (m_2 + 2m_3)\tan\langle\alpha_2, H\rangle\}(\alpha_1 + 2\alpha_2).$$

Hence, $\tau_H = 0$ if and only if

$$(\cot\langle\alpha_2, H\rangle)^2 = \frac{m_2 + 2m_3}{2m_1 + m_2 + 2m_3}.$$

By Theorem 6, the orbit $K_2 \cdot \pi_1(x)$ is biharmonic if and only if

$$0 = \frac{1}{2}\langle\tau_H, \alpha_2\rangle\{(2m_1 + m_2 + 2m_3)(1 - (\cot\langle\alpha_2, H\rangle)^2)$$
$$+ (m_2 + 2m_3)(1 - (\tan\langle\alpha_2, H\rangle)^2) + 4(m_2 + 2m_3)\}(\alpha_1 + 2\alpha_2).$$

Therefore, the orbit $K_2 \cdot \pi_1(x)$ is biharmonic if and only if $\tau_H = 0$ or

$$(2m_1 + m_2 + 2m_3)(1 - (\cot\langle\alpha_2, H\rangle)^2)$$
$$+ (m_2 + 2m_3)(1 - (\tan\langle\alpha_2, H\rangle)^2) + 4(m_2 + 2m_3) = 0$$

holds. The last equation is equivalent to

$$((2m_1 + m_2 + 2m_3)(\cot\langle\alpha_2, H\rangle)^2 - (m_2 + 2m_3))((\cot\langle\alpha_2, H\rangle)^2 - 1)$$
$$= 4(m_2 + 2m_3)(\cot\langle\alpha_2, H\rangle)^2.$$

Since $m_2 + 2m_3 > 0$, the solutions of the equation are not harmonic. Hence the orbit $K_2 \cdot \pi_1(x)$ is proper biharmonic if and only if

$$(\cot\langle\alpha_2, H\rangle)^2 = \frac{m_1 + 3(m_2 + 2m_3) \pm \sqrt{m_1^2 + 4m_1(m_2 + 2m_3) + 8(m_2 + 2m_3)^2}}{2m_1 + m_2 + 2m_3}.$$

In this case, there exist exactly two proper biharmonic orbits.

(3) When $H \in P_0^{\{\alpha_1,\alpha_2\}} = \{tH_{\alpha_1} + (1 - t)H_{\alpha_2} \mid 0 < t < 1\}$, we have $\Sigma_H^+ = \{2e_1\}$ and $\langle\alpha_2, H\rangle = (\pi/2) - \langle\alpha_1, H\rangle$. By Theorem 4, we have

$$\tau_H = -(m_1 + m_3)\tan\langle\alpha_1, H\rangle\alpha_2 + (2m_2 + m_3)\cot\langle\alpha_1, H\rangle\alpha_2.$$

Hence, $\tau_H = 0$ if and only if

$$(\cot\langle\alpha_1, H\rangle)^2 = \frac{m_1 + m_3}{2m_2 + m_3}.$$

By Theorem 6, the orbit $K_2 \cdot \pi_1(x)$ is biharmonic if and only if

$$0 = -\langle\tau_H, \alpha_1\rangle\{(m_3 + 2m_2)(1 - (\cot\langle\alpha_1, H\rangle)^2) + (m_1 + m_3)(1 - (\tan\langle\alpha_1, H\rangle)^2) + 4m_3\}\alpha_2.$$

Therefore, the orbit $K_2 \cdot \pi_1(x)$ is biharmonic if and only if $\tau_H = 0$ or

$$(m_3 + 2m_2)(1 - (\cot\langle\alpha_1, H\rangle)^2) + (m_1 + m_3)(1 - (\tan\langle\alpha_1, H\rangle)^2) + 4m_3 = 0$$

holds. The last equation is equivalent to

$$((m_3 + 2m_2)(\cot\langle\alpha_1, H\rangle)^2 - (m_1 + m_3))((\cot\langle\alpha_1, H\rangle)^2 - 1) = 4m_3(\cot\langle\alpha_1, H\rangle)^2.$$

Since $m_3 > 0$, the solutions of the equation are not harmonic. Hence the orbit $K_2 \cdot \pi_1(x)$ is proper biharmonic if and only if

$$(\cot\langle\alpha_1, H\rangle)^2 = \frac{m_1 + 2m_2 + 6m_3 \pm \sqrt{(m_1 - 2m_2)^2 + 8m_3(m_1 + 2m_2 + 4m_3)}}{2(2m_2 + m_3)}$$

holds. *In this case, there exist exactly two proper biharmonic orbits.*

5.4. Type G_2**.** We set

$$\Sigma^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\}, \qquad W^+ = \varnothing,$$

$$\langle\alpha_1, \alpha_1\rangle = 1, \qquad \langle\alpha_1, \alpha_2\rangle = -\frac{3}{2}, \qquad \langle\alpha_2, \alpha_2\rangle = 3,$$

$$\delta = 3\alpha_1 + 2\alpha_2,$$

and

$$m = m(\alpha_1) = m(\alpha_2).$$

(1) When $H \in P_0^{\{\alpha_1, \delta\}} = \{tH_{\alpha_1} \mid 0 < t < 1\}$, we have $\Sigma_H^+ = \{\alpha_2\}$, $W_H^+ = \varnothing$. By Theorem 4, we have

$$\tau_H = -m\left\{\cot\langle\alpha_1, H\rangle + \cot\langle 2\alpha_1, H\rangle + 3\frac{\cot\langle\alpha_1, H\rangle\cot\langle 2\alpha_1, H\rangle - 1}{\cot\langle\alpha_1, H\rangle + \cot\langle 2\alpha_1, H\rangle}\right\}(2\alpha_1 + \alpha_2).$$

Thus, $\tau_H = 0$ if and only if

$$
\begin{aligned}
0 &= \left\{\cot\langle\alpha_1, H\rangle + \cot\langle 2\alpha_1, H\rangle + 3\frac{\cot\langle\alpha_1, H\rangle\cot\langle 2\alpha_1, H\rangle - 1}{\cot\langle\alpha_1, H\rangle + \cot\langle 2\alpha_1, H\rangle}\right\} \\
&= \frac{1}{4}(15(\cot\langle\alpha_1, H\rangle)^2 - 24 + (\tan\langle\alpha_1, H\rangle)^2).
\end{aligned}
$$

Since $0 < \langle\alpha_1, H\rangle < (\pi/3)$, $\tau_H = 0$ if and only if

$$(\cot\langle\alpha_1, H\rangle)^2 = \frac{12 + \sqrt{129}}{15}.$$

By Theorem 6, the orbit $K_2 \cdot \pi_1(x)$ is biharmonic if and only if

$$
\begin{aligned}
0 = m\langle\tau_H, \alpha_1\rangle\{&(1 - (\cot\langle\alpha_1, H\rangle)^2) \\
&+ 2(1 - (\cot\langle 2\alpha_1, H\rangle)^2) + 9(1 - (\cot\langle 3\alpha_1, H\rangle)^2)\}(2\alpha_1 + \alpha_2).
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
&(1 - (\cot\langle\alpha_1, H\rangle)^2) + 2(1 - (\cot\langle 2\alpha_1, H\rangle)^2) + 9(1 - (\cot\langle 3\alpha_1, H\rangle)^2) \\
&\quad = 12 - \Bigg[(\cot\langle\alpha_1, H\rangle)^2 + 2(\cot\langle 2\alpha_1, H\rangle)^2 \\
&\qquad\qquad + 9\left(\frac{\cot\langle\alpha_1, H\rangle\cot\langle 2\alpha_1, H\rangle - 1}{\cot\langle\alpha_1, H\rangle + \cot\langle 2\alpha_1, H\rangle}\right)^2\Bigg].
\end{aligned}
$$

Thus, the orbit $K_2 \cdot \pi_1(x)$ is biharmonic if and only if

$$
\begin{aligned}
0 &= \{(\cot\langle\alpha_1, H\rangle)^2 + 2(\cot\langle 2\alpha_1, H\rangle)^2\}(\cot\langle\alpha_1, H\rangle + \cot\langle 2\alpha_1, H\rangle)^2 \\
&\quad + 9(\cot\langle\alpha_1, H\rangle\cot\langle 2\alpha_1, H\rangle - 1)^2 - 12(\cot\langle\alpha_1, H\rangle + \cot\langle 2\alpha_1, H\rangle)^2 \\
&= \frac{(\tan\langle\alpha_1, H\rangle)^4}{8}\{45(\cot\langle\alpha_1, H\rangle)^8 - 378(\cot\langle\alpha_1, H\rangle)^6 \\
&\quad + 318(\cot\langle\alpha_1, H\rangle)^4 - 30(\cot\langle\alpha_1, H\rangle)^2 + 1\}.
\end{aligned}
$$

We set $u = (\cot\langle\alpha_1, H\rangle)^2$ and

$$f(u) = 45u^4 - 378u^3 + 318u^2 - 30u + 1.$$

Then,

$$
\begin{aligned}
\frac{df}{du}(u) &= 180u^3 - 1026u^2 + 636u - 30 = 6(u - 5)(30u^2 - 21u + 1) \\
&= 180(u - 5)\left(u - \frac{21 + \sqrt{321}}{60}\right)\left(u - \frac{21 - \sqrt{321}}{60}\right).
\end{aligned}
$$

Since

$$f\left(\frac{1}{3}\right) = \frac{128}{9} > 0, \qquad \frac{df}{du}\left(\frac{1}{3}\right) = \frac{224}{3} > 0,$$

$$f(5) = -6824 < 0 \qquad \text{and} \qquad f(7) = 6112 > 0,$$

the equation $f(u) = 0$ has distinct two solutions for $(1/3) < u$. Therefore, there exist $0 < t_-, t_+ < 1$ such that the orbits $K_2 \cdot \pi_1(\exp(t_\pm H_{\alpha_1}))$ are biharmonic. Since

$$f\left(\frac{12 + \sqrt{129}}{15}\right) \neq 0,$$

the orbits $K_2 \cdot \pi_1(\exp(t_\pm H_{\alpha_1}))$ are proper biharmonic. *In this case, there exist exactly two proper biharmonic orbits.*

(2) When $H \in P_0^{\{\alpha_2, \delta\}} = \{tH_{\alpha_2} \mid 0 < t < 1\}$, we have $\Sigma_H^+ = \{\alpha_1\}$, $W_H^+ = \varnothing$. By Theorem 4, we have

$$\tau_H = -\frac{1}{2} m\{5 \cot\langle\alpha_2, H\rangle - \tan\langle\alpha_2, H\rangle\}(3\alpha_1 + 2\alpha_2).$$

Hence, $\tau_H = 0$ if and only if

$$(\cot\langle\alpha_2, H\rangle)^2 = \frac{1}{5}.$$

By Theorem 6, the orbit $K_2 \cdot \pi_1(x)$ is biharmonic if and only if

$$0 = \frac{1}{2} m\langle\tau_H, \alpha_2\rangle\{5(1 - (\cot\langle\alpha_2, H\rangle)^2) + (1 - (\tan\langle\alpha_2, H\rangle)^2) + 4\}(3\alpha_1 + 2\alpha_2).$$

Therefore, the orbit $K_2 \cdot \pi_1(x)$ is biharmonic if and only if $\tau_H = 0$ or

$$5(1 - (\cot\langle\alpha_2, H\rangle)^2) + (1 - (\tan\langle\alpha_2, H\rangle)^2) + 4 = 0$$

holds. The last equation is equivalent to

$$(5(\cot\langle\alpha_2, H\rangle)^2 - 1)((\cot\langle\alpha_2, H\rangle)^2 - 1) = 4(\cot\langle\alpha_2, H\rangle)^2.$$

Thus, the solutions of the equation are not harmonic. Hence $K_2 \cdot \pi_1(x)$ is proper biharmonic if and only if

$$(\cot\langle\alpha_2, H\rangle)^2 = \frac{5 \pm 2\sqrt{5}}{5}$$

holds. *In this case, there exist exactly two proper biharmonic orbits.*

(3) When $H \in P_0^{\{\alpha_1,\alpha_2\}} = \{tH_{\alpha_1} + (1-t)H_{\alpha_2} \mid 0 < t < 1\}$, we have $\Sigma_H^+ = \{3\alpha_1 + 2\alpha_2\}$, $W_H^+ = \emptyset$. We set $\vartheta = (\pi/6)t$. Then,

$$\langle \alpha_1, H \rangle = 2\vartheta, \qquad \langle \alpha_2, H \rangle = \frac{\pi}{2} - 3\vartheta.$$

By Theorem 4 we have

$$\tau_H = -m\{\cot(2\vartheta) - \tan\vartheta - \tan(3\vartheta)\}\alpha_1.$$

Since

$$\tan(3\vartheta) = \frac{\cot\vartheta + \cot(2\vartheta)}{\cot\vartheta\cot(2\vartheta) - 1},$$

$\tau_H = 0$ if and only if

$$\begin{aligned} 0 &= (\cot(2\vartheta) - \tan\vartheta)(\cot\vartheta\cot(2\vartheta) - 1) - 3(\cot\vartheta + \cot(2\vartheta)) \\ &= \frac{(\cot\vartheta)^4 - 24(\cot\vartheta)^2 + 15}{\cot\vartheta}. \end{aligned}$$

Since $0 < \vartheta < (\pi/6)$, we have $\cot\vartheta > \sqrt{3}$. Hence $\tau_H = 0$ if and only if

$$(\cot\vartheta)^2 = 12 + \sqrt{129}.$$

By Theorem 6, the orbit $K_2 \cdot \pi_1(x)$ is biharmonic if and only if

$$0 = \frac{m}{2}\langle \tau_H, \alpha_1 \rangle \{2(1 - (\cot(2\vartheta))^2) + (1 - (\tan\vartheta)^2) + 9(1 - (\tan(3\vartheta))^2)\}\alpha_1.$$

Therefore, $K_2 \cdot \pi_1(x)$ is biharmonic if and only if $\tau_H = 0$ or

$$\begin{aligned} 0 &= 2(1 - (\cot(2\vartheta))^2) + (1 - (\tan\vartheta)^2) + 9(1 - (\tan(3\vartheta))^2) \\ &= (12 - 2(\cot(2\vartheta))^2 - (\tan\vartheta)^2) - 9\frac{(\cot(2\vartheta) + \cot\vartheta)^2}{((\cot(2\vartheta))(\cot\vartheta) - 1)^2} \end{aligned}$$

holds. Thus $K_2 \cdot \pi_1(x)$ is biharmonic if and only if $\tau_H = 0$ or

$$\begin{aligned} 0 &= \{12 - 2(\cot(2\vartheta))^2 - (\tan\vartheta)^2\}((\cot(2\vartheta))(\cot\vartheta - 1)^2 - 9(\cot(2\vartheta) + \cot\vartheta)^2 \\ &= -\frac{1}{8}(\tan\vartheta)^2\{(\cot\vartheta)^8 - 32(\cot\vartheta)^6 + 330(\cot\vartheta)^4 - 360(\cot\vartheta)^2 + 45\}. \end{aligned}$$

