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Pseudo-Einstein unit tangent sphere bundles

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ABSTRACT. In the present paper, we study the pseudo-Hermitian almost CR structure of unit tangent sphere bundle T_1M over a Riemannian manifold M. Then we prove that if the unit tangent sphere bundle T_1M is pseudo-Einstein, that is, the pseudo-Hermitian Ricci tensor is proportional to the Levi form, then the base manifold M is Einstein. Moreover, when dim $M = 3$ or 4, we prove that T_1M is pseudo-Einstein if and only if M is of constant curvature 1.

1. Introduction

It is well-known that the unit tangent sphere bundle T_1M over a Riemannian manifold M admits a pseudo-Hermitian, strictly pseudo-convex, almost CR structure (η, L) (or (η, J)), where L is the *Levi form* associated with an endomorphism J on $D(=$ kernel of $\eta)$ such that $J^2 = -id$. Here, J defines an almost CR structure $\mathscr{H} = \{\overline{X} - iJ\overline{X} : \overline{X} \in \Gamma(D)\}$, that is $\mathscr{H} \cap \overline{\mathscr{H}} = \{0\}$. We say that the almost CR structure is integrable if $[\mathscr{H}, \mathscr{H}] \subset \mathscr{H}$. For complex analytical considerations, it is desirable to have integrability of the almost complex structure J (on D). If this is the case, we speak of an *(integrable) CR structure* and of a *CR manifold*. Indeed, S. Webster ([16]) introduced the term *pseudo-Hermitian structure* for a CR manifold with a non-degenerate Levi-form. In earlier works [3], [5], [7], we started the intriguing study of the interactions between the contact metric structure and the contact strictly pseudo-convex almost CR structure. In the present paper, we treat the pseudo-Hermitian structure on T_1M as an extension to the case of non-integrable $\mathscr{H}$.

There is a canonical affine connection in a non-degenerate CR manifold, the so-called pseudo-Hermitian connection (or the Tanaka-Webster connection). S. Tanno ([15]) extends the Tanaka-Webster connection for strictly pseudo-convex almost CR manifolds (in which $\mathscr{H}$ is in general non-integrable). We call it the *generalized Tanaka-Webster connection*.

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We define the *pseudo-Hermitian Ricci curvature tensor* in a strictly pseudo-convex almost CR manifold $(\overline{M};\eta,J)$ by

$$\hat{\rho}(\overline{X},\overline{Y}) = \text{trace of } \{\overline{V} \mapsto \hat{R}(\overline{V},\overline{X})\overline{Y}\},$$

where $\overline{X}$, $\overline{Y}$ and $\overline{V}$ are any vector fields on $\overline{M}$.

If the pseudo-Hermitian Ricci curvature tensor is proportional to the Levi form in a strictly pseudo-convex almost CR manifold, then it is said to have the *pseudo-Einstein structure*. In Section 3, we obtain the pseudo-Hermitian curvature tensor and the pseudo-Hermitian Ricci curvature tensor (of generalized Tanaka-Webster connection) on T_1M. In Section 4, we prove that T_1M is pseudo-Einstein, then M is Einstein (Theorem 4). Moreover, when $\dim M = 3$ or 4, we prove that T_1M is pseudo-Einstein if and only if M is of constant curvature 1 (Corollary 5 and Theorem 6).

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2. Preliminaries

First, we review some fundamental facts on contact metric manifolds. We refer to [1] for more details. All manifolds are assumed to be connected and of class C^∞. A $(2n-1)$-dimensional manifold $\overline{M}$ is said to be an *almost contact manifold* if its structure group of the linear frame bundle is reducible to $U(n-1)\times\{1\}$. This is equivalent to the existence of a $(1,1)$-tensor field ϕ, a vector field ξ and a 1-form η satisfying

$$\eta(\xi) = 1 \qquad \text{and} \qquad \phi^2 = -id + \eta\otimes\xi. \tag{1}$$

Here (ϕ,ξ,η) is called an *almost contact structure*. Then one can always find a compatible Riemannian metric $\bar{g}$:

$$\bar{g}(\phi\overline{X},\phi\overline{Y}) = \bar{g}(\overline{X},\overline{Y}) - \eta(\overline{X})\eta(\overline{Y}) \tag{2}$$

for all vector fields $\overline{X}$ and $\overline{Y}$ on $\overline{M}$. Such a metric is called an *associated metric* and $(\overline{M},\phi,\xi,\eta,\bar{g})$ is said to be an *almost contact metric manifold*. The *fundamental* 2-*form* Φ is defined by $\Phi(\overline{X},\overline{Y}) = \bar{g}(\overline{X},\phi\overline{Y})$. If $\overline{M}$ satisfies in addition $d\eta = \Phi$, then $\overline{M}$ is called a *contact metric manifold*, where d is the exterior differential operator. We call the structure vector field ξ the *Reeb vector field* or the *characteristic vector field*. From (1) and (2) it follows that

$$\phi\xi = 0, \qquad \eta\circ\phi = 0, \qquad \eta(\overline{X}) = \bar{g}(\overline{X},\xi). \tag{3}$$