By the same argument as (1) *in* 5.4, *we have that if the orbits* $K_2 \cdot \pi_1(x)$ *is biharmonic, then it is harmonic.*

Cases of $\theta_1 \not\sim \theta_2$. Next, we consider the cases of $\theta_1 \not\sim \theta_2$. Let (G, K_1, K_2) be a compact symmetric triad which satisfies the condition (A), (B) or (C) in

Theorem 2. Then the triple $(\tilde{\Sigma}, \Sigma, W)$ is a symmetric triad of $\mathfrak{a}$ with multiplicities. From (4) and (5), the orbit spaces of K_2-action on N_1 and K_1-action on N_2 are described as $\overline{P_0} = \{H \in \mathfrak{a} \mid \langle \alpha, H \rangle \geq 0, \langle \tilde{\alpha}, H \rangle \leq (\pi/2)$ $(\alpha \in \Pi)\}$ where $\tilde{\alpha}$ is a unique element in W^+ which satisfies $\alpha + \lambda \notin W$ for all $\lambda \in \Pi$. We set a basis $\{H_\alpha\}_{\alpha \in \Pi}$ of $\mathfrak{a}$ as follows;

$$\langle H_\alpha, \beta \rangle = 0, \qquad \langle H_\alpha, \tilde{\alpha} \rangle = \frac{\pi}{2} \qquad (\alpha \neq \beta,\ \alpha, \beta \in \Pi).$$

Then we have

$$P_0 = \left\{ \sum_{\alpha \in \Pi} t_\alpha H_\alpha \,\middle|\, t_\alpha > 0 \ \ (\alpha \in \Pi),\ \sum_{\alpha \in \Pi} t_\alpha < 1 \right\}.$$

We apply Theorem 6 to the following three cases;

(1) $H \in P_0^{\{\alpha_1, \tilde{\alpha}\}} = \{tH_{\alpha_1} \mid 0 < t < 1\}$,

(2) $H \in P_0^{\{\alpha_2, \tilde{\alpha}\}} = \{tH_{\alpha_2} \mid 0 < t < 1\}$,

(3) $H \in P_0^{\{\alpha_1, \alpha_2\}} = \{tH_{\alpha_1} + (1-t)H_{\alpha_2} \mid 0 < t < 1\}$.

In the following, we solve the equation (10) in Theorem 6 for each symmetric triad with multiplicities which satisfies $\dim \mathfrak{a} = 2$.

5.5. Type I-B$_2$ and I-BC$_2$-A$_1^2$. We set

$$\Sigma^+ = \{e_1 \pm e_2, e_1, e_2, 2e_1, 2e_2\}, \qquad W^+ = \{e_1, e_2\},$$

$$\Pi = \{\alpha_1 = e_1 - e_2, \alpha_2 = e_2\}, \qquad \tilde{\alpha} = \alpha_1 + \alpha_2 = e_1$$

and

$$m_1 = m(e_1), \qquad m_2 = m(e_1 + e_2), \qquad m_3 = m(2e_1), \qquad n_1 = n(e_1),$$

Here $m_3 = 0$ when $(\tilde{\Sigma}, \Sigma, W)$ is of type I-B$_2$.

(1) When $H \in P_0^{\{\alpha_1, \tilde{\alpha}\}} = \{tH_{\alpha_1} \mid 0 < t < 1\}$, we have $\Sigma_H^+ = \{\alpha_2, 2\alpha_2\}$ and $W_H^+ = \varnothing$. By Theorem 4, we have

$$\tau_H = \{-(2m_2 + m_1 + m_3) \cot\langle \alpha_1, H \rangle + (n_1 + m_3) \tan\langle \alpha_1, H \rangle\} e_1.$$

Hence we have $\tau_H = 0$ if and only if

$$(\cot\langle \alpha_1, H \rangle)^2 = \frac{n_1 + m_3}{m_1 + 2m_2 + m_3}.$$

By Theorem 6, the orbit $K_2 \cdot \pi_1(x)$ is biharmonic if and only if

$$\begin{aligned} 0 = \langle \tau_H, \alpha_1 \rangle \{ &(m_1 + 2m_2 + m_3)(1 - (\cot\langle \alpha_1, H \rangle)^2) + 4m_3 \\ &+ (n_1 + m_3)(1 - (\tan\langle \alpha_1, H \rangle)^2) \} e_1. \end{aligned}$$

Therefore, $K_2 \cdot \pi_1(x)$ is biharmonic if and only if $\tau_H = 0$ or

$$(m_1 + 2m_2 + m_3)(\cot\langle\alpha_1, H\rangle)^4 \\ - \{(m_1 + 2m_2 + m_3) + (n_1 + m_3) + 4m_3\}(\cot\langle\alpha_1, H\rangle)^2 + n_1 + m_3 = 0 \quad (11)$$

holds. Let H_+ and H_- denote the solutions of the biharmonic equation (11) such that $(\cot\langle\alpha_1, H_-\rangle)^2 \le (\cot\langle\alpha_1, H_+\rangle)^2$. Since $\tau_H = 0$ if and only if

$$(\cot\langle\alpha_1, H\rangle)^2 = \frac{n_1 + m_3}{m_1 + 2m_2 + m_3},$$

$K_2 \cdot \pi_1(x)$ is proper biharmonic if and only if

$$(\cot\langle\alpha_1, H\rangle)^2 \\ = \begin{cases} \frac{-(m_1+2m_2+6m_3+n_1) \pm \sqrt{(m_1+2m_2+6m_3+n_1)^2 - 4(m_1+2m_2+m_3)(n_1+m_3)}}{2(m_1+2m_2+m_3)} & (m_3 > 0) \\ 1 & (m_3 = 0). \end{cases}$$

Let H_0 be a vector in $\mathfrak{a}$ satisfying $\tau_{H_0} = 0$ and $0 < \langle\alpha_1, H_0\rangle < \pi/2$.

- (I-B_2) *If $m_3 = 0$, then there exists a unique proper biharmonic orbit.*
- (I-BC_2-A_1^2) *If $m_3 > 0$, then*

$$\langle\alpha_1, H_-\rangle < \langle\alpha_1, H_0\rangle < \langle\alpha_1, H_+\rangle,$$

hence there exist exactly two proper biharmonic orbits.

(2) When $H \in P_0^{\{\alpha_2, \tilde{\alpha}\}} = \{tH_{\alpha_2} \mid 0 < t < 1\}$, we have $\Sigma_H^+ = \{\alpha_1\}$, $W_H^+ = \emptyset$. By Theorem 4, we have

$$\tau_H = \frac{1}{2}\{-(2m_1 + m_2 + 2m_3)\cot\langle\alpha_2, H\rangle \\ + (2n_1 + m_2 + 2m_3)\tan\langle\alpha_2, H\rangle\}(\alpha_1 + 2\alpha_2).$$

Hence, $\tau_H = 0$ if and only if

$$(\cot\langle\alpha_2, H\rangle)^2 = \frac{2n_1 + m_2 + 2m_3}{2m_1 + m_2 + 2m_3}.$$

By Theorem 6, the orbit $K_2 \cdot \pi_1(x)$ is biharmonic if and only if

$$0 = \langle\tau_H, \alpha_2\rangle\{m_1(1 - (\cot\langle\alpha_2, H\rangle)^2) + (2m_2 + 4m_3)(1 - (\cot\langle 2\alpha_2, H\rangle)^2) \\ + n_1(1 - (\tan\langle\alpha_2, H\rangle)^2)\}(\alpha_1 + 2\alpha_2).$$

Therefore, $K_2 \cdot \pi_1(x)$ is biharmonic if and only if $\tau_H = 0$ or

$$0 = m_1(1 - (\cot\langle\alpha_2, H\rangle)^2) + (2m_2 + 4m_3)(1 - (\cot\langle 2\alpha_2, H\rangle)^2) + n_1(1 - (\tan\langle\alpha_2, H\rangle)^2)$$

holds. The last equation is equivalent to

$$((2m_1 + m_2 + 2m_3)(\cot\langle\alpha_2, H\rangle)^2 - (2n_1 + m_2 + 2m_3))((\cot\langle\alpha_2, H\rangle)^2 - 1) = (2m_2 + 4m_3)(\cot\langle\alpha_2, H\rangle)^2.$$

Since $2m_2 + 4m_3 > 0$, the solutions of the equation are not harmonic. Hence the orbit $K_2 \cdot \pi_1(x)$ is proper biharmonic if and only if

$$(\cot\langle\alpha_2, H\rangle)^2 = \frac{l_1 + l_2 + 2m_2 + 4m_3 \pm \sqrt{(l_1 + l_2 + 2m_2 + 4m_3)^2 - 4l_1l_2}}{2l_1}$$

holds, where $l_1 = 2m_1 + m_2 + 2m_3$ and $l_2 = 2n_1 + m_2 + 2m_3$. *In this case, there exist exactly two proper biharmonic orbits.*

(3) When $H \in P_0^{\{\alpha_1,\alpha_2\}} = \{tH_{\alpha_1} + (1-t)H_{\alpha_2} \mid 0 < t < 1\}$, we have $\Sigma_H^+ = \{2\alpha_2 + 2\alpha_2\}$, $W_H^+ = \{\alpha_1 + \alpha_2\}$. We set $\vartheta = \langle\alpha_1, H\rangle$. Then, $\langle\alpha_2, H\rangle = (\pi/2) - \vartheta$. By Theorem 4, we have

$$\tau_H = \{(2m_2 + m_3 + n_1)\cot\vartheta - (m_1 + m_3)\tan\vartheta\}\alpha_2.$$

Hence, $\tau_H = 0$ if and only if

$$(\cot\vartheta)^2 = \frac{m_1 + m_3}{2m_2 + m_3 + n_1}.$$

By Theorem 6, the orbit $K_2 \cdot \pi_1(x)$ is biharmonic if and only if

$$0 = -\langle\tau_H, \alpha_1\rangle\{(2m_2 + n_1 + m_3)(1 - (\cot\vartheta)^2) + (m_1 + m_3)(1 - (\tan\vartheta)^2) + 4m_3\}\alpha_2.$$

Therefore, $K_2 \cdot \pi_1(x)$ is biharmonic if and only if $\tau_H = 0$ or

$$0 = \{(2m_2 + n_1 + m_3)(1 - (\cot\vartheta)^2) + (m_1 + m_3)(1 - (\tan\vartheta)^2) + 4m_3\}$$

holds. The last equation is equivalent to

$$((2m_2 + m_3 + n_1)(\cot\vartheta)^2 - (m_1 + m_3))((\cot\vartheta)^2 - 1) = 4m_3(\cot\vartheta)^2.$$

Since $m_3 > 0$, the solutions of the equation are not harmonic. Hence the orbit $K_2 \cdot \pi_1(x)$ is proper biharmonic if and only if

$$(\cot \vartheta)^2 = \frac{l_1 + l_2 + 4m_3 \pm \sqrt{(l_1 + l_2 + 4m_3)^2 - 4l_1 l_2}}{2l_1}$$

holds, where $l_1 = 2m_2 + m_3 + n_1$ and $l_2 = m_1 + m_3$. *In this case, there exist exactly two proper biharmonic orbits.*

5.6. Type I-C$_2$. We set

$$\Sigma^+ = \{e_1 \pm e_2, 2e_1, 2e_2\}, \qquad W^+ = \{e_1 - e_2, e_1 + e_2\},$$
$$\Pi = \{\alpha_1 = e_1 - e_2, \alpha_2 = 2e_2\}, \qquad \tilde{\alpha} = \alpha_1 + \alpha_2 = e_1 + e_2,$$

and $m_1 = m(e_1 + e_2)$, $m_2 = m(2e_1)$, $n_1 = n(e_1 + e_2)$. *In this case, we have the same results as cases of Type* I-B$_2$.

5.7. Type I-BC$_2$-B$_2$. We set

$$\Sigma^+ = \{e_1 \pm e_2, e_1, e_2, 2e_1, 2e_2\}, \qquad W^+ = \{e_1 \pm e_2, e_1, e_2\},$$
$$\Pi = \{\alpha_1 = e_1 - e_2, \alpha_2 = e_2\}, \qquad \tilde{\alpha} = \alpha_1 + 2\alpha_2 = e_1 + e_2$$

and

$$m_1 = m(e_1), \qquad m_2 = m(e_1 + e_2), \qquad m_3 = m(2e_1),$$
$$n_1 = n(e_1), \qquad n_2 = n(e_1 + e_2).$$

Since $e_1 \in \Sigma \cap W$, $e_1 - e_2 \in W$ and $2\langle e_1, e_1 - e_2\rangle / \langle e_1 - e_2, e_1 - e_2\rangle$ is odd, by definition of multiplicities, we have $m_1 = m(e_1) = n(e_1) = n_1$.

(1) When $H \in P_0^{\{\alpha_1, \tilde{\alpha}\}} = \{tH_{\alpha_1} \,|\, 0 < t < 1\}$, we have $\Sigma_H^+ = \{\alpha_2, 2\alpha_2\}$, $W_H^+ = \emptyset$. By Theorem 4, we have

$$\tau_H = \{-(m_1 + 2m_2 + m_3)\cot\langle\alpha_1, H\rangle + (m_1 + 2n_2 + m_3)\tan\langle\alpha_1, H\rangle\}e_1.$$

Hence we have $\tau_H = 0$ if and only if

$$(\cot\langle\alpha_1, H\rangle)^2 = \frac{m_1 + 2n_2 + m_3}{m_1 + 2m_2 + m_3}.$$

By Theorem 6, the orbit $K_2 \cdot \pi_1(x)$ is biharmonic if and only if

$$\begin{aligned} 0 = \langle\tau_H, \alpha_1\rangle\{&(m_1 + 2m_2 + m_3)(1 - (\cot\langle\alpha_1, H\rangle)^2) \\ &+ (m_1 + 2n_2 + m_3)(1 - (\tan\langle\alpha_1, H\rangle)^2) + 4m_3\}(\alpha_1 + \alpha_2). \end{aligned}$$

Therefore, $K_2 \cdot \pi_1(x)$ is biharmonic if and only if $\tau_H = 0$ or

$$\begin{aligned}0 &= (m_1 + 2m_2 + m_3)(1 - (\cot\langle\alpha_1, H\rangle)^2)\\ &\quad + (m_1 + 2n_2 + m_3)(1 - (\tan\langle\alpha_1, H\rangle)^2) + 4m_3\end{aligned}$$

holds. The last equation is equivalent to

$$\begin{aligned}&((m_1 + 2m_2 + m_3)(\cot\langle\alpha_2, H\rangle)^2 - (m_1 + 2n_2 + m_3))((\cot\langle\alpha_2, H\rangle)^2 - 1)\\ &\quad = 4m_3(\cot\langle\alpha_2, H\rangle)^2.\end{aligned}$$

Since $m_3 > 0$, the solutions of the equation are not harmonic. Hence the orbit $K_2 \cdot \pi_1(x)$ is proper biharmonic if and only if

$$(\cot\langle\alpha_1, H\rangle)^2 = \frac{l_1 + l_2 + 4m_3 \pm \sqrt{(l_1 + l_2 + 4m_3)^2 - 4l_1 l_2}}{2l_1}$$

holds, where $l_1 = m_1 + 2m_2 + m_3$ and $l_2 = m_1 + 2n_2 + m_3$. *In this case, there exist exactly two proper biharmonic orbits.*

(2) When $H \in P_0^{\{\alpha_2, \tilde{\alpha}\}} = \{tH_{\alpha_2} \mid 0 < t < 1\}$, we have $\Sigma_H^+ = \{\alpha_1\}$, $W_H^+ = \varnothing$. By Theorem 4, we have

$$\tau_H = \{-(2m_1 + m_2 + 2m_3)\cot\langle 2\alpha_2, H\rangle + n_2 \tan\langle 2\alpha_2, H\rangle\}(\alpha_1 + 2\alpha_2).$$

Hence we have $\tau_H = 0$ if and only if

$$(\cot\langle 2\alpha_2, H\rangle)^2 = \frac{n_2}{2m_1 + m_2 + 2m_3}.$$

By Theorem 6, the orbit $K_2 \cdot \pi_1(x)$ is biharmonic if and only if

$$\begin{aligned}0 &= 2\langle\tau_H, \alpha_2\rangle\{(2m_1 + m_2 + 2m_3)(1 - (\cot\langle 2\alpha_2, H\rangle)^2)\\ &\quad + n_2(1 - (\tan\langle 2\alpha_2, H\rangle)^2) - 4m_3\}(\alpha_1 + 2\alpha_2).\end{aligned}$$

Therefore, $K_2 \cdot \pi_1(x)$ is biharmonic if and only if $\tau_H = 0$ or

$$\begin{aligned}0 &= (2m_1 + m_2 + 2m_3)(1 - (\cot\langle 2\alpha_2, H\rangle)^2)\\ &\quad + n_2(1 - (\tan\langle 2\alpha_2, H\rangle)^2) - 4m_3\end{aligned}$$

holds. The last equation is equivalent to

$$\begin{aligned}&((2m_1 + m_2 + 2m_3)(\cot\langle 2\alpha_2, H\rangle)^2 - 2n_2)((\cot\langle\alpha_2, H\rangle)^2 - 1)\\ &\quad = -4m_3(\cot\langle\alpha_2, H\rangle)^2.\end{aligned}$$

Since $m_3 > 0$, the solutions of the equation are not harmonic.

- *When* $(-2m_1 + m_2 + 2m_3 + n_2)^2 - 4(2m_1 + m_2 + 2m_3)n_2 > 0$, *the orbit* $K_2 \cdot \pi_1(x)$ *is proper biharmonic if and only if*

$$(\cot\langle 2\alpha_2, H\rangle)^2 = \frac{l \pm \sqrt{l^2 - 4(2m_1 + m_2 + 2m_3)n_2}}{2(2m_1 + m_2 + 2m_3)}$$

holds, where $l = -2m_1 + m_2 + 2m_3 + n_2$. *In this case, there exist exactly two proper biharmonic orbits.*

- *When* $(-2m_1 + m_2 + 2m_3 + n_2)^2 - 4(2m_1 + m_2 + 2m_3)n_2 < 0$, *if the orbit* $K_2 \cdot \pi_1(x)$ *is biharmonic, then it is harmonic.*
- *When* $(-2m_1 + m_2 + 2m_3 + n_2)^2 - 4(2m_1 + m_2 + 2m_3)n_2 = 0$, *there exists a unique proper biharmonic orbit.*

(3) When $H \in P_0^{\{\alpha_1, \alpha_2\}} = \{tH_{\alpha_1} + (1-t)H_{\alpha_2} \,|\, 0 < t < 1\}$, we have $\Sigma_H^+ = \emptyset$, $W_H^+ = \{\alpha_1 + 2\alpha_2\}$. We set $2\vartheta = \langle \alpha_1, H\rangle$. Then $\langle \alpha_2, H\rangle = (\pi/4) - \vartheta$. By Theorem 4, we have

$$\tau_H = \{-m_2 \cot(2\vartheta) + (2m_1 + m_3 + n_2)\tan(2\vartheta)\}\alpha_1.$$

Hence we have $\tau_H = 0$ if and only if

$$(\cot(2\vartheta))^2 = \frac{2m_1 + 2m_3 + n_2}{m_2}.$$

By Theorem 6, the orbit $K_2 \cdot \pi_1(x)$ is biharmonic if and only if

$$\begin{aligned} 0 = 2\langle \tau_H, \alpha_1 + \alpha_2\rangle &\{m_2(1 - (\cot(2\vartheta))^2) \\ &+ (2m_1 + 2m_3 + n_2)(1 - (\tan(2\vartheta))^2) - 2m_1\}\alpha_1. \end{aligned}$$

Therefore, $K_2 \cdot \pi_1(x)$ is biharmonic if and only if $\tau_H = 0$ or

$$0 = (m_2(1 - (\cot(2\vartheta))^2) + (2m_1 + 2m_3 + n_2)(1 - (\tan(2\vartheta))^2) - 2m_1$$

holds. The last equation is equivalent to

$$\{m_2(\cot(2\vartheta))^2 - (2m_1 + 2m_3 + n_2)\}((\cot(2\vartheta))^2 - 1) = -2m_1(\cot(2\vartheta))^2.$$

Since $m_1 > 0$, the solutions of the equation are not harmonic.

- *When* $(2m_3 + m_2 + m_2)^2 - 4m_2(2m_1 + 2m_3 + n_2) > 0$, *the orbit* $K_2 \cdot \pi_1(x)$ *is proper biharmonic if and only if*

$$\begin{aligned} &(\cot(2\vartheta))^2 \\ &= \frac{2m_3 + m_2 + m_2 \pm \sqrt{(2m_3 + m_2 + m_2)^2 - 4m_2(2m_1 + 2m_3 + n_2)}}{2m_2} \end{aligned}$$

holds. In this case, there exist exactly two proper biharmonic orbits.

- *When* $(2m_3 + m_2 + m_2)^2 - 4m_2(2m_1 + 2m_3 + n_2) < 0$, *if the orbit* $K_2 \cdot \pi_1(x)$ *is biharmonic, then it is harmonic.*
- *When* $(2m_3 + m_2 + m_2)^2 - 4m_2(2m_1 + 2m_3 + n_2) = 0$, *there exists a unique proper biharmonic orbit.*

5.8. Type II-BC$_2$**.** We set

$$\Sigma^+ = \{e_1 \pm e_2, e_1, e_2\}, \qquad W^+ = \{e_1 \pm e_2, e_1, e_2, 2e_1, 2e_2\},$$
$$\Pi = \{\alpha_1 = e_1 - e_2, \alpha_2 = e_2\}, \qquad \tilde{\alpha} = 2\alpha_1 + 2\alpha_2 = 2e_1$$

and

$$m_1 = m(e_1), \qquad m_2 = m(e_1 + e_2), \qquad n_1 = n(e_1),$$
$$n_2 = n(e_1 + e_2), \qquad n_3 = n(2e_1).$$

Since $e_1, e_1 + e_2 \in \Sigma \cap W$, $2e_1 \in W$ and $2\langle e_1, 2e_1\rangle/\langle 2e_1, 2e_1\rangle = 1$ and $2\langle e_1 + e_2, 2e_1\rangle/\langle 2e_1, 2e_1\rangle = 1$ are odd, by definition of multiplicities, we have $m_1 = m(e_1) = n(e_1) = n_1$ and $m_2 = m(e_1 + e_2) = n(e_1 + e_2) = n_2$.

(1) When $H \in P_0^{\{\alpha_1, \tilde{\alpha}\}} = \{tH_{\alpha_1} \mid 0 < t < 1\}$, we have $\Sigma_H^+ = \{\alpha_2\}$, $W_H^+ = \varnothing$. By Theorem 4, we have

$$\tau_H = 2\{-(m_1 + 2m_2)\cot\langle 2\alpha_1, H\rangle + n_3 \tan\langle 2\alpha_1, H\rangle\}e_1.$$

Hence we have $\tau_H = 0$ if and only if

$$(\cot\langle 2\alpha_1, H\rangle)^2 = \frac{n_3}{m_1 + 2m_2}.$$

By Theorem 6, the orbit $K_2 \cdot \pi_1(x)$ is biharmonic if and only if

$$0 = 8\langle \tau_H, \alpha_1\rangle\{-(m_1 + 2m_2)(\cot\langle 2\alpha_1, H\rangle)^2 + n_3(1 - (\tan\langle 2\alpha_1, H\rangle)^2)\}(\alpha_1 + \alpha_2).$$

Therefore, $K_2 \cdot \pi_1(x)$ is biharmonic if and only if $\tau_H = 0$ or

$$0 = -(m_1 + 2m_2)(\cot\langle 2\alpha_1, H\rangle)^2 + n_3(1 - (\tan\langle 2\alpha_1, H\rangle)^2)$$

holds. The last equation is equivalent to

$$\{(m_1 + 2m_2)(\cot\langle 2\alpha_1, H\rangle)^2 - n_3\}((\cot\langle 2\alpha_1, H\rangle)^2 - 1)$$
$$= -(m_1 + 2m_2)(\cot\langle 2\alpha_1, H\rangle)^2.$$

Since $m_1 + 2m_2 > 0$, the solutions of the equation are not harmonic.

- *When* $n_3^2 - 4(m_1 + 2m_2)n_3 > 0$, *the orbit* $K_2 \cdot \pi_1(x)$ *is proper biharmonic if and only if*

$$(\cot\langle 2\alpha_1, H\rangle)^2 = \frac{n_3 \pm \sqrt{n_3^2 - 4(m_1 + 2m_2)n_3}}{2(m_1 + 2m_2)}$$

holds. In this case, there exist exactly two proper biharmonic orbits.

- *When* $n_3^2 - 4(m_1 + 2m_2)n_3 < 0$, *if the orbit* $K_2 \cdot \pi_1(x)$ *is biharmonic, then it is harmonic.*
- *When* $n_3^2 - 4(m_1 + 2m_2)n_3 = 0$, *there exists a unique proper biharmonic orbit.*

(2) When $H \in P_0^{\{\alpha_2, \tilde{\alpha}\}} = \{tH_{\alpha_2} \,|\, 0 < t < 1\}$, we have $\Sigma_H^+ = \{\alpha_1\}$, $W_H^+ = \emptyset$. By Theorem 4, we have

$$\tau_H = \{-(2m_1 + m_2)\cot\langle 2\alpha_2, H\rangle + (m_2 + 2n_3)\tan\langle 2\alpha_2, H\rangle\}(\alpha_1 + 2\alpha_2).$$

Hence we have $\tau_H = 0$ if and only if

$$(\cot\langle 2\alpha_2, H\rangle)^2 = \frac{m_2 + 2n_3}{2m_1 + m_2}.$$

By Theorem 6, the orbit $K_2 \cdot \pi_1(x)$ is biharmonic if and only if

$$\begin{aligned} 0 = \langle \tau_H, \alpha_2\rangle\{&-(4m_1 + 2m_2)(\cot\langle 2\alpha_2, H\rangle)^2 \\ &+ (2m_2 + 4n_3)(1 - (\tan\langle 2\alpha_2, H\rangle)^2) - 4m_1\}(\alpha_1 + 2\alpha_2). \end{aligned}$$

Therefore, $K_2 \cdot \pi_1(x)$ is biharmonic if and only if $\tau_H = 0$ or

$$\begin{aligned} 0 = (2m_1 + m_2)&(1 - (\cot\langle 2\alpha_2, H\rangle)^2) \\ &+ (m_2 + 2n_3)(1 - (\tan\langle 2\alpha_2, H\rangle)^2) - 2m_1 \end{aligned}$$

holds. The last equation is equivalent to

$$\begin{aligned} ((2m_1 + m_2)&(\cot\langle 2\alpha_2, H\rangle)^2 - (m_2 + 2n_3))((\cot\langle 2\alpha_2, H\rangle)^2 - 1) \\ &= -2m_1(\cot\langle 2\alpha_2, H\rangle)^2. \end{aligned}$$

Since $2m_1 > 0$, the solutions of the equation are not harmonic.

- *When* $(m_2 + n_3)^2 - (2m_1 + m_2)(m_2 + 2n_3) > 0$, *the orbit* $K_2 \cdot \pi_1(x)$ *is proper biharmonic if and only if*

$$(\cot\langle 2\alpha_2, H\rangle)^2 = \frac{m_2 + n_3 \pm \sqrt{(m_2 + n_3)^2 - (2m_1 + m_2)(m_2 + 2n_3)}}{(2m_1 + m_2)}$$

holds. In this case, there exist exactly two proper biharmonic orbits.

- *When* $(m_2+n_3)^2-(2m_1+m_2)(m_2+2n_3)<0$, *if the orbit* $K_2\cdot\pi_1(x)$ *is biharmonic, then it is harmonic.*
- *When* $(m_2+n_3)^2-(2m_1+m_2)(m_2+2n_3)=0$, *there exists a unique proper biharmonic orbit.*

(3) When $H\in P_0^{\{\alpha_1,\alpha_2\}}=\{tH_{\alpha_1}+(1-t)H_{\alpha_2}\,|\,0<t<1\}$, we have $\Sigma_H^+=\varnothing$, $W_H^+=\{\tilde\alpha=2\alpha_1+2\alpha_2\}$. We set $\vartheta=\langle 2\alpha_1,H\rangle$. Then $\langle 2\alpha_2,H\rangle=(\pi/2)-\vartheta$. By Theorem 4, we have

$$\tau_H=2\{(2m_2+n_3)\cot\vartheta-m_1\tan\vartheta\}\alpha_2.$$

Hence we have $\tau_H=0$ if and only if

$$(\cot\vartheta)^2=\frac{m_1}{2m_2+n_3}.$$

By Theorem 6, the orbit $K_2\cdot\pi_1(x)$ is biharmonic if and only if

$$0=4\langle\tau_H,\alpha_1\rangle\{(2m_2+n_3)(1-(\cot\vartheta)^2)+m_1(1-(\tan\vartheta)^2)-(2m_2+m_1)\}\alpha_2.$$

Therefore, $K_2\cdot\pi_1(x)$ is biharmonic if and only if $\tau_H=0$ or

$$0=(2m_2+n_3)(1-(\cot\vartheta)^2)+m_1(1-(\tan\vartheta)^2)-(2m_2+m_1)$$

holds. The last equation is equivalent to

$$\{(2m_2+n_3)(\cot\vartheta)^2-m_1\}((\cot\vartheta)^2-1)=-(2m_2+m_1)(\cot(2\vartheta))^2.$$

Since $2m_2+m_1>0$, the solutions of the equation are not harmonic.