Given a contact metric manifold $\overline{M}$, we define the *structural operator* h by $h = \frac{1}{2}L_\xi\phi$, where L_ξ denotes Lie differentiation for ξ. Then we may observe that h is self-adjoint and satisfies

$$h\xi = 0 \qquad \text{and} \qquad h\phi = -\phi h, \tag{4}$$

$$\overline{\nabla}_{\overline{X}}\xi = -\phi\overline{X} - \phi h\overline{X}, \tag{5}$$

where $\overline{\nabla}$ is the Levi-Civita connection on $\overline{M}$. From (4) and (5) we see that each trajectory of ξ is a geodesic. For a contact metric manifold $\overline{M}$ one may define naturally an almost complex structure $\tilde{J}$ on $\overline{M} \times \mathbb{R}$;

$$\tilde{J}\left(\overline{X}, f\frac{d}{dt}\right) = \left(\phi\overline{X} - f\xi, \eta(\overline{X})\frac{d}{dt}\right),$$

where $\overline{X}$ is a vector field tangent to $\overline{M}$, t the coordinate of $\mathbb{R}$ and f a function on $\overline{M} \times \mathbb{R}$. If the almost complex structure $\tilde{J}$ is integrable, $\overline{M}$ is said to be normal or Sasakian. It is known that $\overline{M}$ is normal if and only if $\overline{M}$ satisfies

$$[\phi, \phi] + 2\, d\eta \otimes \xi = 0,$$

where $[\phi, \phi]$ is the Nijenhuis torsion of ϕ. A Sasakian manifold is characterized by a condition

$$(\overline{\nabla}_{\overline{X}}\phi)\overline{Y} = \bar{g}(\overline{X}, \overline{Y})\xi - \eta(\overline{Y})\overline{X} \tag{6}$$

for all vector fields $\overline{X}$ and $\overline{Y}$ on $\overline{M}$.

Next, we recall the natural relation of contact metric manifolds with CR manifolds ([3], [5], [7]). For a contact metric manifold $\overline{M}$, the tangent space $T_p\overline{M}$ of $\overline{M}$ at each point $p \in \overline{M}$ is decomposed as the direct sum $T_p\overline{M} = D_p \oplus \{\xi\}_p$, where we denote $D_p = \{v \in T_p\overline{M} \mid \eta(v) = 0\}$. Then $D : p \to D_p$ defines a $(2n-2)$-dimensional distribution orthogonal to ξ, which is called the *contact distribution.* For a given contact metric manifold $\overline{M} = (\overline{M}; \eta, g)$, its associated almost CR-structure is given by the holomorphic subbundle

$$\mathscr{H} = \{\overline{X} - iJ\overline{X} : \overline{X} \in D\}$$

of the complexification $T\overline{M}^{\mathbb{C}}$ of the tangent bundle $T\overline{M}$, where $J = \phi|_D$, the restriction of ϕ to D. We see that each fiber $\mathscr{H}_x$, $x \in \overline{M}$, is of complex dimension $n-1$, $\mathscr{H} \cap \overline{\mathscr{H}} = \{0\}$ and $\mathbb{C}D = \mathscr{H} \oplus \overline{\mathscr{H}}$.

We define the *Levi form* L by

$$L : D \times D \to \mathscr{F}(\overline{M}), \qquad L(\overline{X}, \overline{Y}) = -d\eta(\overline{X}, J\overline{Y}),$$

where $\mathscr{F}(\overline{M})$ denotes the algebra of differential functions on $\overline{M}$. Since $d\eta(\overline{X}, \overline{Y}) = \bar{g}(\overline{X}, \phi\overline{Y})$ on $\overline{M}$, the Levi form is Hermitian and positive definite. So, the pair (η, L) is a *strictly pseudo-convex (pseudo-Hermitian) almost CR structure* on $\overline{M}$.

The associated CR structure is *integrable* if $[\mathscr{H}, \mathscr{H}] \subset \mathscr{H}$. This property does not hold for a general contact metric manifold. In terms of the structure tensors, integrability is equivalent to the condition $\Omega = 0$, where Ω is the $(1,2)$-tensor field on $\overline{M}$ defined as

$$\Omega(\overline{X}, \overline{Y}) = (\overline{\nabla}_{\overline{X}}\phi)\overline{Y} - \bar{g}(\overline{X} + h\overline{X}, \overline{Y})\xi + \eta(\overline{Y})(\overline{X} + h\overline{X}) \tag{7}$$

for all vector fields $\overline{X}$ and $\overline{Y}$ on $\overline{M}$ (see [14, Proposition 2.1]). In this case, the pair (η, L) is called a *strictly pseudo-convex (integrable) CR structure* and $(\overline{M}; \eta, L)$ is called a *strictly pseudo-convex CR manifold.* From (6) and (7), we see that the associated CR structure of a Sasakian manifold is strictly pseudo-convex integrable. The same is true for the associated CR structure of any three-dimensional contact metric space.

We review the *generalized Tanaka-Webster connection* $\hat{\nabla}$ ([14]) on a contact strictly pseudo-convex almost CR manifold $\overline{M} = (\overline{M}; \eta, L)$. It is defined by

$$\hat{\nabla}_{\overline{X}}\overline{Y} = \overline{\nabla}_{\overline{X}}\overline{Y} + \eta(\overline{X})\phi\overline{Y} + (\overline{\nabla}_{\overline{X}}\eta)(\overline{Y})\xi - \eta(\overline{Y})\overline{\nabla}_{\overline{X}}\xi$$

for all vector fields $\overline{X}$ and $\overline{Y}$ on $\overline{M}$. Together with (5), $\hat{\nabla}$ may be rewritten as

$$\hat{\nabla}_{\overline{X}}\overline{Y} = \overline{\nabla}_{\overline{X}}\overline{Y} + A(\overline{X}, \overline{Y}), \tag{8}$$

where we put

$$A(\overline{X}, \overline{Y}) = \eta(\overline{X})\phi\overline{Y} + \eta(\overline{Y})(\phi\overline{X} + \phi h\overline{X}) - \bar{g}(\phi\overline{X} + \phi h\overline{X}, \overline{Y})\xi. \tag{9}$$