- *When* $n_3^2-4(2m_2+n_3)m_1>0$, *the orbit* $K_2\cdot\pi_1(x)$ *is proper biharmonic if and only if*

$$(\cot\vartheta)^2=\frac{n_3\pm\sqrt{n_3^2-4(2m_2+n_3)m_1}}{2(2m_2+n_3)}$$

  holds. In this case, there exist exactly two proper biharmonic orbits.
- *When* $n_3^2-4(2m_2+n_3)m_1<0$, *if the orbit* $K_2\cdot\pi_1(x)$ *is biharmonic, then it is harmonic.*
- *When* $n_3^2-4(2m_2+n_3)m_1=0$, *there exists a unique proper biharmonic orbit.*

5.9. Type III-A$_2$**.** We set

$$\mathfrak{a}=\{x_1e_1+x_2e_2+x_3e_3\,|\,x_i\in\mathbb{R},\ x_1+x_2+x_3=0\},$$

and

$$\Sigma^+ = W^+ = \{e_1 - e_2, e_2 - e_3, e_1 - e_3\},$$
$$\Pi = \{\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3\}, \qquad \tilde{\alpha} = \alpha_1 + \alpha_2,$$
$$m := m(\lambda) = n(\lambda) \qquad (\lambda \in \tilde{\Sigma}).$$

(1) When $H \in P_0^{\{\alpha_1, \tilde{\alpha}\}} = \{tH_{\alpha_1} \mid 0 < t < 1\}$, we have $\Sigma_H^+ = \{\alpha_2\}$, $W_H^+ = \emptyset$. By Theorem 4, we have

$$\tau_H = m\{-\cot\langle\alpha_1, H\rangle + \tan\langle\alpha_1, H\rangle\}(2\alpha_1 + \alpha_2).$$

Hence we have that $\tau_H = 0$ if and only if $\langle\alpha_1, H\rangle = \pi/4$. By Theorem 6, the orbit $K_2 \cdot \pi_1(x)$ is biharmonic if and only if

$$0 = -m\langle\tau_H, \alpha_1\rangle(\cot\langle\alpha_1, H\rangle - \tan\langle\alpha_1, H\rangle)^2(2\alpha_1 + \alpha_2).$$

Hence, the orbit $K_2 \cdot \pi_1(x)$ is biharmonic if and only if $\langle\alpha_1, H\rangle = \pi/4$. *Therefore, if the orbit $K_2 \cdot \pi_1(x)$ is biharmonic, then it is harmonic.*

(2) When $H \in P_0^{\{\alpha_2, \tilde{\alpha}\}} = \{tH_{\alpha_2} \mid 0 < t < 1\}$, we have $\Sigma_H^+ = \{\alpha_1\}$, $W_H^+ = \emptyset$. By the same calculation as (1), we have that the orbit $K_2 \cdot \pi_1(x)$ is biharmonic if and only if $\langle\alpha_2, H\rangle = \pi/4$. *Therefore, if the orbit is biharmonic, then it is harmonic.*

(3) When $H \in P_0^{\{\alpha_1, \alpha_2\}} = \{tH_{\alpha_1} + (1-t)H_{\alpha_2} \mid 0 < t < 1\}$, we have $\Sigma_H^+ = \emptyset$, $W_H^+ = \{\alpha_1 + \alpha_2\}$. By the same calculation as (1), we have that the orbit $K_2 \cdot \pi_1(x)$ is biharmonic if and only if $\langle\alpha_1, H\rangle = \pi/4$. *Therefore, if the orbit is biharmonic, then it is harmonic.*

5.10. Type III-B$_2$ **and** III-C$_2$**.** We set

$$\Sigma^+ = \{e_1 \pm e_2, e_1, e_2\}, \qquad W^+ = \{e_1 \pm e_2, e_1, e_2\},$$
$$\Pi = \{\alpha_1 = e_1 - e_2, \alpha_2 = e_2\}, \qquad \tilde{\alpha} = \alpha_1 + 2\alpha_2 = e_1 + e_2$$

and

$$m_1 = m(e_1), \qquad m_2 = m(e_1 + e_2), \qquad n_1 = n(e_1), \qquad n_2 = n(e_1 + e_2).$$

Since $e_1 \in \Sigma \cap W$, $e_1 + e_2 \in W$ and $2\langle e_1, e_1 + e_2\rangle / \langle e_1 + e_2, e_1 + e_2\rangle = 1$ is odd, by definition of multiplicities, we have $m_1 = m(e_1) = n(e_1) = n_1$.

(1) When $H \in P_0^{\{\alpha_1, \tilde{\alpha}\}} = \{tH_{\alpha_1} \mid 0 < t < 1\}$, we have $\Sigma_H^+ = \{\alpha_2\}$, $W_H^+ = \emptyset$. By Theorem 4, we have

$$\tau_H = -(2m_2 + m_1)\cot\langle\alpha_1, H\rangle(\alpha_1 + \alpha_2) + (2n_2 + m_1)\tan\langle\alpha_1, H\rangle(\alpha_1 + \alpha_2).$$

Hence we have $\tau_H = 0$ if and only if

$$(\cot\langle 2\alpha_1, H\rangle)^2 = \frac{2m_2 + m_1}{2n_2 + m_1}.$$

By Theorem 6, the orbit $K_2 \cdot \pi_1(x)$ is biharmonic if and only if

$$\begin{aligned} 0 = \langle \tau_H, \alpha_1\rangle \{ &(2m_2 + m_1)(1 - (\cot\langle \alpha_1, H\rangle)^2) \\ &+ (2n_2 + m_1)(1 - (\tan\langle \alpha_1, H\rangle)^2)\}(\alpha_1 + 2\alpha_2). \end{aligned}$$

Therefore, $K_2 \cdot \pi_1(x)$ is biharmonic if and only if $\tau_H = 0$ or

$$0 = (2m_2 + m_1)(1 - (\cot\langle \alpha_1, H\rangle)^2) + (2n_2 + m_1)(1 - (\tan\langle \alpha_1, H\rangle)^2)$$

holds. The last equation is equivalent to

$$\{(2m_2 + m_1)(\cot\langle \alpha_1, H\rangle)^2 - (2n_2 + m_1)\} + ((\cot\langle \alpha_1, H\rangle)^2 - 1) = 0.$$

- *When $m_2 \neq n_2$, the orbit $K_2 \cdot \pi_1(x)$ is proper biharmonic if and only if*

$$(\cot\langle 2\alpha_1, H\rangle)^2 = 1 \qquad (\textit{i.e. } \langle 2\alpha_1, H\rangle = (\pi/4))$$

holds. In this case, there exists a unique proper biharmonic orbit.
- *When $m_2 = n_2$, if the orbit $K_2 \cdot \pi_1(x)$ is biharmonic, then it is harmonic.*

(2) When $H \in P_0^{\{\alpha_2, \tilde{\alpha}\}} = \{tH_{\alpha_2} \mid 0 < t < 1\}$, we have $\Sigma_H^+ = \{\alpha_1\}$, $W_H^+ = \emptyset$. By Theorem 4, we have

$$\tau_H = \{-(2m_1 + m_2)\cot\langle 2\alpha_2, H\rangle + n_2 \tan\langle 2\alpha_2, H\rangle\}(\alpha_1 + 2\alpha_2).$$

Hence we have $\tau_H = 0$ if and only if

$$(\cot\langle 2\alpha_2, H\rangle)^2 = \frac{n_2}{2m_1 + m_2}.$$

By Theorem 6, the orbit $K_2 \cdot \pi_1(x)$ is biharmonic if and only if

$$\begin{aligned} 0 = \langle \tau_H, 2\alpha_2\rangle \{ &(2m_1 + m_2)(1 - (\cot\langle 2\alpha_2, H\rangle)^2) \\ &+ n_2(1 - (\tan\langle 2\alpha_2, H\rangle)^2) - 2m_1\}(\alpha_1 + 2\alpha_2). \end{aligned}$$

Therefore, $K_2 \cdot \pi_1(x)$ is biharmonic if and only if $\tau_H = 0$ or

$$0 = (2m_1 + m_2)(1 - (\cot\langle 2\alpha_2, H\rangle)^2) + n_2(1 - (\tan\langle 2\alpha_2, H\rangle)^2) - 2m_1$$

holds. The last equation is equivalent to

$$((2m_1 + m_2)(\cot\langle 2\alpha_2, H\rangle)^2 - n_2)((\cot\langle 2\alpha_2, H\rangle)^2 - 1) = -2m_1(\cot\langle 2\alpha_2, H\rangle)^2.$$

Since $2m_1 > 0$, the solutions of the equation are not harmonic.

- *When* $(m_2+n_2)^2-4(2m_1+m_2)n_2>0$, *the orbit* $K_2\cdot\pi_1(x)$ *is proper biharmonic if and only if*

$$(\cot\langle 2\alpha_2,H\rangle)^2=\frac{m_2+n_2\pm\sqrt{(m_2+n_2)^2-4(2m_1+m_2)n_2}}{2(2m_1+m_2)}$$

  holds. In this case, there exist exactly two proper biharmonic orbits.
- *When* $(m_2+n_2)^2-4(2m_1+m_2)n_2<0$, *if the orbit* $K_2\cdot\pi_1(x)$ *is biharmonic, then it is harmonic.*
- *When* $(m_2+n_2)^2-4(2m_1+m_2)n_2=0$, *there exists a unique proper biharmonic orbit.*

(3) When $H\in P_0^{\{\alpha_1,\alpha_2\}}=\{tH_{\alpha_1}+(1-t)H_{\alpha_2}\,|\,0<t<1\}$, we have $\Sigma_H^+=\emptyset$, $W_H^+=\{\tilde{\alpha}=\alpha_1+2\alpha_2\}$. We set $\vartheta=\langle\alpha_1,H\rangle$. Then $\langle 2\alpha_2,H\rangle=(\pi/2)-\vartheta$. By Theorem 4, we have

$$\tau_H=\{-m_2\cot(\vartheta)+(n_2+2m_1)\tan(\vartheta)\}\alpha_1.$$

Hence we have $\tau_H=0$ if and only if

$$(\cot\vartheta)^2=\frac{n_2+2m_1}{m_2}.$$

By Theorem 6, the orbit $K_2\cdot\pi_1(x)$ is biharmonic if and only if

$$0=\langle\tau_H,\alpha_1\rangle\{m_2(1-(\cot\langle\alpha_1,H\rangle)^2)+(n_2+2m_1)(1-(\tan\langle\alpha_1,H\rangle)^2)-2m_1\}\alpha_1.$$

Therefore, $K_2\cdot\pi_1(x)$ is biharmonic if and only if $\tau_H=0$ or

$$0=m_2(1-(\cot\langle\alpha_1,H\rangle)^2)+(n_2+2m_1)(1-(\tan\langle\alpha_1,H\rangle)^2)-2m_1$$

holds. The last equation is equivalent to

$$\{m_2(\cot\vartheta)^2-(n_2+2m_1)\}((\cot(\vartheta))^2-1)=-2m_1(\cot(\vartheta))^2.$$

Since $m_1>0$, the solutions of the equation are not harmonic.

- *When* $(m_2+n_2)^2-4m_2(n_2+2m_1)>0$, *the orbit* $K_2\cdot\pi_1(x)$ *is proper biharmonic if and only if*

$$(\cot\vartheta)^2=\frac{(m_2+n_2)\pm\sqrt{(m_2+n_2)^2-4m_2(n_2+2m_1)}}{2m_2}$$

  holds. In this case, there exist exactly two proper biharmonic orbits.
- *When* $(m_2+n_2)^2-4m_2(n_2+2m_1)<0$, *if the orbit* $K_2\cdot\pi_1(x)$ *is biharmonic, then it is harmonic.*

- *When* $(m_2+n_2)^2 - 4m_2(n_2+2m_1) = 0$, *there exists a unique proper biharmonic orbit.*

5.11. Type III-BC$_2$**.** We set

$$\Sigma^+ = W^+ = \{e_1 \pm e_2, e_1, e_2, 2e_1, 2e_2\},$$

$$\Pi = \{\alpha_1 = e_1 - e_2, \alpha_2 = e_2\}, \qquad \tilde{\alpha} = 2\alpha_1 + 2\alpha_2 = 2e_1$$

and

$$m_1 = m(e_1), \qquad m_2 = m(e_1+e_2), \qquad m_3 = (2e_1),$$

$$n_1 = n(e_1), \qquad n_2 = n(e_1+e_2), \qquad n_3 = (2e_1).$$

Since $e_1, e_1+e_2 \in \Sigma \cap W$, $2e_1 \in W$ and $2\langle e_1, 2e_1\rangle/\langle 2e_1, 2e_1\rangle = 1$ and $2\langle e_1+e_2, 2e_1\rangle/\langle 2e_1, 2e_1\rangle = 1$ are odd, by definition of multiplicities, we have $m_1 = m(e_1) = n(e_1) = n_1$, $m_2 = m(e_1+e_2) = n(e_1+e_2) = n_2$.

(1) When $H \in P_0^{\{\alpha_1, \tilde{\alpha}\}} = \{tH_{\alpha_1} \mid 0 < t < 1\}$, we have $\Sigma_H^+ = \{\alpha_2, 2\alpha_2\}$, $W_H^+ = \emptyset$. By Theorem 4, we have

$$\tau_H = 2\{-(2m_2+m_1+m_3)\cot\langle 2\alpha_1, H\rangle + n_3 \tan\langle 2\alpha_1, H\rangle\}(\alpha_1+\alpha_2).$$

Hence we have $\tau_H = 0$ if and only if

$$(\cot\langle 2\alpha_1, H\rangle)^2 = \frac{n_3}{m_1+2m_2+m_3}.$$

By Theorem 6, the orbit $K_2 \cdot \pi_1(x)$ is biharmonic if and only if

$$\begin{aligned} 0 = 4\langle \tau_H, \alpha_1\rangle\{&(2m_2+m_1+m_3)(1-(\cot\langle 2\alpha_1, H\rangle)^2) \\ &+ n_3(1-(\tan\langle 2\alpha_1, H\rangle)^2) - (2m_2+m_1)\}(\alpha_1+\alpha_2). \end{aligned}$$

Therefore, $K_2 \cdot \pi_1(x)$ is biharmonic if and only if $\tau_H = 0$ or

$$\begin{aligned} 0 = &(2m_2+m_1+m_3)(1-(\cot\langle 2\alpha_1, H\rangle)^2) \\ &+ n_3(1-(\tan\langle 2\alpha_1, H\rangle)^2) - (2m_2+m_1) \end{aligned}$$

holds. The last equation is equivalent to

$$\begin{aligned} &\{(2m_2+m_1+m_3)(\cot\langle 2\alpha_1, H\rangle)^2 - n_3\}((\cot\langle 2\alpha_1, H\rangle)^2 - 1) \\ &\quad = -(2m_2+m_1)(\cot\langle 2\alpha_1, H\rangle)^2. \end{aligned}$$

Since $(2m_2+m_1) > 0$, the solutions of the equation are not harmonic.

- *When* $(m_3+n_3)^2-4(2m_2+m_1+m_3)n_3>0$, *the orbit* $K_2\cdot\pi_1(x)$ *is proper biharmonic if and only if*

$$(\cot\langle 2\alpha_1, H\rangle)^2=\frac{m_3+n_3\pm\sqrt{(m_3+n_3)^2-4(2m_2+m_1+m_3)n_3}}{2(2m_2+m_1+m_3)}$$

holds. In this case, there exist exactly two proper biharmonic orbits.
- *When* $(m_3+n_3)^2-4(2m_2+m_1+m_3)n_3<0$, *if the orbit* $K_2\cdot\pi_1(x)$ *is biharmonic, then it is harmonic.*
- *When* $(m_3+n_3)^2-4(2m_2+m_1+m_3)n_3=0$, *there exists a unique proper biharmonic orbit.*

(2) When $H\in P_0^{\{\alpha_2,\tilde{\alpha}\}}=\{tH_{\alpha_2}\,|\,0<t<1\}$, we have $\Sigma_H^+=\{\alpha_1\}$, $W_H^+=\emptyset$. By Theorem 4, we have

$$\tau_H=\{-(2m_1+m_2+2m_3)\cot\langle 2\alpha_2,H\rangle+(m_2+2n_2)\tan\langle 2\alpha_2,H\rangle\}(\alpha_1+2\alpha_2).$$

Hence we have $\tau_H=0$ if and only if

$$(\cot\langle 2\alpha_2,H\rangle)^2=\frac{m_2+2n_2}{2m_1+m_2+2m_3}.$$

By Theorem 6, the orbit $K_2\cdot\pi_1(x)$ is biharmonic if and only if

$$\begin{aligned}0=\langle\tau_H,2\alpha_2\rangle\{&(2m_1+m_2+2m_3)(1-(\cot\langle 2\alpha_2,H\rangle)^2)\\&+(n_2+2n_3)(1-(\tan\langle 2\alpha_2,H\rangle)^2)-2m_1\}(\alpha_1+2\alpha_2).\end{aligned}$$

Therefore, $K_2\cdot\pi_1(x)$ is biharmonic if and only if $\tau_H=0$ or

$$\begin{aligned}0=&(2m_1+m_2+2m_3)(1-(\cot\langle 2\alpha_2,H\rangle)^2)\\&+(n_2+2n_3)(1-(\tan\langle 2\alpha_2,H\rangle)^2)-2m_1\end{aligned}$$

holds. The last equation is equivalent to

$$\begin{aligned}&((2m_1+m_2+2m_3)(\cot\langle 2\alpha_2,H\rangle)^2-(m_2+2n_3))((\cot\langle 2\alpha_2,H\rangle)^2-1)\\&\quad=-2m_1(\cot\langle 2\alpha_2,H\rangle)^2.\end{aligned}$$

Since $2m_1>0$, the solutions of the equation are not harmonic.

- *When* $(m_2+m_3+n_3)^2-(2m_1+m_2+2m_3)(m_2+2n_2)>0$, *the orbit* $K_2\cdot\pi_1(x)$ *is proper biharmonic if and only if*

$$(\cot\langle 2\alpha_2, H\rangle)^2 = \frac{m_2+m_3+n_3 \pm \sqrt{(m_2+m_3+n_3)^2-(2m_1+m_2+2m_3)(m_2+2n_2)}}{2m_1+m_2+2m_3}$$

holds. In this case, there exist exactly two proper biharmonic orbits.

- *When* $(m_2+m_3+n_3)^2-(2m_1+m_2+2m_3)(m_2+2n_2)<0$, *if the orbit* $K_2\cdot\pi_1(x)$ *is biharmonic, then it is harmonic.*
- *When* $(m_2+m_3+n_3)^2-(2m_1+m_2+2m_3)(m_2+2n_2)=0$, *there exists a unique proper biharmonic orbit.*

(3) When $H \in P_0^{\{\alpha_1,\alpha_2\}} = \{tH_{\alpha_1}+(1-t)H_{\alpha_2}\,|\,0<t<1\}$, we have $\Sigma_H^+=\emptyset$, $W_H^+=\{\tilde\alpha = 2\alpha_1+2\alpha_2 = 2e_1\}$. We set $\vartheta = \langle 2\alpha_1, H\rangle$. Then $\langle 2\alpha_2, H\rangle = (\pi/2)-\vartheta$. By Theorem 4, we have

$$\tau_H = \{(4m_2+2n_3)\cot\vartheta - (2m_1+2m_3)\tan((\pi/2)-\vartheta)\}\alpha_2.$$

Hence we have $\tau_H=0$ if and only if

$$(\cot\vartheta)^2 = \frac{m_1+m_3}{2m_2+n_3}.$$

By Theorem 6, the orbit $K_2\cdot\pi_1(x)$ is biharmonic if and only if

$$\begin{aligned}0 = 4\langle\tau_H,\alpha_2\rangle\{&(2m_2+n_3)(1-(\cot\vartheta)^2)\\ &+(m_1+m_3)(1-(\tan\vartheta)^2)-(2m_2+m_1)\}\alpha_2.\end{aligned}$$

Therefore, $K_2\cdot\pi_1(x)$ is biharmonic if and only if $\tau_H=0$ or

$$0 = (2m_2+n_3)(1-(\cot\vartheta)^2)+(m_1+m_3)(1-(\tan\vartheta)^2)-(2m_2+m_1)$$

holds. The last equation is equivalent to

$$\{(2m_2+n_3)(\cot\vartheta)^2-(m_1+m_3)\}((\cot\vartheta)^2-1) = -(m_1+2m_2)(\cot\vartheta)^2.$$

Since $m_1+2m_2>0$, the solutions of the equation are not harmonic.

- *When* $(m_3+n_3)^2-4(2m_2+n_3)(m_1+m_3)>0$, *the orbit* $K_2\cdot\pi_1(x)$ *is proper biharmonic if and only if*

$$(\cot\vartheta)^2 = \frac{(m_3+n_3)\pm\sqrt{(m_3+n_3)^2-4(2m_2+n_3)(m_1+m_3)}}{2(2m_2+n_3)}$$

  holds. In this case, there exist exactly two proper biharmonic orbits.
- *When* $(m_3+n_3)^2-4(2m_2+n_3)(m_1+m_3)<0$, *if the orbit* $K_2\cdot\pi_1(x)$ *is biharmonic, then it is harmonic.*

- *When* $(m_3+n_3)^2 - 4(2m_2+n_3)(m_1+m_3) = 0$, *there exists a unique proper biharmonic orbit.*

5.12. Type III-G_2**.** We set

$$\Sigma^+ = W^+ = \{\alpha_1, \alpha_2, \alpha_1+\alpha_2, 2\alpha_1+\alpha_2, 3\alpha_1+\alpha_2, 3\alpha_1+2\alpha_2\},$$

$$\langle \alpha_1, \alpha_1\rangle = 1, \qquad \langle \alpha_1, \alpha_2\rangle = -\frac{3}{2}, \qquad \langle \alpha_2, \alpha_2\rangle = 3, \qquad \tilde{\alpha} = 3\alpha_1 + 2\alpha_2,$$

and

$$m_1 = m(\alpha_1), \qquad m_2 = m(\alpha_2).$$

(1) When $H \in P_0^{\{\alpha_1,\tilde{\alpha}\}} = \{tH_{\alpha_1} \mid 0 < t < 1\}$, we have $\Sigma_H^+ = \{\alpha_2\}$, $W_H^+ = \emptyset$. By Theorem 4, we have

$$\tau_H = 2[-m_1\{\cot\langle 2\alpha_1, H\rangle + \cot\langle 4\alpha_1, H\rangle\} - 3m_2 \cot\langle 6\alpha_1, H\rangle](2\alpha_1+\alpha_2).$$

Hence we have $\tau_H = 0$ if and only if

$$-m_1\{\cot\langle 2\alpha_1, H\rangle + \cot\langle 4\alpha_1, H\rangle\} - 3m_2 \cot\langle 6\alpha_1, H\rangle = 0.$$

By Theorem 6, the orbit $K_2 \cdot \pi_1(x)$ is biharmonic if and only if

$$\begin{aligned} 0 = &-4\langle \tau_H, \alpha_1\rangle\{m_1(\cot\langle 2\alpha_1, H\rangle)^2 + 2m_1(\cot\langle 4\alpha_1, H\rangle)^2 \\ &+ 9m_2(\cot\langle 6\alpha_1, H\rangle)^2\}(2\alpha_1+\alpha_2). \end{aligned}$$

Therefore, $K_2 \cdot \pi_1(x)$ is biharmonic if and only if $\tau_H = 0$ or

$$0 = m_1(\cot\langle 2\alpha_1, H\rangle)^2 + 2m_1(\cot\langle 4\alpha_1, H\rangle)^2 + 9m_2(\cot\langle 6\alpha_1, H\rangle)^2$$

holds. Clearly,

$$m_1(\cot\langle 2\alpha_1, H\rangle)^2 + 2m_1(\cot\langle 4\alpha_1, H\rangle)^2 + 9m_2(\cot\langle 6\alpha_1, H\rangle)^2 > 0$$

for $0 < t < 1$. *Therefore, if the orbit* $K_2 \cdot \pi_1(x)$ *is biharmonic, then it is harmonic.*

(2) When $H \in P_0^{\{\alpha_2,\tilde{\alpha}\}} = \{tH_{\alpha_2} \mid 0 < t < 1\}$, we have $\Sigma_H^+ = \{\alpha_1\}$, $W_H^+ = \emptyset$. Then $K_2 \cdot \pi_1(x)$ is biharmonic if and only if $\tau_H = 0$ or

$$0 = (m_1+m_2)(\cot\langle 2\alpha_2, H\rangle)^2 + 2m_2(\cot\langle 4\alpha_1, H\rangle)^2$$

holds. Clearly,

$$(m_1+m_2)(\cot\langle 2\alpha_2, H\rangle)^2 + 2m_2(\cot\langle 4\alpha_1, H\rangle)^2 > 0$$

for $0 < t < 1$. *Therefore, if the orbit* $K_2 \cdot \pi_1(x)$ *is biharmonic, then it is harmonic.*

(3) When $H \in P_0^{\{\alpha_1, \alpha_2\}} = \{tH_{\alpha_1} + (1-t)H_{\alpha_2} \mid 0 < t < 1\}$, we have $\Sigma_H^+ = \emptyset$, $W_H^+ = \{3\alpha_1 + 2\alpha_2\}$. We set $\vartheta = \langle \alpha_1, H \rangle$. Then $\langle 2\alpha_2, H \rangle = (\pi/2) - 3\vartheta$ and $0 < \vartheta < (\pi/6)$. Then $K_2 \cdot \pi_1(x)$ is biharmonic if and only if $\tau_H = 0$ or

$$0 = m_1(\cot(2\vartheta))^2 + \frac{9}{2} m_2(\tan(3\vartheta))^2 + \frac{1}{2} m_1(\tan \vartheta)^2$$

holds. Clearly,

$$m_1(\cot(2\vartheta))^2 + \frac{9}{2} m_2(\tan(3\vartheta))^2 + \frac{1}{2} m_1(\tan \vartheta)^2 > 0$$

for $0 < t < 1$. *Therefore, if the orbit* $K_2 \cdot \pi_1(x)$ *is biharmonic, then it is harmonic.*

5.13. Tables of proper biharmonic orbits. By the above arguments, we classify all the proper biharmonic submanifolds which are singular orbits of commutative Hermann actions whose cohomogeneity is two. The co-dimensions of these submanifolds are greater than two, since we consider singular orbits of cohomogeneity two actions.

Theorem 8. *Let* (G, K_1, K_2) *be a compact symmetric triad which satisfies*

- $\theta_1 \sim \theta_2$ *and* G/K_1 *is an irreducible symmetric space, or*
- *one of the conditions* (A), (B) *and* (C) *in Theorem* 2.

Assume that the K_2*-action on* $N_1 = G/K_1$ *is cohomogeneity two. Then, all singular orbit types which are one parameter families in the orbit space are divided into the following three cases:*

(i) *There exists a unique proper biharmonic orbit.*

(ii) *There exist exactly two distinct proper biharmonic orbits.*

(iii) *Any biharmonic orbit is harmonic.*

We list below all the results of the above computations. In the following tables, the first column shows compact symmetric triads which induce Hermann actions; the second column shows multiplicities of symmetric triads which are induced by compact symmetric triads in the first column; the third column shows a subset Δ in $\Pi \cup \{\delta\}$ or $\Pi \cup \{\tilde{\alpha}\}$ where P_0^{Δ} gives a singular orbit type which is one parameter family in the orbit space; the fourth column represents the result (i), (ii) or (iii) in Theorem 8 for the orbit type of P_0^{Δ}; the fifth column shows the co-dimension of orbits $K_2 \cdot \pi_1(x)$ in N_1 and $K_1 \cdot \pi_2(x)$ in N_2.

Cases of $\theta_1 = \theta_2$ (Isotropy actions).