We see that the generalized Tanaka-Webster connection $\hat{\nabla}$ has the torsion

$$\hat{T}(\overline{X}, \overline{Y}) = 2\bar{g}(\overline{X}, \phi\overline{Y})\xi + \eta(\overline{Y})\phi h\overline{X} - \eta(\overline{X})\phi h\overline{Y}. \tag{10}$$

In particular, for a K-contact manifold we get

$$A(\overline{X}, \overline{Y}) = \eta(\overline{X})\phi\overline{Y} + \eta(\overline{Y})\phi\overline{X} - \bar{g}(\phi\overline{X}, \overline{Y})\xi. \tag{11}$$

The generalized Tanaka-Webster connection can also be characterized differently.

Proposition 1 ([14]). *The generalized Tanaka-Webster connection $\hat{\nabla}$ on a contact metric manifold $\overline{M} = (\overline{M}; \eta, g)$ is the unique linear connection satisfying the following conditions:*

(i) $\hat{\nabla}\eta = 0$, $\hat{\nabla}\xi = 0$;
(ii) $\hat{\nabla}g = 0$;
(iii-1) $\hat{T}(\overline{X}, \overline{Y}) = 2L(\overline{X}, J\overline{Y})\xi$, $\overline{X}, \overline{Y} \in D$;
(iii-2) $\hat{T}(\xi, \phi\overline{Y}) = -\phi\hat{T}(\xi, \overline{Y})$, $\overline{Y} \in D$;
(iv) $(\hat{\nabla}_{\overline{X}}\phi)\overline{Y} = \Omega(\overline{X}, \overline{Y})$, $\overline{X}, \overline{Y} \in TM$.

We note that the Tanaka-Webster connection ([13], [16]) was originally defined for a non-degenerate integrable CR manifold, in which case condition (iv) reduces to $\hat{\nabla} J = 0$.

The curvature tensor $\hat{R}$ of generalized Tanaka-Webster connection $\hat{\nabla}$ is defined by $\hat{R}(\overline{X}, \overline{Y})\overline{Z} = [\hat{\nabla}_{\overline{X}}, \hat{\nabla}_{\overline{Y}}]\overline{Z} - \hat{\nabla}_{[\overline{X}, \overline{Y}]}\overline{Z}$ for all vector fields $\overline{X}$, $\overline{Y}$ and $\overline{Z}$ on $\overline{M}$. First we have quite generally

Proposition 2.

$$\hat{R}(\overline{X}, \overline{Y})\overline{Z} = -\hat{R}(\overline{Y}, \overline{X})\overline{Z},$$

$$L(\hat{R}(\overline{X}, \overline{Y})\overline{Z}, \overline{W}) = -L(\hat{R}(\overline{X}, \overline{Y})\overline{W}, \overline{Z}).$$

The first identity follows trivially from the definition of $\hat{R}$. Since the connection is metrical with respect to its associated metric g, $\hat{\nabla} g = 0$, the second identity is proved in a similar way as for the case of Riemanian curvature tensor. Since the generalized Tanaka-Webster connection is not torsion-free, the Jacobi- or Bianchi-identies do not hold, in general. Before we study the curvature tensor $\hat{R}$, from (4), (8) and (9) we have

$$\begin{aligned}(\hat{\nabla}_{\overline{X}} h)\overline{Y} &= (\overline{\nabla}_{\overline{X}} h)\overline{Y} + A(\overline{X}, h\overline{Y}) - hA(\overline{X}, \overline{Y}) \\ &= (\overline{\nabla}_{\overline{X}} h)\overline{Y} + 2\eta(\overline{X})\phi h\overline{Y} + \bar{g}((\phi h + \phi h^2)\overline{X}, \overline{Y})\xi \\ &\quad + \eta(\overline{Y})(\phi h\overline{X} + \phi h^2\overline{X}). \end{aligned} \tag{12}$$

We denote by $\overline{R}$ the Riemannian curvature tensor of $\overline{M}$. Then, from the definition of $\hat{R}$, together with (8), taking account of $\hat{\nabla}\eta = 0$, $\hat{\nabla}\xi = 0$, $\hat{\nabla} g = 0$ and (12), straightforward computations yield

$$\begin{aligned}\hat{R}(\overline{X}, \overline{Y})\overline{Z} = \overline{R}(\overline{X}, \overline{Y})\overline{Z} &\\ &+ \eta(\overline{Z})(\Omega(\overline{X}, \overline{Y}) - \Omega(\overline{Y}, \overline{X}) + \Omega(\overline{X}, h\overline{Y}) - \Omega(\overline{Y}, h\overline{X}) \\ &+ \phi P(\overline{X}, \overline{Y}) + \phi(A(\overline{X}, \overline{Y}) - A(\overline{Y}, \overline{X})) \\ &+ \phi(A(\overline{X}, h\overline{Y}) - A(\overline{Y}, h\overline{X}))) \\ &- \bar{g}(\Omega(\overline{X}, \overline{Y}) - \Omega(\overline{Y}, \overline{X}) + \Omega(\overline{X}, h\overline{Y}) - \Omega(\overline{Y}, h\overline{X}) \\ &+ \phi P(\overline{X}, \overline{Y}) + \phi(A(\overline{X}, \overline{Y}) - A(\overline{Y}, \overline{X})) \\ &+ \phi(A(\overline{X}, h\overline{Y}) - A(\overline{Y}, h\overline{X})), \overline{Z})\xi \\ &- 2\bar{g}(\phi\overline{X}, \overline{Y})\phi\overline{Z} - \eta(\overline{X})(\Omega(\overline{Y}, \overline{Z}) + \phi A(\overline{Y}, \overline{Z})) \\ &+ \eta(\overline{Y})(\Omega(\overline{X}, \overline{Z}) + \phi A(\overline{X}, \overline{Z}))\end{aligned}$$