Type A_2

(G, K_1)	$m(\alpha)$	Δ	Theorem 8	codim
$(\mathrm{SU}(3), \mathrm{SO}(3))$	1	$\{\alpha_1, \delta\}$	(ii)	3
		$\{\alpha_2, \delta\}$	(ii)	3
		$\{\alpha_1, \alpha_2\}$	(ii)	3
$(\mathrm{SU}(3) \times \mathrm{SU}(3), \mathrm{SU}(3))$	2	$\{\alpha_1, \delta\}$	(ii)	4
		$\{\alpha_2, \delta\}$	(ii)	4
		$\{\alpha_1, \alpha_2\}$	(ii)	4
$(\mathrm{SU}(6), \mathrm{Sp}(3))$	4	$\{\alpha_1, \delta\}$	(ii)	6
		$\{\alpha_2, \delta\}$	(ii)	6
		$\{\alpha_1, \alpha_2\}$	(ii)	6
(E_6, F_4)	8	$\{\alpha_1, \delta\}$	(ii)	10
		$\{\alpha_2, \delta\}$	(ii)	10
		$\{\alpha_1, \alpha_2\}$	(ii)	10

Type B_2

(G, K_1)	(m_1, m_2)	Δ	Theorem 8	codim
$(\mathrm{SO}(5) \times \mathrm{SO}(5), \mathrm{SO}(5))$	$(2, 2)$	$\{\alpha_1, \delta\}$	(ii)	4
		$\{\alpha_2, \delta\}$	(ii)	4
		$\{\alpha_1, \alpha_2\}$	(ii)	4
$(\mathrm{SO}(4+n), \mathrm{SO}(2) \times \mathrm{SO}(2+n))$	$(n, 1)$	$\{\alpha_1, \delta\}$	(ii)	$n+2$
		$\{\alpha_2, \delta\}$	(ii)	3
		$\{\alpha_1, \alpha_2\}$	(ii)	3

Type C_2

(G, K_1)	(m_1, m_2)	Δ	Theorem 8	codim
$(Sp(2), U(2))$	$(1, 1)$	$\{\alpha_1, \delta\}$	(ii)	3
		$\{\alpha_2, \delta\}$	(ii)	3
		$\{\alpha_1, \alpha_2\}$	(ii)	3
$(Sp(2) \times Sp(2), Sp(2))$	$(2, 2)$	$\{\alpha_1, \delta\}$	(ii)	4
		$\{\alpha_2, \delta\}$	(ii)	4
		$\{\alpha_1, \alpha_2\}$	(ii)	4
$(Sp(4), Sp(2) \times Sp(2))$	$(4, 3)$	$\{\alpha_1, \delta\}$	(ii)	5
		$\{\alpha_2, \delta\}$	(ii)	6
		$\{\alpha_1, \alpha_2\}$	(ii)	5
$(SU(4), S(U(2) \times U(2)))$	$(2, 1)$	$\{\alpha_1, \delta\}$	(ii)	3
		$\{\alpha_2, \delta\}$	(ii)	4
		$\{\alpha_1, \alpha_2\}$	(ii)	3
$(SO(8), U(4))$	$(4, 1)$	$\{\alpha_1, \delta\}$	(ii)	3
		$\{\alpha_2, \delta\}$	(ii)	6
		$\{\alpha_1, \alpha_2\}$	(ii)	3

Type BC_2

(G, K_1)	(m_1, m_2, m_3)	Δ	Theorem 8	codim
$(\mathrm{SU}(4+n), \mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(2+n)))$	$(2n, 2, 1)$	$\{\alpha_1, \delta\}$	(ii)	5
		$\{\alpha_2, \delta\}$	(ii)	$2n+2$
		$\{\alpha_1, \alpha_2\}$	(ii)	3
$(\mathrm{SO}(10), \mathrm{U}(5))$	$(4, 4, 1)$	$\{\alpha_1, \delta\}$	(ii)	7
		$\{\alpha_2, \delta\}$	(ii)	6
		$\{\alpha_1, \alpha_2\}$	(ii)	3
$(\mathrm{Sp}(4+n), \mathrm{Sp}(2) \times \mathrm{Sp}(2+n))$	$(4n, 4, 3)$	$\{\alpha_1, \delta\}$	(ii)	9
		$\{\alpha_1, \delta\}$	(ii)	$4n+2$
		$\{\alpha_1, \alpha_2\}$	(ii)	5
$(E_6, \mathrm{T}^1 \cdot \mathrm{Spin}(10))$	$(8, 6, 1)$	$\{\alpha_1, \delta\}$	(ii)	9
		$\{\alpha_1, \delta\}$	(ii)	10
		$\{\alpha_1, \alpha_2\}$	(ii)	3

Type G_2

(G, K_1)	(m_1, m_2)	Δ	Theorem 8	codim
$(G_2, \mathrm{SO}(4))$	$(1, 1)$	$\{\alpha_1, \delta\}$	(ii)	3
		$\{\alpha_2, \delta\}$	(ii)	3
		$\{\alpha_1, \alpha_2\}$	(iii)	3
$(G_2 \times G_2, G_2)$	$(2, 2)$	$\{\alpha_1, \delta\}$	(ii)	4
		$\{\alpha_2, \delta\}$	(ii)	4
		$\{\alpha_1, \alpha_2\}$	(iii)	4

Cases of $\theta_1 \not\sim \theta_2$

Type I-B_2

(G, K_1, K_2)	(m_1, m_2, n_1)	Δ	Theorem 8	codim
$(SO(2+a+b), SO(2+a)\times SO(b), SO(2)\times SO(a+b))$	$(b-2, 1, a)$	$\{\alpha_1, \tilde{\alpha}\}$	(i)	3
		$\{\alpha_2, \tilde{\alpha}\}$	(ii)	b
		$\{\alpha_1, \alpha_2\}$	(ii)	$a+2$
$(SO(6)\times SO(6), \Delta(SO(6)\times SO(6)), K_2)$ with $(G_\sigma)_0 \cong SO(3)\times SO(3)$ (C)	$(2, 2, 2)$	$\{\alpha_1, \tilde{\alpha}\}$	(i)	4
		$\{\alpha_2, \tilde{\alpha}\}$	(ii)	4
		$\{\alpha_1, \alpha_2\}$	(ii)	4
$(SO(6)\times SO(6), \Delta(SO(6)\times SO(6)), K_2)$ with $(G_\sigma)_0 \cong SO(5)$ (C)	$(2, 2, 2)$	$\{\alpha_1, \tilde{\alpha}\}$	(i)	4
		$\{\alpha_2, \tilde{\alpha}\}$	(ii)	4
		$\{\alpha_1, \alpha_2\}$	(ii)	4

Here $(2 < b,\ 1 \leq a)$.

Type I-C_2

(G, K_1, K_2)	(m_1, m_2, n_1)	Δ	Theorem 8	codim
$(SO(8), SO(4)\times SO(4), U(4))$	$(2, 1, 2)$	$\{\alpha_1, \tilde{\alpha}\}$	(i)	3
		$\{\alpha_2, \tilde{\alpha}\}$	(ii)	4
		$\{\alpha_1, \alpha_2\}$	(ii)	4
$(SU(4), SO(4), S(U(2)\times U(2)))$	$(1, 1, 1)$	$\{\alpha_1, \tilde{\alpha}\}$	(i)	3
		$\{\alpha_2, \tilde{\alpha}\}$	(ii)	3
		$\{\alpha_1, \alpha_2\}$	(ii)	3
$(SU(4)\times SU(4), \Delta(SU(4)\times SU(4)), K_2)$ with $(G_\sigma)_0 \cong SO(4)$ (C)	$(2, 2, 2)$	$\{\alpha_1, \tilde{\alpha}\}$	(i)	4
		$\{\alpha_2, \tilde{\alpha}\}$	(ii)	4
		$\{\alpha_1, \alpha_2\}$	(ii)	4
$(SU(4)\times SU(4), \Delta(SU(4)\times SU(4)), K_2)$ with $(G_\sigma)_0 \cong Sp(2)$ (C)	$(2, 2, 2)$	$\{\alpha_1, \tilde{\alpha}\}$	(i)	4
		$\{\alpha_2, \tilde{\alpha}\}$	(ii)	4
		$\{\alpha_1, \alpha_2\}$	(ii)	4

Type I-BC_2-A_1^2

(G, K_1, K_2)	(m_1, m_2, m_3, n_1)	Δ	Theorem 8	codim
$(SU(2+a+b)$, $S(U(2+a) \times U(b))$, $S(U(2) \times U(a+b)))$	$(2(b-2), 2, 1, 2a)$	$\{\alpha_1, \tilde{\alpha}\}$	(ii)	$2b-1$
		$\{\alpha_2, \tilde{\alpha}\}$	(ii)	4
		$\{\alpha_1, \alpha_2\}$	(ii)	$2(a+1)$
$(Sp(2+a+b)$, $Sp(2+s) \times Sp(t)$, $Sp(2) \times Sp(s+t))$	$(4(b-1), 4, 3, 4a)$	$\{\alpha_1, \tilde{\alpha}\}$	(ii)	$4b+1$
		$\{\alpha_2, \tilde{\alpha}\}$	(ii)	6
		$\{\alpha_1, \alpha_2\}$	(ii)	$4a+2$
$(SO(12), U(6), U(6)')$	$(4, 4, 1, 4)$	$\{\alpha_1, \tilde{\alpha}\}$	(ii)	7
		$\{\alpha_2, \tilde{\alpha}\}$	(ii)	6
		$\{\alpha_1, \alpha_2\}$	(ii)	6

Here $2 < b, 1 \leq a$.

Type I-BC_2-B_2

(G, K_1, K_2)	(m_1, m_2, m_3, n_2)	Δ	Theorem 8	codim
$(SO(4+2a)$, $SO(4) \times SO(2a)$, $U(2+a))$	$(2(a-2), 2, 1, 2)$	$\{\alpha_1, \tilde{\alpha}\}$	(ii)	$2a-1$
		$\{\alpha_2, \tilde{\alpha}\}$	(ii)	4
		$\{\alpha_1, \alpha_2\}$	(iii)	4
$(E_6$, $SU(6) \cdot SU(2)$, $SO(10) \cdot U(1))$	$(4, 4, 1, 2)$	$\{\alpha_1, \tilde{\alpha}\}$	(ii)	7
		$\{\alpha_2, \tilde{\alpha}\}$	(iii)	6
		$\{\alpha_1, \alpha_2\}$	(iii)	4
$(E_7$, $SO(12) \cdot SU(2)$, $E_6 \cdot U(1))$	$(8, 6, 1, 2)$	$\{\alpha_1, \tilde{\alpha}\}$	(ii)	11
		$\{\alpha_2, \tilde{\alpha}\}$	(iii)	8
		$\{\alpha_1, \alpha_2\}$	(iii)	4

Here $2 \leq a$.

Type II-BC$_2$

(G, K_1, K_2)	(m_1, m_2, m_3, n_3)	Δ	Theorem 8	codim
$(\mathrm{SU}(2+a),$ $\mathrm{SO}(2+a),$ $\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(a)))$	$(a-2, 1, 1)$	$\{\alpha_1, \tilde{\alpha}\}$	(iii)	3
		$\{\alpha_2, \tilde{\alpha}\}$	(iii)	a
		$\{\alpha_1, \alpha_2\}$	(iii)	3
$(\mathrm{SO}(10),$ $\mathrm{SO}(5) \times \mathrm{SO}(5),$ $\mathrm{U}(5))$	$(2, 2, 1)$	$\{\alpha_1, \tilde{\alpha}\}$	(iii)	4
		$\{\alpha_2, \tilde{\alpha}\}$	(iii)	4
		$\{\alpha_1, \alpha_2\}$	(iii)	3
$(E_6,$ $\mathrm{Sp}(4),$ $\mathrm{SO}(10) \cdot \mathrm{U}(1))$	$(4, 3, 1)$	$\{\alpha_1, \tilde{\alpha}\}$	(iii)	5
		$\{\alpha_2, \tilde{\alpha}\}$	(iii)	6
		$\{\alpha_1, \alpha_2\}$	(iii)	3
$(\mathrm{SU}(5) \times \mathrm{SU}(5), \Delta(\mathrm{SU}(5) \times \mathrm{SU}(5)),$ $K_2)$ with $(G_\sigma)_0 \cong \mathrm{SO}(5)$ (C)	$(2, 2, 2)$	$\{\alpha_1, \tilde{\alpha}\}$	(iii)	4
		$\{\alpha_2, \tilde{\alpha}\}$	(iii)	4
		$\{\alpha_1, \alpha_2\}$	(iii)	4

Here $2 \leq a$.

Type III-A$_2$

(G, K_1, K_2)	(m_1, n_1)	Δ	Theorem 8	codim
$(\mathrm{SU}(6), \mathrm{Sp}(3), \mathrm{SO}(6))$	$(2, 2)$	$\{\alpha_1, \tilde{\alpha}\}$	(iii)	4
		$\{\alpha_2, \tilde{\alpha}\}$	(iii)	4
		$\{\alpha_1, \alpha_2\}$	(iii)	4
$(E_6, \mathrm{Sp}(4), F_4)$	$(4, 4)$	$\{\alpha_1, \tilde{\alpha}\}$	(iii)	6
		$\{\alpha_2, \tilde{\alpha}\}$	(iii)	6
		$\{\alpha_1, \alpha_2\}$	(iii)	6
$(U \times U,$ $\Delta(U \times U),$ $\bar{K} \times \bar{K})$, (B)	(m, m)	$\{\alpha_1, \tilde{\alpha}\}$	(iii)	$m+2$
		$\{\alpha_2, \tilde{\alpha}\}$	(iii)	$m+2$
		$\{\alpha_1, \alpha_2\}$	(iii)	$m+2$

Here m is the multiplicity of the root system of the symmetric pair $(U, \overline{K})$ of type A$_2$.

Type III-B_2

(G, K_1, K_2)	(m_1, m_1, n_2)	Δ	Theorem 8	codim
$(U \times U,$ $\Delta(U \times U),$ $\bar{K} \times \bar{K})$, (B)	(m, n, n)	$\{\alpha_1, \tilde{\alpha}\}$	(iii)	$m+2$
		$\{\alpha_2, \tilde{\alpha}\}$	(iii)	$n+2$
		$\{\alpha_1, \alpha_2\}$	(iii)	$n+2$

Here (m, n) is the multiplicity of the root system of the symmetric pair $(U, \bar{K})$ of type B_2.

Type III-C_2

(G, K_1, K_2)	(m_1, m_1, n_2)	Δ	Theorem 8	codim
(SU(8), S(U(4) × U(4)), Sp(4))	$(4, 3, 1)$	$\{\alpha_1, \tilde{\alpha}\}$	(i)	6
		$\{\alpha_2, \tilde{\alpha}\}$	(iii)	5
		$\{\alpha_1, \alpha_2\}$	(iii)	3
(Sp(4), U(4), Sp(2) × Sp(2))	$(2, 1, 2)$	$\{\alpha_1, \tilde{\alpha}\}$	(i)	4
		$\{\alpha_2, \tilde{\alpha}\}$	(iii)	3
		$\{\alpha_1, \alpha_2\}$	(iii)	4
$(U \times U,$ $\Delta(U \times U),$ $\bar{K} \times \bar{K})$, (B)	(m, n, n)	$\{\alpha_1, \tilde{\alpha}\}$	(iii)	$m+2$
		$\{\alpha_2, \tilde{\alpha}\}$	(iii)	$n+2$
		$\{\alpha_1, \alpha_2\}$	(iii)	$n+2$

Here (m, n) is the multiplicity of the root system of the symmetric pair $(U, \bar{K})$ of type C_2.