$$+\eta(A(\bar{X},\bar{Z}))(\phi\bar{Y}+\phi h\bar{Y})-\eta(A(\bar{Y},\bar{Z}))(\phi\bar{X}+\phi h\bar{X})$$
$$+\bar{g}(\phi\bar{X}+\phi h\bar{X},A(\bar{Y},\bar{Z}))\xi-\bar{g}(\phi\bar{Y}+\phi h\bar{Y},A(\bar{X},\bar{Z}))\xi, \quad (13)$$

where we put $P(\bar{X},\bar{Y})=(\bar{\nabla}_{\bar{X}}h)\bar{Y}-(\bar{\nabla}_{\bar{Y}}h)\bar{X}$. By using (3), (4) and (9), we have

$$\hat{R}(\bar{X},\bar{Y})\bar{Z}=\bar{R}(\bar{X},\bar{Y})\bar{Z}+B(\bar{X},\bar{Y})\bar{Z}, \quad (14)$$

where

$$\begin{aligned}B(\bar{X},\bar{Y})\bar{Z}&=\eta(\bar{Z})(\Omega(\bar{X},\bar{Y})-\Omega(\bar{Y},\bar{X})+\Omega(\bar{X},h\bar{Y})-\Omega(\bar{Y},h\bar{X})+\phi P(\bar{X},\bar{Y}))\\&\quad-\bar{g}(\Omega(\bar{X},\bar{Y})-\Omega(\bar{Y},\bar{X})+\Omega(\bar{X},h\bar{Y})-\Omega(\bar{Y},h\bar{X})+\phi P(\bar{X},\bar{Y}),\bar{Z})\xi\\&\quad-\eta(\bar{Z})\{\eta(\bar{Y})(\bar{X}+h\bar{X})-\eta(\bar{X})(\bar{Y}+h\bar{Y})\}\\&\quad-\eta(\bar{X})\Omega(\bar{Y},\bar{Z})+\eta(\bar{Y})\Omega(\bar{X},\bar{Z})\\&\quad+\eta(\bar{Y})g(\bar{X}+h\bar{X},\bar{Z})\xi-\eta(\bar{X})g(\bar{Y}+h\bar{Y},\bar{Z})\xi\\&\quad+\bar{g}(\phi\bar{Y}+\phi h\bar{Y},\bar{Z})(\phi\bar{X}+\phi h\bar{X})-\bar{g}(\phi\bar{X}+\phi h\bar{X},Z)(\phi\bar{Y}+\phi h\bar{Y})\\&\quad-2\bar{g}(\phi\bar{X},\bar{Y})\phi\bar{Z} \qquad (15)\end{aligned}$$

for all vector fields $\bar{X}$, $\bar{Y}$ and $\bar{Z}$ on $\bar{M}$. The pseudo-Hermitian Ricci curvature tensor $\hat{\rho}$ is given by

$$\hat{\rho}(\bar{X},\bar{Y})=\bar{\rho}(\bar{X},\bar{Y})+\sum_{i=1}^{2n-1}\bar{g}(B(E_i,\bar{X})\bar{Y},E_i), \quad (16)$$

where $\{E_i\}$ $(1\le i\le 2n-1)$ is an orthonormal basis on $\bar{M}$ and $\bar{\rho}$ denotes the Ricci curvature tensor of the Levi-Civita connection.

Definition 1 ([6]). Let $(\bar{M};\eta,J)$ be a strictly pseudo-convex almost CR manifold. Then the pseudo-Hermitian structure (η,J) is said to be pseudo-Einstein if the pseudo-Hermitian Ricci tensor is proportional to the Levi form, namely,

$$\hat{\rho}(\bar{X},\bar{Y})=\lambda L(\bar{X},\bar{Y}),$$

where $\bar{X},\bar{Y}\in\Gamma(D)$ and $\lambda=\hat{r}/(2n-2)$. Here $\hat{r}$ is the scalar curvature of generalized Tanaka-Webster connection.

3. Unit tangent sphere bundles

The basic facts and fundamental formulas about tangent bundle and unit tangent sphere bundle are well-known ([2], [7], [8]). Let (M,g) be an

n-dimensional Riemannian manifold and ∇ the associated Levi-Civita connection. The tangent bundle over (M, g) is denoted by TM and consists of pairs (p, u), where p is a point in M and u a tangent vector to M at p. The mapping $\pi : TM \to M$, $\pi(p, u) = p$, is the natural projection from TM onto M. For a vector field X on M, its *vertical lift* X^v on TM is the vector field defined by $X^v \omega = \omega(X) \circ \pi$, where ω is a 1-form on M. For the Levi-Civita connection ∇ on M, the *horizontal lift* X^h of X is defined by $X^h \omega = \nabla_X \omega$. The tangent bundle TM can be endowed in a natural way with a Riemannian metric $\tilde{g}$, the so-called *Sasaki metric*, depending only on the Riemannian metric g on M. It is determined by

$$\tilde{g}(X^h, Y^h) = \tilde{g}(X^v, Y^v) = g(X, Y) \circ \pi, \qquad \tilde{g}(X^h, Y^v) = 0$$

for all vector fields X and Y on M. Also, TM admits an almost complex structure tensor J defined by $JX^h = X^v$ and $JX^v = -X^h$. Then $\tilde{g}$ is a Hermitian metric for the almost complex structure J.

The unit tangent sphere bundle $\bar{\pi} : T_1M \to M$ is a hypersurface of TM given by $g_p(u, u) = 1$. Note that $\bar{\pi} = \pi \circ i$, where i is the immersion of T_1M into TM. A unit normal vector field $N = u^v$ to T_1M is given by the vertical lift of u for (p, u). The horizontal lift of a vector is tangent to T_1M, but the vertical lift of a vector is not tangent to T_1M in general. So, we define the *tangential lift* of X to $(p, u) \in T_1M$ by

$$X^t_{(p,u)} = (X - g(X, u)u)^v.$$

Clearly, the tangent space $T_{(p,u)}T_1M$ is spanned by vectors of the form X^h and X^t, where $X \in T_pM$. We now define the standard contact metric structure of the unit tangent sphere bundle T_1M over a Riemannian manifold (M, g). The metric g' on T_1M is induced from the Sasaki metric $\tilde{g}$ on TM. Using the almost complex structure J on TM, we define a unit vector field ξ', a 1-form η' and a $(1,1)$-tensor field ϕ' on T_1M by

$$\xi' = -JN, \qquad \phi' = J - \eta' \otimes N.$$

Since $g'(\bar{X}, \phi'\bar{Y}) = 2\, d\eta'(\bar{X}, \bar{Y})$, (η', g', ϕ', ξ') is not a contact metric structure. If we rescale this structure by

$$\xi = 2\xi', \qquad \eta = \frac{1}{2}\eta', \qquad \phi = \phi', \qquad \bar{g} = \frac{1}{4}g',$$

we get the standard contact metric structure $(\eta, \bar{g}, \phi, \xi)$. Here the tensor ϕ is explicitly given by

$$\phi X^t = -X^h + \frac{1}{2}g(X, u)\xi, \qquad \phi X^h = X^t, \tag{17}$$

where X and Y are vector fields on M. From now on, we consider $T_1M = (T_1M, \eta, \bar{g}, \phi, \xi)$ with the standard contact metric structure. The Levi-Civita connection $\bar{\nabla}$ of T_1M is described by

$$\begin{aligned}
\bar{\nabla}_{X^t} Y^t &= -g(Y,u)X^t,\\
\bar{\nabla}_{X^t} Y^h &= \frac{1}{2}(R(u,X)Y)^h,\\
\bar{\nabla}_{X^h} Y^t &= (\nabla_X Y)^t + \frac{1}{2}(R(u,Y)X)^h,\\
\bar{\nabla}_{X^h} Y^h &= (\nabla_X Y)^h - \frac{1}{2}(R(X,Y)u)^t
\end{aligned} \tag{18}$$

for all vector fields X and Y on M. The Riemann curvature tensor $\bar{R}$ of T_1M is given by

$$\begin{aligned}
\bar{R}(X^t, Y^t)Z^t &= -(g(X,Z) - g(X,u)g(Z,u))Y^t\\
&\quad + (g(Y,Z) - g(Y,u)g(Z,u))X^t,\\
\bar{R}(X^t, Y^t)Z^h &= \{R(X - g(X,u)u, Y - g(Y,u)u)Z\}^h\\
&\quad + \frac{1}{4}\{[R(u,X), R(u,Y)]Z\}^h,\\
\bar{R}(X^h, Y^t)Z^t &= -\frac{1}{2}\{R(Y - g(Y,u)u, Z - g(Z,u)u)X\}^h\\
&\quad - \frac{1}{4}\{R(u,Y)R(u,Z)X\}^h,\\
\bar{R}(X^h, Y^t)Z^h &= \frac{1}{2}\{R(X,Z)(Y - g(Y,u)u)\}^t - \frac{1}{4}\{R(X, R(u,Y)Z)u\}^t\\
&\quad + \frac{1}{2}\{(\nabla_X R)(u,Y)Z\}^h,\\
\bar{R}(X^h, Y^h)Z^t &= \{R(X,Y)(Z - g(Z,u)u)\}^t\\
&\quad + \frac{1}{4}\{R(Y, R(u,Z)X)u - R(X, R(u,Z)Y)u\}^t\\
&\quad + \frac{1}{2}\{(\nabla_X R)(u,Z)Y - (\nabla_Y R)(u,Z)X\}^h,\\
\bar{R}(X^h, Y^h)Z^h &= (R(X,Y)Z)^h + \frac{1}{2}\{R(u, R(X,Y)u)Z\}^h
\end{aligned} \tag{19}$$

$$-\frac{1}{4}\{R(u, R(Y, Z)u)X - R(u, R(X, Z)u)Y\}^h$$
$$+\frac{1}{2}\{(\nabla_Z R)(X, Y)u\}^t$$

for all vector fields X, Y and Z on M.