Type III-BC_2

(G, K_1, K_2)	(m_1, m_2, m_3, n_3)	Δ	Theorem 8	codim
$(SU(4+2s), S(U(4) \times U(2s)), Sp(2+s))$	$(4(s-2), 4, 3, 1)$	$\{\alpha_1, \tilde{\alpha}\}$	(iii)	6
		$\{\alpha_2, \tilde{\alpha}\}$	(iii)	$4s-3$
		$\{\alpha_1, \alpha_2\}$	(iii)	3
$(SU(10), S(U(5) \times U(5)), Sp(5))$	$(4, 4, 1, 3)$	$\{\alpha_1, \tilde{\alpha}\}$	(iii)	6
		$\{\alpha_2, \tilde{\alpha}\}$	(iii)	7
		$\{\alpha_1, \alpha_2\}$	(iii)	5
$(Sp(2+s), U(2+s), Sp(2) \times Sp(s))$ $(s > 2)$	$(2(s-2), 2, 1, 2)$	$\{\alpha_1, \tilde{\alpha}\}$	(iii)	$2s-1$
		$\{\alpha_2, \tilde{\alpha}\}$	(iii)	4
		$\{\alpha_1, \alpha_2\}$	(iii)	4
$(U \times U, \Delta(U \times U), \bar{K} \times \bar{K})$, (B)	(m, n, l, l)	$\{\alpha_1, \tilde{\alpha}\}$	(iii)	$n+2$
		$\{\alpha_2, \tilde{\alpha}\}$	(iii)	$m+l+2$
		$\{\alpha_1, \alpha_2\}$	(iii)	$l+2$

Here (m, n, l) is the multiplicity of the root system of the symmetric pair $(U, \bar{K})$ of type BC_2.

Type III-G_2

(G, K_1, K_2)	(m_1, m_1, n_1, n_2)	Δ	Theorem 8	codim
$(U \times U, \Delta(U \times U), \bar{K} \times \bar{K})$, (B)	(m, n, m, n)	$\{\alpha_1, \tilde{\alpha}\}$	(iii)	$n+2$
		$\{\alpha_2, \tilde{\alpha}\}$	(iii)	$m+2$
		$\{\alpha_1, \alpha_2\}$	(iii)	$n+2$

Here (m, n) is the multiplicity of the root system of the symmetric pair $(U, \bar{K})$ of type G_2.

6. Biharmonic homogeneous hypersurfaces in compact Lie groups

In this section, applying Corollary 4 we will study biharmonic regular orbits of cohomogeneity one $(K_2 \times K_1)$-actions on compact Lie groups.

Let (G, K_1, K_2) be a commutative compact symmetric triad where G is semisimple. Hereafter we assume that $\dim \mathfrak{a} = 1$ and G is simply-connected. Then the orbit space of $(K_2 \times K_1)$-action is homeomorphic to a closed interval. A point in the interior of the orbit space corresponds to a regular orbit, and there exists a unique minimal (harmonic) orbit among regular orbits. On the other hand, two endpoints of the orbit space correspond to singular orbits. These singular orbits are minimal (harmonic), moreover these are weakly reflective ([IST2]). For $H \in \mathfrak{a}$, we set $x = \exp(H)$ and consider the orbit $(K_2 \times K_1) \cdot x$ of the $(K_2 \times K_1)$-action on G through x. For simplicity, we denote the tension field $dL_x^{-1}(\tau_H)_x$ of the orbit $(K_2 \times K_1) \cdot x$ in G by τ_H for $H \in \mathfrak{a}$.

Cases of $\theta_1 \sim \theta_2$. First, we consider the cases of $\theta_1 = \theta_2$. Then $(G, K_1) = (G, K_2)$ is a compact symmetric pair of rank one. The restricted root type system of (G, K_1) is of type B_1 or BC_1. Let $\vartheta := \langle \delta, H \rangle$ for $H \in \mathfrak{a}$, where δ is the highest root of $\tilde{\Sigma}$. Then, by (6), $P_0 = \{H \in \mathfrak{a} \,|\, 0 < \vartheta < \pi\}$ is a cell in these types.

6.1. Type B_1**.** We set $\Sigma^+ = \{\alpha\}$, $W^+ = \emptyset$ and $m = m(\alpha)$. In this case, ϑ satisfies $0 < \vartheta < \pi$. By Corollary 2 we have

$$\tau_H = -m \cot \vartheta \alpha.$$

Hence $\tau_H = 0$ if and only if $\vartheta = \pi/2$ holds. By Corollary 4, $(K_2 \times K_1) \cdot x$ in G is biharmonic if and only if

$$0 = m\langle \tau_H, \alpha \rangle \left(\frac{3}{2} - (\cot \vartheta)^2 \right)$$

holds. Therefore, the orbit $(K_2 \times K_1) \cdot x$ is proper biharmonic if and only if

$$(\cot \vartheta)^2 = \frac{3}{2}$$

holds. *In this case, there exist exactly two proper biharmonic hypersurfaces which are regular orbits of the $(K_2 \times K_1)$-action on G.*

6.2. Type BC_1**.** We set $\Sigma^+ = \{\alpha, 2\alpha\}$, $W^+ = \emptyset$ and $m_1 = m(\alpha)$, $m_2 = m(2\alpha)$. In this case ϑ satisfies $0 < \vartheta < \pi$. By Corollary 2 we have

$$\begin{aligned} \tau_H &= -m_1 \cot\left(\frac{\vartheta}{2}\right)\alpha - m_2(\cot \vartheta)2\alpha \\ &= -\left\{ (m_1 + m_2) \cot\left(\frac{\vartheta}{2}\right) - m_2 \cot\left(\frac{\vartheta}{2}\right) \right\}\alpha. \end{aligned}$$

Hence $\tau_H = 0$ if and only if

$$\left(\cot \frac{\vartheta}{2}\right)^2 = \frac{m_2}{m_1 + m_2}$$

holds. By Corollary 4, $(K_2 \times K_1) \cdot x$ in G is biharmonic if and only if

$$0 = \langle \tau_H, \alpha \rangle \left\{ m_1 \left(\frac{3}{2} - \left(\cot \frac{\vartheta}{2} \right)^2 \right) + 4m_2 \left(\frac{3}{2} - (\cot \vartheta)^2 \right) \right\}$$

holds. Therefore, the orbit $(K_2 \times K_1) \cdot x$ is biharmonic if and only if $\tau_H = 0$ or

$$0 = m_1 \left(\frac{3}{2} - \left(\cot \frac{\vartheta}{2} \right)^2 \right) + 4m_2 \left(\frac{3}{2} - (\cot \vartheta)^2 \right)$$

holds. The last equation is equivalent to

$$\left((m_1 + m_2) \left(\cot \frac{\vartheta}{2} \right)^2 - m_2 \right) \left(\left(\cot \frac{\vartheta}{2} \right)^2 - 1 \right) = \left(\frac{m_1}{2} + 6m_2 \right) \left(\cot \frac{\vartheta}{2} \right)^2.$$

Since $(m_1/2) + 6m_2 > 0$, the solutions of the equation are not harmonic. Hence the orbit $(K_2 \times K_1) \cdot x$ is proper biharmonic if and only if

$$\left(\cot \left(\frac{\vartheta}{2} \right) \right)^2 = \frac{3m_1 + 16m_2 \pm \sqrt{(3m_1 + 16m_2)^2 - 16(m_1 + m_2)m_2}}{4(m_1 + m_2)}$$

holds. *In this case, there exist exactly two proper biharmonic hypersurfaces which are regular orbits of the $(K_2 \times K_1)$-action on G.*

Cases of $\theta_1 \not\sim \theta_2$. If G is simple and $\theta_1 \not\sim \theta_2$, then for a commutative compact symmetric triad (G, K_1, K_2) the triple $(\tilde{\Sigma}, \Sigma, W)$ is a symmetric triad with multiplicities $m(\lambda)$ and $n(\alpha)$ (cf. Theorem 2).

All the symmetric triads with $\dim \mathfrak{a} = 1$ are classified into the following four types ([I1]):

	Σ^+	W^+	$\tilde{\alpha}$
III-B_1	$\{\alpha\}$	$\{\alpha\}$	α
I-BC_1	$\{\alpha, 2\alpha\}$	$\{\alpha\}$	α
II-BC_1	$\{\alpha\}$	$\{\alpha, 2\alpha\}$	2α
III-BC_1	$\{\alpha, 2\alpha\}$	$\{\alpha, 2\alpha\}$	2α

Let $\vartheta := \langle \tilde{\alpha}, H \rangle$ for $H \in \mathfrak{a}$. Then, by (4), $P_0 = \{H \in \mathfrak{a} \mid 0 < \vartheta < \pi/2\}$ is a cell in these types. If G is simply connected and K_1 and K_2 are connected, then the orbit space of the $(K_2 \times K_1)$-action on G is identified with $\overline{P_0} = \{H \in \mathfrak{a} \mid 0 \leq \vartheta \leq \pi/2\}$, more precisely, each orbit meets $\exp \overline{P_0}$ at one point. For each orbit $(K_2 \times K_1) \cdot x$, $dL_x^{-1}(\tau_H)_x \in \mathfrak{a}$ holds.

6.3. Type III-B$_1$. We set $\Sigma^+ = \{\alpha\}$, $W^+ = \{\alpha\}$ and $m = m(\alpha)$, $n = n(\alpha)$. By Corollary 2 we have

$$\tau_H = -m \cot \vartheta\alpha + n \tan \vartheta\alpha.$$

Hence $\tau_H = 0$ if and only if

$$(\cot \vartheta)^2 = \frac{n}{m}$$

holds. By Corollary 4, $(K_2 \times K_1) \cdot x$ in G is biharmonic if and only if

$$0 = \langle \tau_H, \alpha \rangle \left\{ m\left(\frac{3}{2} - (\cot \vartheta)^2\right) + n\left(\frac{3}{2} - (\tan \vartheta)^2\right) \right\}$$

holds. Therefore, the orbit $(K_2 \times K_1) \cdot x$ is biharmonic if and only if $\tau_H = 0$ or

$$0 = m\left(\frac{3}{2} - (\cot \vartheta)^2\right) + n\left(\frac{3}{2} - (\tan \vartheta)^2\right)$$

holds. The last equation is equivalent to

$$\begin{aligned} 0 &= m(\cot \vartheta)^4 - \frac{3}{2}(m+n)(\cot \vartheta)^2 + n \\ &= (m(\cot \vartheta)^2 - n)(m(\cot \vartheta)^2 - 1) - \left(\frac{m+n}{2}\right)(\cot \vartheta)^2. \end{aligned}$$

Since $(m+n)/2 > 0$, the solutions of the equation are not harmonic. Hence the orbit $(K_2 \times K_1) \cdot x$ is proper biharmonic if and only if

$$(\cot \vartheta)^2 = \frac{3(m+n) \pm \sqrt{9(m+n)^2 - 16mn}}{4m}$$

holds. *In this case, there exist exactly two proper biharmonic hypersurfaces which are regular orbits of the* $(K_2 \times K_1)$*-action on* G.

6.4. Type I-BC$_1$. We set $\Sigma^+ = \{\alpha, 2\alpha\}$, $W^+ = \{\alpha\}$ and $m_1 = m(\alpha)$, $m_2 = m(2\alpha)$, $n = n(\alpha)$. By Corollary 2 we have

$$\begin{aligned} \tau_H &= -m_1 \cot \vartheta\alpha - m_2 \cot(2\vartheta)2\alpha + n \tan \vartheta\alpha \\ &= \{-(m_1 + m_2) \cot \vartheta + (m_2 + n) \tan \vartheta\}\alpha. \end{aligned}$$

Hence $\tau_H = 0$ if and only if

$$(\cot \vartheta)^2 = \frac{m_2 + n}{m_1 + m_2}$$

holds. By Corollary 4, $(K_2 \times K_1) \cdot x$ in G is biharmonic if and only if

$$0 = \langle \tau_H, \alpha \rangle \left\{ m_1 \left(\frac{3}{2} - (\cot \vartheta)^2 \right) + 4m_2 \left(\frac{3}{2} - (\cot(2\vartheta))^2 \right) + n \left(\frac{3}{2} - (\tan \vartheta)^2 \right) \right\}$$

holds. Therefore, the orbit $(K_2 \times K_1) \cdot x$ is biharmonic if and only if $\tau_H = 0$ or

$$0 = m_1 \left(\frac{3}{2} - (\cot \vartheta)^2 \right) + 4m_2 \left(\frac{3}{2} - (\cot(2\vartheta))^2 \right) + n \left(\frac{3}{2} - (\tan \vartheta)^2 \right)$$

holds. The last equation is equivalent to

$$\begin{aligned} 0 &= (m_1 + m_2)(\cot \vartheta)^4 - \left(\frac{3}{2}(m_1 + n) + 8m_2 \right)(\cot \vartheta)^2 + (m_2 + n) \\ &= ((m_2 + m_2)(\cot \vartheta)^2 - (m_2 + n))((\cot \vartheta)^2 - 1) - \left(\frac{m_1 + n}{2} + 6m_2 \right)(\cot \vartheta)^2. \end{aligned}$$

Since $(m_1 + n)/2 + 6m_2 > 0$, the solutions of the equation are not harmonic. Hence the orbit $(K_2 \times K_1) \cdot x$ is proper biharmonic if and only if

$$(\cot \vartheta)^2 = \frac{3(m + n) + 16m_2 \pm \sqrt{(3(m + n) + 16m_2)^2 - 16(m_1 + m_2)(m_2 + n)}}{4(m_1 + m_2)}$$

holds. *In this case, there exist exactly two proper biharmonic hypersurfaces which are regular orbits of the* $(K_2 \times K_1)$*-action on* G.