Now, using (14) and (15), we calculate the curvature tensor $\hat{R}$ of generalized Tanaka-Webster connection of T_1M. Then we have

$$\hat{R}(X^t, Y^t)Z^t = \bar{R}(X^t, Y^t)Z^t,$$

$$\begin{aligned}
\hat{R}(X^t, Y^t)Z^h \\
&= \bar{R}(X^t, Y^t)Z^h - g(X, Z)\left(Y^h - \frac{1}{2}g(Y, u)\xi - \frac{1}{2}(R_u Y)^h\right) \\
&\quad + g(Y, Z)\left(X^h - \frac{1}{2}g(X, u)\xi - \frac{1}{2}(R_u X)^h\right) + \frac{1}{2}g(R_u X, Z)\left(Y^h - \frac{1}{2}(R_u Y)^h\right) \\
&\quad - \frac{1}{2}g(R_u Y, Z)\left(X^h - \frac{1}{2}(R_u X)^h\right) \\
&\quad - g(Z, u)\left\{(R(X, Y)u)^h - \frac{1}{4}(R(u, X)R_u Y)^h + \frac{1}{4}(R(u, Y)R_u X)^h\right. \\
&\quad \left. - g(X, u)\left(Y^h - \frac{3}{2}(R_u Y)^h\right) + g(Y, u)\left(X^h - \frac{3}{2}(R_u X)^h\right)\right\} \\
&\quad + \left\{\frac{1}{2}g(R(X, Y)u, Z) - \frac{1}{8}g(R(u, X)R_u Y, Z) + \frac{1}{8}g(R(u, Y)R_u X, Z)\right. \\
&\quad \left. + \frac{3}{4}g(X, u)g(R_u Y, Z) - \frac{3}{4}g(Y, u)g(R_u X, Z)\right\}\xi,
\end{aligned}$$

$$\begin{aligned}
\hat{R}(X^h, Y^t)Z^t \\
&= \bar{R}(X^h, Y^t)Z^t + \frac{1}{2}(g(X, Y) - g(X, u)g(Y, u))\left(Z^h - \frac{1}{2}g(Z, u)\xi\right) \\
&\quad + \frac{1}{4}g(X, u)\{(R(u, Y)Z)^h + g(Z, u)(R_u Y)^h\} \\
&\quad + \frac{1}{4}g(R_u X, Z)\{2Y^h - g(Y, u)\xi - (R_u Y)^h\} \\
&\quad + \frac{1}{4}\left\{g(R(X, u)Y, Z) + g(Z, u)g(R_u X, Y) - g(Y, u)g(R_u X, Z)\right. \\
&\quad \left. - \frac{1}{2}g(X, u)g(R_u Y, Z) + \frac{1}{2}g(R(X, R_u Y)u, Z)\right\}\xi, \qquad (20)
\end{aligned}$$

$$\hat{R}(X^h, Y^t)Z^h$$
$$= \bar{R}(X^h, Y^t)Z^h - \frac{1}{2}(g(X, Y) - g(X,u)g(Y,u))Z^t + \frac{1}{4}g(X,u)(R(u, Y)Z)^t$$
$$- \frac{1}{2}\left\{g(Y,Z) - g(Y,u)g(Z,u) - \frac{1}{2}g(R_u Y, Z)\right\}(R_u X)^t$$
$$- \frac{1}{4}g(Z,u)\{2(R(X,u)Y)^t + (R(X, R_u Y)u)^t - g(X,u)(R_u Y)^t$$
$$- 2((\nabla_X R)(Y,u)u)^h\} - \frac{1}{4}g((\nabla_X R)(Y,u)u, Z)\xi,$$
$$\hat{R}(X^h, Y^h)Z^t$$
$$= \bar{R}(X^h, Y^h)Z^t + \frac{1}{4}g(Y,u)\{(R(u,X)Z)^t - g(Z,u)(R_u X)^t\}$$
$$- \frac{1}{4}g(X,u)\{(R(u, Y)Z)^t - g(Z,u)(R_u Y)^t\} - \frac{1}{4}g(R_u X, Z)(R_u Y)^t$$
$$+ \frac{1}{4}g(R_u Y, Z)(R_u X)^t - \frac{1}{4}\{g((\nabla_X R)(Y,u)u, Z) - g((\nabla_Y R)(X,u)u, Z)\}\xi,$$
$$\hat{R}(X^h, Y^h)Z^h$$
$$= \bar{R}(X^h, Y^h)Z^h + \frac{1}{4}g(Y,u)(R(u,X)Z)^h - \frac{1}{4}g(X,u)(R(u, Y)Z)^h$$
$$- \frac{1}{2}g(Z,u)\left\{2(R(X, Y)u)^h - (R_u(R(X, Y)u))^h - \frac{1}{2}(R(u, R_u Y)X)^h\right.$$
$$+ \frac{1}{2}(R(u, R_u X)Y)^h + \frac{1}{2}g(X,u)(R_u Y)^h - \frac{1}{2}g(Y,u)(R_u X)^h$$
$$\left. - ((\nabla_X R)(Y,u)u)^t + ((\nabla_Y R)(X,u)u)^t\right\}$$
$$+ \frac{1}{8}\{4g(R(X, Y)u, Z) - g(R(u, R_u Y)X, Z) + g(R(u, R_u X)Y, Z)$$
$$- 2g(R_u(R(X, Y)u), Z) + g(X,u)g(R_u Y, Z) - g(Y,u)g(R_u X, Z)\}\xi$$

for all vector fields X, Y and Z on M. From (19) and (20), we have the pseudo-Hermitian Ricci curvature tensor $\hat{\rho}$ of $T_1 M$

$$\hat{\rho}(X^t, Y^t) = \left(n - \frac{3}{2}\right)(g(X, Y) - g(X,u)g(Y,u)) + \frac{1}{4}\sum_{i=1}^{n} g(R(u, X)e_i, R(u, Y)e_i)$$
$$+ \frac{1}{2}g(R_u X, Y) - \frac{1}{2}g(R_u^2 X, Y),$$

$$
\begin{aligned}
\hat{\rho}(X^t, Y^h) &= \frac{1}{2}\{(\nabla_u\rho)(X, Y) - (\nabla_X\rho)(u, Y)\} \\
&\quad - \frac{1}{2}g(Y, u)\{(\nabla_u\rho)(X, u) - (\nabla_X\rho)(u, u)\} - \frac{1}{2}g(R'_u X, Y), \\
\hat{\rho}(X^h, Y^t) &= \frac{1}{2}\{(\nabla_u\rho)(X, Y) - (\nabla_Y\rho)(u, X)\} - \frac{1}{2}g(R'_u X, Y), \\
\hat{\rho}(X^h, Y^h) &= \rho(X, Y) + \frac{1}{2}(g(X, Y) - g(X, u)g(Y, u)) - g(Y, u)\rho(X, u) \\
&\quad - \frac{1}{2}\sum_{i=1}^{n} g(R(u, e_i)X, R(u, e_i)Y) + \frac{1}{2}g(Y, u)\sum_{i=1}^{n} g(R(u, e_i)X, R(u, e_i)u) \\
&\quad - \frac{1}{2}g(R_u X, Y) + \frac{1}{2}g(R_u^2 X, Y) \qquad (21)
\end{aligned}
$$

for all vector fields X, Y and Z on M.

4. Pseudo-Einstein unit tangent sphere bundles

In this section, we study the pseudo-Einstein structure of unit tangent sphere bundle T_1M. First, we prove

THEOREM 1. *Let $M = (M, g)$ be an n-dimensional Riemannian manifold of constant curvature c and let T_1M be the unit tangent sphere bundle with the standard contact metric structure $(\eta, \bar{g}, \phi, \xi)$ over M. Then T_1M is pseudo-Einstein if and only if M is a* 2*-dimensional manifold or a space of constant curvature* 1.

PROOF. Let M be a space of constant curvature c and T_1M has pseudo-Einstein structure, i.e., $\hat{\rho}(\overline{X}, \overline{Y}) = \lambda\bar{g}(\overline{X}, \overline{Y})$ for any vector fields $\overline{X}$ and $\overline{Y}$ orthogonal to ξ. Then from the definition of pseudo-Einstein and (21), we have two equations;

$$n + \frac{c}{2} - \frac{3}{2} - \frac{\lambda}{4} = 0, \qquad (22)$$

$$cn - \frac{3}{2}c + \frac{1}{2} - \frac{\lambda}{4} = 0. \qquad (23)$$

From the above two equations, we obtain $n = 2$ or $c = 1$. Using (21), the converse is easily proved. □

THEOREM 2. *Let M be an $n(\geq 3)$-dimensional Riemannian manifold and let T_1M be the unit tangent sphere bundle with the standard contact metric* *structure* $(\eta, \bar{g}, \phi, \xi)$ *over* M. *If* T_1M *admits a pseudo-Einstein structure, then* M *is Einstein.*

Proof. Suppose that T_1M admits a pseudo-Einstein structure. Then from (21), we obtain two equations;

$$\left(n-\frac{3}{2}-\frac{\lambda}{4}\right)(g(X,Y)-g(X,u)g(Y,u))+\frac{1}{4}\sum_{i=1}^{n} g(R(u,X)e_i, R(u,Y)e_i) + \frac{1}{2}g(R_uX,Y)-\frac{1}{2}g(R_u^2X,Y)=0, \tag{24}$$

$$\rho(X,Y)+\left(\frac{1}{2}-\frac{\lambda}{4}\right)g(X,Y)-\frac{1}{2}g(X,u)g(Y,u)-g(Y,u)\rho(X,u) - \frac{1}{2}\sum_{i=1}^{n} g(R(u,e_i)X, R(u,e_i)Y)+\frac{1}{2}g(Y,u)\sum_{i=1}^{n} g(R(u,e_i)X, R(u,e_i)u) - \frac{1}{2}g(R_uX,Y)+\frac{1}{2}g(R_u^2X,Y)=0. \tag{25}$$

Combining (24) and (25), we have

$$\rho(X,Y)+\left(n-1-\frac{\lambda}{2}\right)g(X,Y)-\left(n-1-\frac{\lambda}{4}\right)g(X,u)g(Y,u)-g(Y,u)\rho(X,u) - \frac{1}{2}\sum_{i=1}^{n} g(R(u,e_i)X, R(u,e_i)Y)+\frac{1}{2}g(Y,u)\sum_{i=1}^{n} g(R(u,e_i)X, R(u,e_i)u) + \frac{1}{4}\sum_{i=1}^{n} g(R(u,X)e_i, R(u,Y)e_i)=0. \tag{26}$$

Let $\{e_i\}$ $(1 \le i \le n)$ be an orthonormal basis of the tangent space of M at any point $p \in M$. Putting $X = Y = e_a$ and $u = e_b$ $(a \neq b)$ in (26), we get

$$\rho_{aa}+\left(n-1-\frac{\lambda}{2}\right)\delta_{aa}-\frac{1}{2}\sum_{i,j=1}^{n}(R_{biaj})^2+\frac{1}{4}\sum_{i,j=1}^{n}(R_{baij})^2=0, \tag{27}$$

where δ_{ab} denotes the Kronecker's delta, $R_{ijkl} = g(R(e_i,e_j)e_k,e_l)$ and $\rho_{ij} = \rho(e_i,e_j)$ for $1 \le i,j,k,l,a,b \le n$. Also, we put $X = Y = e_b$ and $u = e_a$ $(a \neq b)$ in (26). Then we have

$$\rho_{bb}+\left(n-1-\frac{\lambda}{2}\right)\delta_{bb}-\frac{1}{2}\sum_{i,j=1}^{n}(R_{aibj})^2+\frac{1}{4}\sum_{i,j=1}^{n}(R_{abij})^2=0. \tag{28}$$

Comparing (27) and (28), we obtain $\rho_{aa} = \rho_{bb}$ for all a, b $(a \neq b)$, that is, M is Einstein. □

A 3-dimensional Einstein manifold has a constant curvature, by Theorem 1 and Theorem 2, we have the following.