6.5. Type II-BC$_1$**.** We set $\Sigma^+ = \{\alpha\}$, $W^+ = \{\alpha, 2\alpha\}$ and $m = m(\alpha) = n(\alpha)$, $n = n(2\alpha)$. By Corollary 2 we have

$$\begin{aligned} \tau_H &= -m \cot\left(\frac{\vartheta}{2}\right)\alpha + m \tan\left(\frac{\vartheta}{2}\right)\alpha + n \tan(\vartheta)2\alpha \\ &= 2\{-m \cot \vartheta + n \tan \vartheta\}\alpha. \end{aligned}$$

Hence $\tau_H = 0$ if and only if

$$(\cot \vartheta)^2 = \frac{n}{m}$$

holds. By Corollary 4, $(K_2 \times K_1) \cdot x$ in G is biharmonic if and only if

$$0 = \langle \tau_H, \alpha \rangle \left\{ m \left(\frac{3}{2} - \left(\cot \frac{\vartheta}{2} \right)^2 \right) + m \left(\frac{3}{2} - \left(\tan \frac{\vartheta}{2} \right)^2 \right) + 4n \left(\frac{3}{2} - (\tan \vartheta)^2 \right) \right\}$$

holds. Therefore, the orbit $(K_2 \times K_1) \cdot x$ is biharmonic if and only if $\tau_H = 0$ or

$$0 = m\left(\frac{3}{2} - \left(\cot \frac{\vartheta}{2}\right)^2\right) + m\left(\frac{3}{2} - \left(\tan \frac{\vartheta}{2}\right)^2\right) + 4n\left(\frac{3}{2} - (\tan \vartheta)^2\right)$$

holds. The last equation is equivalent to

$$\begin{aligned} 0 &= 4m(\cot \vartheta)^4 - (m + 6n)(\cot \vartheta)^2 + 4n \\ &= 4(m(\cot \vartheta)^2 - n)((\cot \vartheta)^2 - 1) - (-3m + 2n)(\cot \vartheta)^2. \end{aligned}$$

If $-3m + 2n = 0$, then the orbit $(K_2 \times K_1) \cdot x$ is proper biharmonic if and only if $(\cot \vartheta)^2 = 1$ holds. When $-3m + 2n \neq 0$, the solutions of the equation are not harmonic.

- *When* $(m + 6n)^2 - 64mn > 0$, *then the orbit* $(K_2 \times K_1) \cdot x$ *is proper biharmonic if and only if*

$$(\cot \vartheta)^2 = \frac{m + 6n \pm \sqrt{(m + 6n)^2 - 64mn}}{8m}$$

  holds. The condition $(m + 6n)^2 - 64mn > 0$ *is equivalent to* $m < (26 - 8\sqrt{10})n$ *or* $(26 + 8\sqrt{10})n < m$. *In this case, if* $(m + 6n)^2 - 64mn > 0$, *then there exist exactly two proper biharmonic hypersurfaces which are regular orbits of the* $(K_2 \times K_1)$*-action on* G.
- *When* $(m + 6n)^2 - 64mn < 0$, *any biharmonic regular orbits of the* $(K_2 \times K_1)$*-action on* G *is harmonic.*

It should be remarked that by the classification of compact symmetric triads, we can see that there is no compact symmetric triad which satisfies $(m + 6n)^2 - 64mn = 0$.

6.6. Type III-BC$_1$**.** We set $\Sigma^+ = \{\alpha, 2\alpha\}$, $W^+ = \{\alpha, 2\alpha\}$ and $m_1 = m(\alpha) = n(\alpha)$, $m_2 = m(2\alpha)$, $n = n(2\alpha)$. By Corollary 2 we have

$$\begin{aligned} \tau_H &= -m_1 \cot\left(\frac{\vartheta}{2}\right)\alpha + m_1 \tan\left(\frac{\vartheta}{2}\right)\alpha + n \tan \vartheta(2\alpha) - m_2 \cot \vartheta(2\alpha) \\ &= 2\{-(m_1 + m_2) \cot \vartheta + n \tan \vartheta\}\alpha. \end{aligned}$$

Hence $\tau_H = 0$ if and only if

$$(\cot \vartheta)^2 = \frac{n}{m_1 + m_2}$$

holds. By Corollary 4, $(K_2 \times K_1) \cdot x$ in G is biharmonic if and only if

$$0 = \langle \tau_H, \alpha \rangle \left\{ m_1 \left(\frac{3}{2} - \left(\cot \frac{\vartheta}{2} \right)^2 \right) + m_1 \left(\frac{3}{2} - \left(\tan \frac{\vartheta}{2} \right)^2 \right) + 4n \left(\frac{3}{2} - (\tan \vartheta)^2 \right) + 4m_2 \left(\frac{3}{2} - (\cot \vartheta)^2 \right) \right\}$$

holds. Therefore, the orbit $(K_2 \times K_1) \cdot x$ is biharmonic if and only if $\tau_H = 0$ or

$$0 = m_1 \left(\frac{3}{2} - \left(\cot \frac{\vartheta}{2} \right)^2 \right) + m_1 \left(\frac{3}{2} - \left(\tan \frac{\vartheta}{2} \right)^2 \right) + 4n \left(\frac{3}{2} - (\tan \vartheta)^2 \right) + 4m_2 \left(\frac{3}{2} - (\cot \vartheta)^2 \right)$$

holds. The last equation is equivalent to

$$\begin{aligned} 0 &= 4(m_1 + m_2)(\cot \vartheta)^4 - (m_1 + 6m_2 + 6n)(\cot \vartheta)^2 + 4n \\ &= 4((m_1 + m_2)(\cot \vartheta)^2 - n)((\cot \vartheta)^2 - 1) - (-3m_1 + m_2 + 2n)(\cot \vartheta)^2. \end{aligned}$$

If $-3m_1 + m_2 + 2n = 0$, then the orbit $(K_2 \times K_1) \cdot x$ is proper biharmonic if and only if $(\cot \vartheta)^2 = 1$ holds. When $-3m_1 + m_2 + 2n \neq 0$, the solutions of the equation are not harmonic.

- *When* $(m_1 + 6m_2 + 6n)^2 - 64(m_1 + m_2)n > 0$, *then the orbit* $(K_2 \times K_1) \cdot x$ *is proper biharmonic if and only if*

$$(\cot \vartheta)^2 = \frac{m_1 + 6m_2 + 6n \pm \sqrt{(m_1 + 6m_2 + 6n)^2 - 64(m_1 + m_2)n}}{8(m_1 + m_2)}$$

holds. In this case, if $(m_1 + 6m_2 + 6n)^2 - 64(m_1 + m_2)n > 0$, *then there exist exactly two proper biharmonic hypersurfaces which are regular orbits of the* $(K_2 \times K_1)$*-action on* G.
- *When* $(m_1 + 6m_2 + 6n)^2 - 64(m_1 + m_2)n < 0$, *any biharmonic regular orbits of the* $(K_2 \times K_1)$*-action on* G *is harmonic.*

It should be remarked that by the classification of compact symmetric triads, we can see that there is no compact symmetric triad which satisfies $(m_1 + 6m_2 + 6n)^2 - 64(m_1 + m_2)n = 0$.

Let $b > 0$, $c > 1$ and $q > 1$. Each commutative compact symmetric triad (G, K_1, K_2) where G is simple, $\theta_1 \not\sim \theta_2$ and $\dim \mathfrak{a} = 1$ is one of the following (see [I2]):

Type III-B_1

(G, K_1, K_2)	$(m(\alpha), n(\alpha))$
$(\mathrm{SO}(1+b+c), \mathrm{SO}(1+b) \times \mathrm{SO}(c), \mathrm{SO}(b+c))$	$(c-1, b)$
$(\mathrm{SU}(4), \mathrm{Sp}(2), \mathrm{SO}(4))$	$(2, 2)$
$(\mathrm{SU}(4), \mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(2)), \mathrm{Sp}(2))$	$(3, 1)$
$(\mathrm{Sp}(2), \mathrm{U}(2), \mathrm{Sp}(1) \times \mathrm{Sp}(1))$	$(1, 2)$

Type I-BC_1

(G, K_1, K_2)	$(m(\alpha), m(2\alpha), n(\alpha))$
$(\mathrm{SO}(2+2q), \mathrm{SO}(2) \times \mathrm{SO}(2q), \mathrm{U}(1+q))$	$(2(q-1), 1, 2(q-1))$
$(\mathrm{SU}(1+b+c), \mathrm{S}(\mathrm{U}(1+b) \times \mathrm{U}(c)), \mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(b+c))$	$(2(c-1), 1, 2b)$
$(\mathrm{Sp}(1+b+c), \mathrm{Sp}(1+b) \times \mathrm{Sp}(c), \mathrm{Sp}(1) \times \mathrm{Sp}(b+c))$	$(4(c-1), 3, 4b)$
$(\mathrm{SO}(8), \mathrm{U}(4), \mathrm{U}(4)')$	$(4, 1, 1)$

Type II-BC_1

(G, K_1, K_2)	$(m(\alpha), n(\alpha), n(2\alpha))$
$(\mathrm{SO}(6), \mathrm{U}(3), \mathrm{SO}(3) \times \mathrm{SO}(3))$	$(2, 2, 1)$
$(\mathrm{SU}(1+q), \mathrm{SO}(1+q), \mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(q)))$	$(q-1, q-1, 1)$

Type III-BC_1

(G, K_1, K_2)	$(m(\alpha), m(2\alpha), n(\alpha), n(2\alpha))$
$(\mathrm{SU}(2+2q), \mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(2q)), \mathrm{Sp}(1+q))$	$(4(q-1), 3, 4(q-1), 1)$
$(\mathrm{Sp}(1+q), \mathrm{U}(1+q), \mathrm{Sp}(1) \times \mathrm{Sp}(q))$	$(2(q-1), 1, 2(q-1), 2)$
$(\mathrm{E}_6, \mathrm{SU}(6) \cdot \mathrm{SU}(2), \mathrm{F}_4)$	$(8, 3, 8, 5)$
$(\mathrm{E}_6, \mathrm{SO}(10) \cdot \mathrm{U}(1), \mathrm{F}_4)$	$(8, 7, 8, 1)$
$(\mathrm{F}_4, \mathrm{Sp}(3) \cdot \mathrm{Sp}(1), \mathrm{Spin}(9))$	$(4, 3, 4, 4)$

Here, we put $\mathrm{U}(4)' = \{g \in \mathrm{SO}(8) \mid JgJ^{-1} = g\}$, where

$$J = \left[\begin{array}{cc|cc} & & I_3 & \\ & & & -1 \\ \hline -I_3 & & & \\ & 1 & & \end{array}\right]$$

and I_l denotes the identity matrix of $l \times l$.

Summing up the above, we obtain the following theorem.

THEOREM 9. *Let (G, K_1, K_2) be a commutative compact symmetric triad where G is simple, and suppose that the $(K_2 \times K_1)$-action on G is of cohomogeneity one. Then all the proper biharmonic hypersurfaces which are regular orbits of the $(K_2 \times K_1)$-action in the compact Lie group G are classified into the following list:*

(1) *When (G, K_1, K_2) is one of the following cases, there exist exactly two distinct proper biharmonic hypersurfaces which are regular orbits of the $(K_2 \times K_1)$-action on G.*
- (1-1) $(\mathrm{SO}(1+b+c), \mathrm{SO}(1+b) \times \mathrm{SO}(c), \mathrm{SO}(b+c))$ $(b > 0,\ c > 1)$
- (1-2) $(\mathrm{SU}(4), \mathrm{Sp}(2), \mathrm{SO}(4))$
- (1-3) $(\mathrm{SU}(4), \mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(2)), \mathrm{Sp}(2))$
- (1-4) $(\mathrm{Sp}(2), \mathrm{U}(2), \mathrm{Sp}(1) \times \mathrm{Sp}(1))$
- (1-5) $(\mathrm{SO}(2+2q), \mathrm{SO}(2) \times \mathrm{SO}(2q), \mathrm{U}(1+q))$ $(q > 1)$
- (1-6) $(\mathrm{SU}(1+b+c), \mathrm{S}(\mathrm{U}(1+b) \times \mathrm{U}(c)), \mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(b+c))$ $(b \geq 0,\ c > 1)$
- (1-7) $(\mathrm{Sp}(1+b+c), \mathrm{Sp}(1+b) \times \mathrm{Sp}(c), \mathrm{Sp}(1) \times \mathrm{Sp}(b+c))$ $(b \geq 0,\ c > 1)$
- (1-8) $(\mathrm{SO}(8), \mathrm{U}(4), \mathrm{U}(4)')$
- (1-9) $(\mathrm{SU}(1+q), \mathrm{SO}(1+q), \mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(q)))$ $(q > 52)$
- (1-10) $(\mathrm{SU}(2+2q), \mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(2q)), \mathrm{Sp}(1+q))$ $(q > 1)$
- (1-11) $(\mathrm{Sp}(1+q), \mathrm{U}(1+q), \mathrm{Sp}(1) \times \mathrm{Sp}(q))$ $(q = 2$ *or* $q > 45)$
- (1-12) $(\mathrm{E}_6, \mathrm{SO}(10) \cdot \mathrm{U}(1), \mathrm{F}_4)$
- (1-13) $(\mathrm{F}_4, \mathrm{Sp}(3) \cdot \mathrm{Sp}(1), \mathrm{Spin}(9))$
- (1-14) $(\mathrm{SO}(1+q), \mathrm{SO}(q), \mathrm{SO}(q))$ $(q > 1)$
- (1-15) $(\mathrm{F}_4, \mathrm{Spin}(9), \mathrm{Spin}(9))$

(2) *When (G, K_1, K_2) is one of the following cases, any biharmonic regular orbit of the $(K_2 \times K_1)$-action on G is harmonic.*
- (2-1) $(\mathrm{SO}(6), \mathrm{U}(3), \mathrm{SO}(3) \times \mathrm{SO}(3))$
- (2-2) $(\mathrm{SU}(1+q), \mathrm{SO}(1+q), \mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(q)))$ $(52 \geq q > 1)$
- (2-3) $(\mathrm{Sp}(1+q), \mathrm{U}(1+q), \mathrm{Sp}(1) \times \mathrm{Sp}(q))$ $(45 \geq q > 2)$
- (2-4) $(\mathrm{E}_6, \mathrm{SU}(6) \cdot \mathrm{SU}(2), \mathrm{F}_4)$

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Shinji Ohno
Osaka City University
Advanced Mathematical Institute (OCAMI)
3-3-138, *Sugimoto, Sumiyoshi-ku*
Osaka, 558-8585 *Japan*
(Present address: Department of Mathematics College of Humanities and Sciences, Nihon University
3-25-40, *Sakurajosui, Setagaya*
Tokyo, 156-8550 *Japan)*
E-mail: kudamono.shinji@gmail.com

Takashi Sakai
Department of Mathematics and Information Sciences
Tokyo Metropolitan University
Minami-Osawa 1-1, *Hachioji*
Tokyo, 192-0397, *Japan*

Hajime Urakawa
Institute for International Education
Tohoku University
Kawauchi 41
Sendai, 980-8576, *Japan*