Corollary 1. *Let $M = (M, g)$ be a 3-dimensional Riemannian manifold. Then T_1M is pseudo-Einstein if and only if M is of constant curvature 1.*

Theorem 3. *Let $M = (M, g)$ be a 4-dimensional Riemannian manifold and let T_1M be the unit tangent sphere bundle with the standard contact metric structure $(\eta, \bar{g}, \phi, \xi)$ over M. Then T_1M is pseudo-Einstein if and only if M is of constant curvature 1.*

Proof. From the result of Theorem 2, we see that M is Einstein $(\rho = \alpha g)$. Then we may choose an orthonormal basis $\{e_i\}$ $(1 \leq i \leq 4)$ (known as the Singer-Thorpe basis) at each point $p \in M$ such that

$$\begin{cases} R_{1212} = R_{3434} = \lambda_1, \quad R_{1313} = R_{2424} = \lambda_2, \quad R_{1414} = R_{2323} = \lambda_3, \\ R_{1234} = \mu_1, \quad R_{1342} = \mu_2, \quad R_{1423} = \mu_3, \\ R_{ijkl} = 0 \quad \text{whenever just three of the indices} \\ \quad i, j, k, l \text{ are distinct (cf. [12]).} \end{cases} \tag{29}$$

Note that

$$\mu_1 + \mu_2 + \mu_3 = 0 \tag{30}$$

by the first Bianchi identity and

$$\lambda_1 + \lambda_2 + \lambda_3 = -\frac{\tau}{4}, \tag{31}$$

where τ is the scalar curvature of M.

We put $X = Y = e_1$, $u = e_2$ in (26). Then we have

$$\alpha + 3 - \frac{\lambda}{2} + \frac{1}{2}(\mu_1^2 - \mu_2^2 - \mu_3^2) = 0. \tag{32}$$

Similarly, if we put $X = Y = e_1$, $u = e_3$ in (26), then we have

$$\alpha + 3 - \frac{\lambda}{2} + \frac{1}{2}(\mu_2^2 - \mu_1^2 - \mu_3^2) = 0. \tag{33}$$

We put $X = Y = e_1$, $u = e_4$ in (26) to have

$$\alpha + 3 - \frac{\lambda}{2} + \frac{1}{2}(\mu_3^2 - \mu_1^2 - \mu_2^2) = 0. \tag{34}$$

From (32)~(34) and (30), we obtain $\mu_1 = \mu_2 = \mu_3 = 0$.

On the other hand, if we put $X = Y = e_1$, $u = e_2$ and $X = Y = e_1$, $u = e_3$ in (25), we have

$$\begin{aligned} &\alpha + \frac{1}{2} - \frac{\lambda}{4} + \frac{1}{2}\lambda_1 - \frac{1}{2}(\mu_2^2 + \mu_3^2) = 0, \\ &\alpha + \frac{1}{2} - \frac{\lambda}{4} + \frac{1}{2}\lambda_2 - \frac{1}{2}(\mu_1^2 + \mu_3^2) = 0. \end{aligned} \tag{35}$$

Similarly, put $X = Y = e_1$, $u = e_4$ in (25) to have

$$\alpha + \frac{1}{2} - \frac{\lambda}{4} + \frac{1}{2}\lambda_3 - \frac{1}{2}(\mu_1^2 + \mu_2^2) = 0 \tag{36}$$

Since $\mu_1 = \mu_2 = \mu_3 = 0$, from (31), (35) and (36), we obtain $\lambda_1 = \lambda_2 = \lambda_3 = -\tau/12$. Next, we put $X = Y = e_1$, $u = e_2$ in (24), we have

$$\frac{5}{2} - \frac{\lambda}{4} - \frac{1}{2}\lambda_1 + \frac{1}{2}\mu_1^2 = 0. \tag{37}$$

From (37), we obtain $\lambda = 10 + \tau/6$ and from (36), we see that M is of constant curvature 1. Conversely, if M is of constant curvature 1, then by Theorem 1, we see easily that T_1M has the pseudo-Einstein structure. □

Remark 1. *Some authors adopt the pseudo-Einstein structure in almost contact metric geometry by the condition* $\bar{\rho}(\bar{X}, \bar{Y}) = \alpha\bar{g}(\bar{X}, \bar{Y}) + \beta\eta(\bar{X})\eta(\bar{Y})$ *for some functions* α *and* β (*cf.* [11]). *Indeed, the unit tangent sphere bundle satisfying the above condition* ([4]) *and the related condition* ([9]) *was studied. Another notable notion is the so-called* ϕ*-Einstein structure which is defined in* [10]. *In this context, it is interesting to study the unit tangent sphere bundle with* ϕ*-Einstein structure.*

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